An Implementation of Binomial Method of Option Pricing using Parallel Computing

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Abstract

The Binomial method of option pricing is based on iterating over discounted option payoffs in a recursive fashion to calculate the present value of an option. Implementing the Binomial method to exploit the resources of a parallel computing cluster is non-trivial as the method is not easily parallelizable. We propose a procedure to transform the method into an "embarrassingly parallel" problem by mapping Binomial probabilities to Bernoulli paths. We have used the parallel computing capabilities in `R` with the `Rmpi` package to implement the methodology on the cluster tara in the UMBC High Performance Computing Facility, which has 82 compute nodes with two quad-core Intel Nehalem processors and 24 GB of memory on a quad-data rate InfiniBand interconnect. With high-performance clusters and multi-core desktops becoming increasingly accessible, we believe that our method will have practical appeal to financial trading firms.

1 Introduction

Options are a class of popular financial contracts that fall under the category of financial derivatives, which derive their value from a less complicated, often elementary asset called *underlying*, in addition to other factors. For this paper, it will suffice to think of a stock (e.g. Google trading on NASDAQ) as an asset. Financial derivatives are traded between two parties: a buyer and a seller. A buyer is the one who buys the financial derivative and a seller sells the contract.

An option is the right (but not an obligation) to buy or sell a certain number of shares at a prespecified fixed price within a prespecified time period. In other words, an option allows one to bet on the future movement of a stock. There are two types of options: call and put. A call (put) gives the buyer the right to buy (sell) a certain number of shares at a fixed price within a fixed time period. We denote the prespecified fixed price, called strike price, as K, and the time limit as T, which is also called time to maturity or expiration time. If the buyer of a call (put) decides to buy (sell) the shares at time $t \leq T$, we say that the buyer has chosen to exercise the option. At time $t \leq T$ the buyer can also decide to sell the option itself. The buyer may also choose to not do anything until T, thereby letting the option expire. After T, an option is worthless. An option is called either an American or European depending on the time period during which it can be exercised. An American option can be exercised anytime before T. A European option, on the other hand, can be exercised only at time T. Since an American option gives more flexibility to the buyer, it typically tends to be more expensive than its European counterpart.

The value of an option is the amount a seller (buyer) is willing to receive (pay) when the option is sold (bought). Valuation of an option is not a trivial problem to solve as the future movements of a stock are stochastic. Intuitively, for a call, the closer the current stock price is to K, the higher is the chance that it might exceed K at T, and therefore one would be willing to pay a higher value *now* to buy the call. Intuition also tells us that the call's value must also depend on the time remaining before it expires as more time to maturity means higher probability of the stock exceeding K at T. Similar arguments can be made for a put. Therefore, the value of an option depends on two factors (in addition to others, which will be mentioned later): current stock price and time to maturity. There are several popular approaches that practitioners use to calculate the value, which is also called premium, of an option. We denote this value as $V(S_t, t)$, where S_t is the price of a stock at time t.

Although $V(S_t, t)$ for $t < T$ is not known, $V(S_T, T)$, called payoff, is known with certainty at T, where S_T is the future price of the stock at the time of maturity. The value $V(S_T, T)$ of a call option at the time of maturity T is given by

$$V(S_T, T) = \begin{cases} 0 & \text{if } S_T \leq K \\ S_T - K & \text{if } S_T > K \end{cases} = \max\{S_T - K, 0\}. \tag{1.1}$$

For a put option, the value at the time of maturity T is given by

$$V(S_T, T) = \begin{cases} K - S_T & \text{if } S_T < K \\ 0 & \text{if } S_T \geq K \end{cases} = \max\{K - S_T, 0\}. \tag{1.2}$$

Going back to the two factors mentioned earlier, stock price and time to maturity, the value of an option at $t < T$ therefore has two components: value associated with the stock price and value associated with the time to maturity. The first of these components is called intrinsic value and the second component is called time value. Clearly, as we get closer to T, much of the option's value comes from its intrinsic value. This phenomenon is called the time decay of an option.

In addition to the strike price K and time remaining to maturity $T - t$, the value of an option $V(S_t, t)$ also depends on the risk-free interest rate r and the volatility (standard derivation) σ of the stock price. For simplicity, we assume that both r and σ stay constant during the life of the option $T - t$. We also assume that the stock does not pay dividends during the life of the option. Dependence on r is motivated by the riskfree-hedging concept, which is related to building a portfolio of assets in such a way that its value grows at the rate offered by US Treasury bonds (since it is common practice to assume that US Treasury does not default on the bonds it issues, we consider the rate of return as riskfree). In this paper we will not go into the details of riskfree-hedging. We will also not discuss how σ of a stock is computed. Both the quantities are assumed to be given. Both r and σ are measured per year (time t is also measured in years). We assume that time starts at $t = 0$, the time the option is to be bought or sold. Therefore, the range of time is $0 \leq t \leq T$. To summarize, the value of a European option depends on the following five factors: expiration time (T), risk-free rate (r), volatility (σ), current stock price (S_t), and strike price (K).

In this paper we are concerned with a popular approach called Binomial method [2] or [5]. We will restrict our discussion to European options for simplicity.

Path-dependent options are a class of options whose payoff depends on the path the underlying takes until maturity. For example, the payoff of an Asian option depends on the average underlying price over the option's lifetime. Our method could be attractive to value such path-dependent options, especially when the value is a complicated function of the paths.

The rest of the paper is organized as follows. Section 2 introduces the Binomial model and presents an algorithm to implement the method. In Section 3, we introduce the procedure to map Binomial probabilities to Bernoulli paths and present the formulation to value a European option using Bernoulli paths. Section 4 briefly discusses the implementation details on a parallel computing cluster. Section 5 presents results from the new procedure and finally, Section 6 gives some concluding remarks.

2 The Binomial Method

The Binomial method is based on simulating an evolution of the future stock price between $t = 0$ and $t = T$ on a grid of possible stock prices. An option's value is calculated starting at T, using (1.1) for calls or (1.2) for puts, and stepping back in time by applying appropriate rules at each time step. Interested readers may refer to [2] or [5] for details.

Our goal is to calculate the value of an option at $t = 0$, i.e. $V(S_t, t = 0)$. We first discretize $0 \leq t \leq T$ into equidistant time steps of size δt. Let N be the number of time steps and $\delta t = T/N$. Let us denote points in time between 0 and T as t_i. Therefore, $t_i = i\delta t$ for $i = 0, \ldots, N$. Imagine a two-dimensional grid with t on the X-axis and stock price S_t on the Y-axis; by discretizing time, we slice the X-axis into equidistant time steps. As we describe below, we next discretize S_t at each $t = t_i$ resulting in discrete values $S_{t_i j}$, where j is the index on Y-axis. For notational convenience, we will write $S_{t_i j}$ as S_{ij}. The Binomial method makes the following assumptions:

A1 The stock price S_{t_i} at t_i over time step δt can only take two possible values: either go up to $S_{t_i}u$ or go down to $S_{t_i}d$ at t_{i+1} with $0 < d < u$ where u is the factor of upward movement and d is the factor of downward movement.

A2 The probability of moving up between time t_i and t_{i+1} is p (and therefore the probability of moving down is $1 - p$).

A3 $E(S_{t_{i+1}} \mid S_{t_i}) = S_{t_i} e^{r\delta t}$

The probability p does not reflect the true probability of a stock moving up. It is an artificial probability reflecting the assumption A3. From assumptions A1 and A2, we have $E(S_{t_{i+1}} \mid S_{t_i}) = pS_{t_i}u + (1-p)S_{t_i}d$. Equating this to $E(S_{i+1} \mid S_{t_i})$ in assumption A3 we get,

$$e^{r\delta t} = pu + (1-p)d, \tag{2.1}$$

and solving for p,

$$p = (e^{r\delta t} - d)/(u - d). \tag{2.2}$$

Algorithm 1 Build the grid of stock prices and calculate option payoffs for Binomial method.

for $i = 1, 2, \ldots, N$ **do**
  $S_{ij} = S_0 u^j d^{i-j}$ for $j = 0, 1, \ldots, i$
end for
for $j = 0, \ldots, N$ **do**
  $V_{Nj} \leftarrow \max\{S_{Nj} - K, 0\}$
end for

Since p is a probability, $0 \leq p \leq 1$ implies that $d \leq e^{r\delta t} \leq u$. Equating variances of the stock price process in the above discrete model and the continuous model (where stock price is assumed to be lognormally distributed), we get

$$e^{2r\delta t + \sigma^2 \delta t} = pu^2 + (1-p)d^2. \tag{2.3}$$

To enforce symmetry in the simulated stock price structure, we assume

$$ud = 1. \tag{2.4}$$

Solving (2.1), (2.3) and (2.4) we get

$$\begin{aligned}
&u = \beta + \sqrt{\beta^2 + 1}, \\
&d = 1/u, \\
&p = (e^{r\delta t} - d)/(u - d), \\
&\text{where } \beta = \frac{1}{2}(e^{-r\delta t} + e^{(r+\sigma^2)\delta t}).
\end{aligned}$$

Starting with the current stock price in the market S_0, a grid of possible future stock prices S_{ij} is built using u, d and p. The procedure is shown as Algorithm 1. For a call option, the value $V(S_T, T)$ is given using (1.1) at each S_{Nj} at T. Therefore, $V_{Nj} = \max\{S_{Nj} - K, 0\}, j = 0, .., N$, where V_{ij} represents $V(S_{ij}, t_i)$. Figure 2.1 shows a two step Binomial tree which starts with a stock price S, and where the value for the last time step has been computed.

Now, to calculate the current value $V_{00} = V(S_0, 0)$ of the option, a backward induction phase recursively computes V_{ij} for time steps $t_i = t_{N-1}, t_{N-2}, \ldots, t_0$. The recursion is based on the equation

$$V_{ij} = e^{-r\delta t}(pV_{i+1,j+1} + (1-p)V_{i+1,j}), \tag{2.5}$$

which represents the expectation of the option value at step $i+1$. The procedure to compute V_{00} is summarized as Algorithm 2.

3 Valuation using Bernoulli Paths

A careful look at Algorithm 2 reveals the difficulty in implementing the Binomial method in parallel on a cluster. The method of backward induction uses option values computed in subsequent

Algorithm 2 Compute option value using Binomial method.

$\delta t \leftarrow T/N$
for $i = N-1, \dots, 0$ **do**
  for $j = 0, \dots, i$ **do**
    $V_{ij} \leftarrow e^{-r\delta t}(pV_{i+1,j+1} + (1-p)V_{i+1,j})$
  end for
end for
return V_{00}

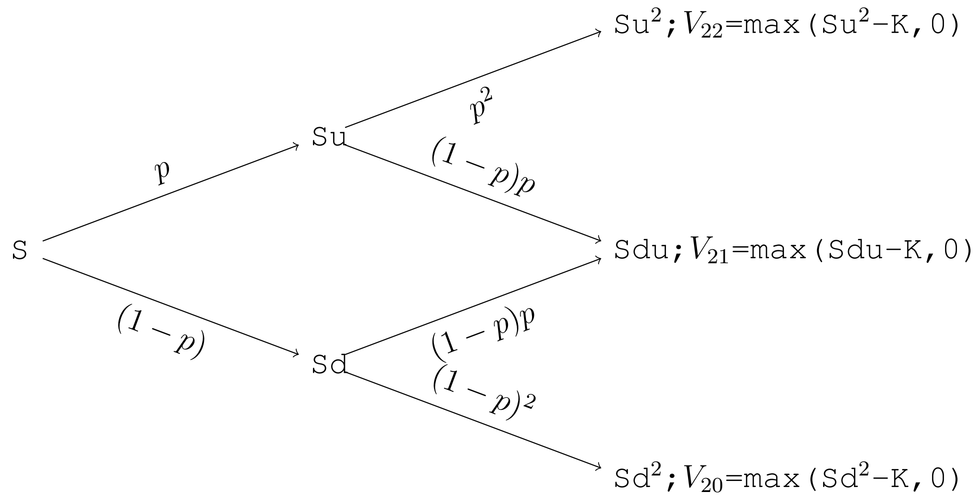


Figure 2.1: A Two Step Binomial Tree

time steps, and implies that data be shared and communicated among the parallel tasks at each time step. If one were to parallelize the computation at each time step t_i by grouping the calculations into parallel tasks, it is clear that the results from the iteration for t_{i+1} must be shared across the tasks. This sharing of intermediary results at each iteration complicates the algorithm. Also, inter-task communication is generally orders of magnitude slower than calculation within a task, so the overall performance of a parallel implementation depends on making efficient use of the interconnect between compute nodes. The implementation can be greatly simplified and the communication cost can be removed if the problem is transformed into an "embarrassingly parallel problem", which is defined as a problem in parallel computing that requires no communication among parallel tasks.

Several parallel implementation methods were proposed in recent years. Multi-threaded parallel implementations to price several options simultaneously on GPUs using the CUDA programmaing language were proposed by Kolb & Pharr [3]. Ganesan et al. [1] proposed another parallel implementation by concurrently processing multiple time steps using a symbolic dependence structure. To the best of our knowledge, none of these procedures have taken the approach of making the Binomial method embarrassingly parallel. We propose a method to price European style options by mapping Binomial probabilities to Bernoulli asset paths, thereby transforming the Binomial method into an embarrassingly parallel problem readily amenable for implementation on a parallel computing cluster.

Consider the backward induction step (2.5) in the Binomial method

$$V_{ij} \leftarrow e^{-r\delta t}(pV_{i+1,j+1} + (1-p)V_{i+1,j})$$

for the two step tree shown in Figure 2.1. V_{22}, V_{21}, and V_{20} are payoffs at $T = 2$. At $T = 1$, option values are computed as

$$V_{10} = e^{-r\delta t}(pV_{21} + (1-p)V_{20}) \tag{3.1}$$

and

$$V_{11} = e^{-r\delta t}(pV_{22} + (1-p)V_{21}) \tag{3.2}$$

At $T = 0$, the option value is computed as

$$V_{00} = e^{-r\delta t}(pV_{11} + (1-p)V_{10}) \tag{3.3}$$

Substituting (3.1) and (3.2) in (3.3) yields

$$V_{00} = V_{01} = e^{-rT}(p^2V_{22} + 2p(1-p)V_{21} + (1-p)^2V_{20})$$

Note that $p^2 = \binom{2}{0}p^2$, $2p(1-p) = \binom{2}{1}p(1-p)$, and $(1-p)^2 = \binom{2}{2}(1-p)^2$. Generalizing to N time steps, the option value V_{00} can be calculated as

$$V_{00} = e^{-rT}\sum_{i=0}^{N}\binom{N}{i}p^i(1-p)^{N-i}V_{Ni}.$$

In vector form, V_{00} can be represented as

$$V_{00} = e^{-rT}\boldsymbol{P}'\boldsymbol{V}, \tag{3.4}$$

where $\boldsymbol{P}$ is an $(N+1)$ dimensional vector of probabilities and $\boldsymbol{V}$ is an $(N+1)$ dimensional vector of payoffs at time $t = T$. All vectors are assumed to be column vectors. Each element P_i in the vector $\boldsymbol{P}$ represents the probability of reaching the i^{th} node at time $t = T$, which can be reached in $\binom{N}{i}$ ways through sequences of ups and downs, where i is from 0 to N with $i = 0$ representing the leaf node reached after all down moves. We represent each such path by an N dimensional vector whose elements take either 1 (up) or 0 (down) as values. For example, $\boldsymbol{x} = (1, 1, \ldots, 1)'$ represents a path of N up moves reaching the top most node at time $t = T$. Figure 3.1 shows the two step Binomial tree from Figure 2.1 with Bernoulli paths to terminal leaf nodes shown as vectors.

The probability of reaching a leaf node via the path $\boldsymbol{x}$ is given by

$$P(\boldsymbol{X} = \boldsymbol{x}) = p^{\boldsymbol{x}'\boldsymbol{1}}(1-p)^{N-\boldsymbol{x}'\boldsymbol{1}} \tag{3.5}$$

where $\boldsymbol{1}$ is a vector N ones. Since there are $\binom{N}{i}$ ways of reaching the leaf node i,

$$P_i = \binom{N}{i}p^i(1-p)^{N-i} = \sum_{\boldsymbol{x}\in\mathbb{B}^N:\boldsymbol{x}'\boldsymbol{1}=i} p^{\boldsymbol{x}'\boldsymbol{1}}(1-p)^{N-\boldsymbol{x}'\boldsymbol{1}} \tag{3.6}$$

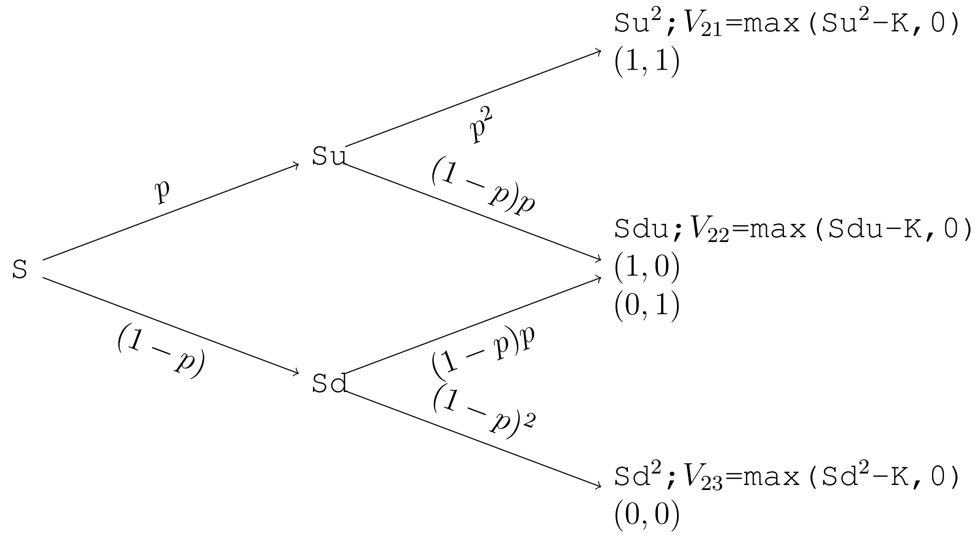


Figure 3.1: Two Step Binomial Tree with Bernoulli Paths

where $\mathbb{B} = \{0, 1\}$. Using (3.6) in (3.4) and after expanding the inner product, we obtain that

$$V_{00} = e^{-rT} \sum_{i=0}^{N} V_{Ni} \sum_{\boldsymbol{x} \in \mathbb{B}^N : \boldsymbol{x}'\boldsymbol{1} = i} p^{\boldsymbol{x}'\boldsymbol{1}} (1-p)^{N-\boldsymbol{x}'\boldsymbol{1}} \tag{3.7}$$

Observe that by transforming the backward induction procedure into an expected option value at time T as a summation over payoffs from each Bernoulli path, the problem becomes embarrassingly parallel.

4 Parallel Bernoulli Path Algorithm

Since the components of the summation in (3.7) are independent of each other, the calculation of the summation can be easily delegated to a set of processors (i.e. cores for multi-core CPUs) on a cluster. Let M be the number of parallel tasks, which is equal to the number of processors we intend to use on a cluster. A compute node for our discussion contains either single or a multiple processors. It is assumed that compute nodes may communicate by an interconnect. We use a master-worker paradigm for our implementation. The master first builds the lattice and calculates the option payoff V_{Nj} (although workers can compute their own payoffs once paths are delegated to them), where $j = 0, .., N$, at time T. Note that if N is the number of time steps used to build a lattice, the total number of paths leading to terminal payoffs is 2^N. It then divides these 2^N paths into M sets with each set containing $\lfloor 2^N/M \rfloor$ number of paths. Paths remaining, if any, after grouping are added to the last set. Let B_m be the m^{th} set of $\lfloor 2^N/M \rfloor$ paths leading to K_m distinct payoffs. Let $\boldsymbol{P}_m$ be a vector of dimension K_m with each element P_m^i representing the probability of reaching the i^{th} terminal payoff, $i = 1, \ldots, K_m$. Recall the vector of probabilities $\boldsymbol{P}$ mentioned

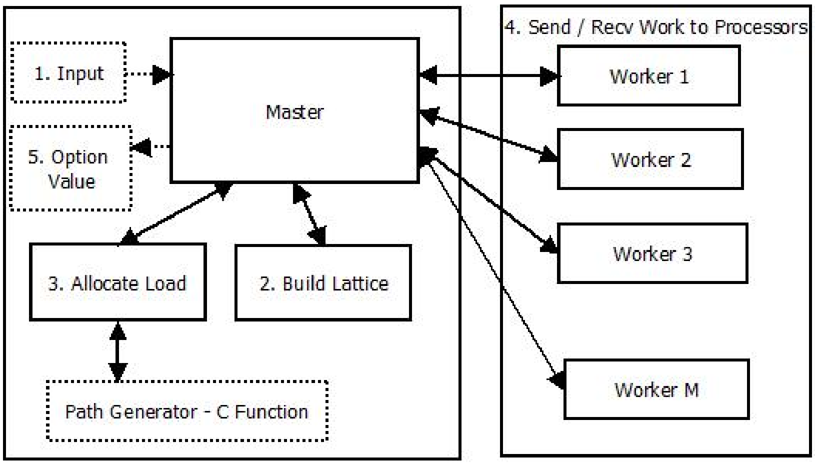


Figure 4.1: Program Structure

in 3. Assuming no payoff is assigned to more than one parallel task, note that $\boldsymbol{P}'$ can be partitioned as $\left[\boldsymbol{P}_1'\middle|\boldsymbol{P}_2'\middle|\cdots\middle|\boldsymbol{P}_M'\right]$. The master then sends each such set B_m along with their payoffs $\boldsymbol{V}_N^m$, $\boldsymbol{P}_m$, interest rate r, and time to maturity T to the m^{th} worker compute node. $\boldsymbol{V}_N^m$ represents a vector of payoffs from the paths in B_m.

The m^{th} parallel task evaluates the expected value of the option as $e^{-rT}\boldsymbol{P}_m'\boldsymbol{V}_N^m$ and returns it to the master. Once the computed components of the option value are received from all workers, the final option value is computed by summing the individual components. A schematic depiction of the algorithm is presented in Figure 4.1. All the parameters needed to build the lattice and to evaluate the value of the option are provided to the master as input. The master then builds the lattice, allocates the load by calculating all the Bernoulli paths (generation of all such paths is not discussed in this paper) to the payoffs and grouping these paths into M sets. It then distributes these sets to M workers, which calculate and return the discounted expected values of the payoffs assigned to them. The master then calculates the option value by summing the values returned. Note that the workers do not need to communicate with each other once the master delegates the computation. The ease and simplicity in the implementation is a result of making the problem embarrassingly parallel. Figures 4.2 and 4.3 show code snippets in `R` for master and worker tasks respectively.

5 Results

We have tested the implementation on the cluster tara in the UMBC High Performance Computing Facility, which has 82 computing nodes with two quad-core Intel Nehalem processors (therefore 8 cores per compute node) and 24 GB of memory on a quad-data rate InfiniBand interconnect, and have used `R` [4, 6] as the programming environment.

We take a European put as an example to illustrate our results. The put has a strike price of $K = 10$. Current price and volatility of the asset is $S = 5$ and 30% respectively. Risk-free interest

```
mpi.spawn.Rslaves(nslaves=M,needlog=FALSE)
mpi.bcast.cmd(source("Worker.R"))
mpi.bcast.cmd(slave())
mpi.bcast(as.integer(N),type=1,rank=0)
.
.
for(i in 1:length(W)) {
    mpi.send.Robj(W[[i]],dest=i,tag=88,comm=1)
}
atag <- mpi.any.tag()
asource <- mpi.any.source()
for(nm in 1:length(W)) {
    retsl <- mpi.recv.Robj(source=asource,tag=atag,comm=1)
    optval <- optval + retsl[2]
}
```

Figure 4.2: Rmpi code for master

```
N <- mpi.bcast(integer(1),type=1,rank=0,comm=1)
P <- mpi.bcast.Robj(rank=0,comm=1)
V <- mpi.recv.Robj(tag=88,source=0,comm=1)
.
.
myrank <- mpi.comm.rank()
retwkr <- c(myrank,sumtotal)
mpi.send.Robj(retwkr,dest=0,tag=2)
```

Figure 4.3: Rmpi code for worker

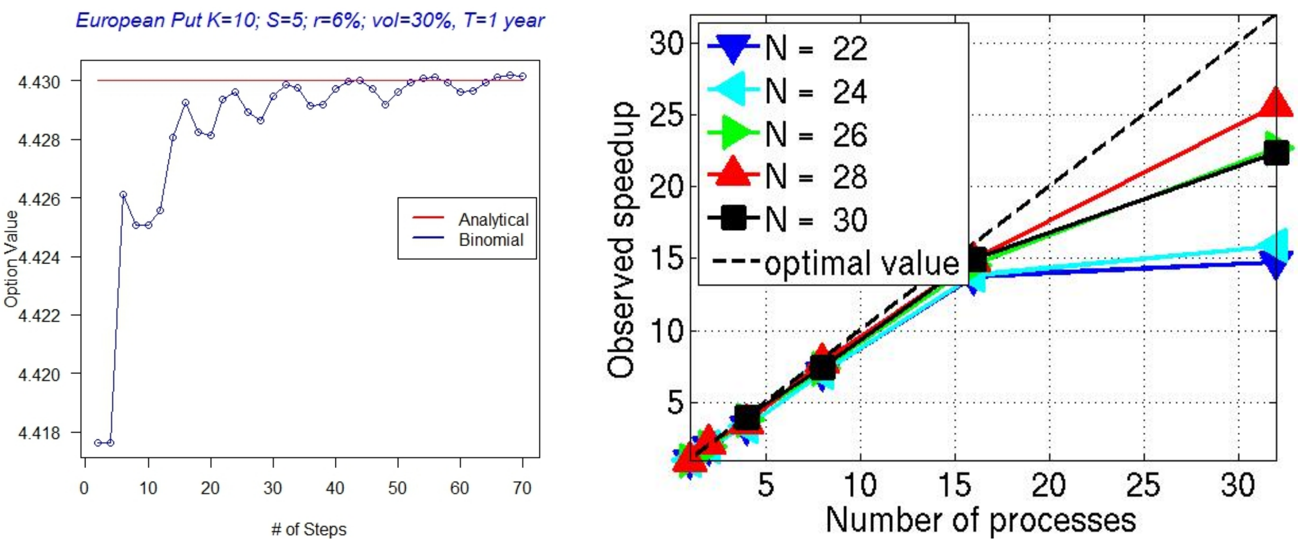
rate is 6% and time to maturity is one year.

Table 5.1 shows the runtime for number of time steps $N = 22$, $N = 24$, $N = 26$, and $N = 28$. Although the runtime drastically decreases as the number of parallel tasks increase for a given number of time steps, the scale of runtime itself also increases sharply as the number of time steps increase. This behavior suggests that the Bernoulli path procedure might be more suitable to price a certain class of complicated exotic options, possibly illiquid, that do not require large number of time steps in a lattice setting.

Figure 5.1 (a) shows the convergence rate of our method to the analytical solution (as the number of time steps N increases), which is available for European options. Figure 5.1 (b) shows speedup rates as the number of parallel processes increase. If $T_p(N)$ denotes the wall clock time for a problem of a fixed size parametrized by N using p processes, then the quantity $S_p(N) = T_1(N)/T_p(N)$ measures the speedup of the code from 1 to p processes, whose optimal value is $S_p = p$. Note that for a fixed problem size, there is a reduced advantage in the speedup beyond a certain number of tasks. This is because the overhead of coordinating tasks begins to dominate the

Table 5.1: Runtime for different number of time steps

(a) $N = 22$ Wall clock time in HH:MM:SS on tara		(b) $N = 24$ Wall clock time in HH:MM:SS on tara	
# of Processors		# of Processors	
2	00:05:27	2	00:22:51
4	00:03:01	4	00:12:46
6	00:01:57	6	00:07:57
8	00:01:25	8	00:06:03
16	00:00:44	16	00:03:04
32	00:00:40	32	00:1:40
(c) $N = 26$ Wall clock time in HH:MM:SS on tara		**(d) $N = 28$ Wall clock time in HH:MM:SS on tara**	
# of Processors		# of Processors	
2	01:38:24	2	06:26:41
4	00:49:51	4	03:49:10
6	00:34:22	6	02:16:52
8	00:25:21	8	01:45:38
16	00:12:52	16	00:55:17
32	00:08:15	32	00:32:06



(a) (b)

Figure 5.1: (a) Convergence to analytical solution and (b) Speedup rate

time spent doing useful calculations. This issue is commonly encountered in parallel computing, and further discussion is beyond the scope of this paper.

6 Concluding Remarks

We have discussed a novel procedure to implement the Binomial method of option pricing in a parallel computing framework by mapping the Binomial probabilities of stock price evolution to Bernoulli paths, thereby transforming the problem into an embarrassingly parallel problem. The valuation based on our method is consistent with the value calculated by the traditional Binomial method. We have provided the outline of an `R` based implementation using the `Rmpi` package on a cluster, which can be as small as a multi-core laptop with necessary parallel computing infrastructure installed. In the future, we plan to evaluate our method in comparison with some of the known numerical methods to value path-dependent options.

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