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Characterizing Key Developmental Understandings and Pedagogically Powerful Ideas within a Statistical Knowledge for Teaching Framework

Abstract

A hypothetical framework to characterize statistical knowledge for teaching (SKT) is described. Empirical grounding for the framework is provided by artifacts from an undergraduate course for prospective teachers that concentrated on the development of SKT. The theoretical notion of "key developmental understanding" (KDU) is used to identify landmarks in the development of SKT subject matter knowledge. Sample KDUs are given for the subject matter knowledge categories of common content knowledge, specialized content knowledge, and horizon knowledge. The theoretical notion of "pedagogically powerful idea" is used to describe how KDUs must be transformed to become useful in teaching. Examples of pedagogically powerful ideas for the pedagogical content knowledge categories of knowledge of content and teaching and curriculum knowledge are provided. Knowledge of content and students is hypothesized as a basis for the development of pedagogically powerful ideas.

Characterizing Key Developmental Understandings and Pedagogically Powerful Ideas within a Statistical Knowledge for Teaching Framework

Recently, much attention has been given to mapping mathematical knowledge for teaching (MKT). A robust conclusion from the research is that it is useful for teachers to have both subject matter knowledge and pedagogical content knowledge (Hill, Ball, & Schilling, 2008). Subject matter knowledge pertains to knowing the content to be taught, and pedagogical content knowledge involves making it understandable to students (Shulman, 1987). Studies that conceptualize MKT as consisting of both subject matter knowledge and pedagogical content knowledge suggest that teachers' MKT is linked to student achievement (Hill, Rowan, & Ball, 2005), the ability to extend students' mathematical thinking (Cengiz, Kline, & Grant, 2011), the ability to make sense of and respond to students' mathematical difficulties (Johnson & Larsen, 2012) and the overall mathematical quality of instruction teachers offer (Hill, Blunk et al., 2008). Such findings portray MKT as a worthwhile research domain.

Theoretical characterizations of MKT to guide research and teacher education have been slow to materialize. This is reflected, in part, by mathematics examinations required for teaching certification over the past century. Hill, Sleep, Lewis, and Ball (2007) observed that many such examinations tested subject matter knowledge while neglecting pedagogical content knowledge. Examinations attempting to assess pedagogical content knowledge have not always been guided by coherent theoretical frameworks. These problems are symptomatic of the larger issue of little theoretical development of MKT up until the recent past. Although development of MKT theory has gained momentum during the past decade, Hill, Ball, and Schilling (2008) noted that "mapping this knowledge is likely to be a long and time-intensive process" (p. 396).

A complicating factor for researchers mapping statistical knowledge for teaching (SKT) is that statistics and mathematics can be considered distinct disciplines. The *Guidelines for Assessment and Instruction in Statistics Education* (GAISE) report (Franklin et al., 2007) emphasized the distinctiveness of statistics by observing that statistical questions are stochastic in nature rather than deterministic, and that variability is a central object of study in statistics. Additionally, although mathematics and statistics are both associated with real world contexts at times, the role of context differs in each discipline. Cobb and Moore (1997) stated,

> Although mathematicians often rely on applied context both for motivation and as a source of problems for research, the ultimate focus in mathematical thinking is on abstract patterns: the context is part of the irrelevant detail that must be boiled off over the flame of abstraction in order to reveal the previously hidden crystal of pure structure. *In mathematics, context obscures structure*. Like mathematicians, data analysts also look for patterns, but ultimately, in data analysis, whether the patterns have meaning, and whether they have any value, depends on how the threads of those patterns interweave with the complementary threads of the story line. *In data analysis, context provides meaning* (p. 803, italics in original).

Statistical reasoning is distinctive in often involving reasoning simultaneously about data and context and drawing qualified conclusions about questions of interest based upon knowledge of both (delMas, 2004). Wild and Pfannkuch (1999) referred to this as "shuttling back and forth" between data and context when engaging in statistical reasoning. Furthermore, some statistical tasks, such as designing survey questions and choosing appropriate study designs, have significant non-mathematical components (Groth, 2007).

The purpose of this paper is to develop a theoretical framework to help guide research on SKT. I do so by building on and integrating existing theories of MKT, SKT, and how MKT develops. The framework includes hypothesized elements of SKT, hypotheses about how the elements relate to one another, and potential landmarks in SKT development. Empirical grounding for the framework is provided by artifacts produced during an undergraduate course I taught that focused on the development of prospective teachers' SKT. I considered conceptualization of SKT theory a worthwhile endeavor because coherent theories can guide researchers in asking questions, formulating hypotheses, and determining variables and relationships to investigate (Johnson, 1980).

An Overview of the Theoretical Framework's Structure

The SKT framework developed herein draws on several existing theories. It starts by revisiting Groth's (2007) working theory of SKT, which is based on the premise that because statistics is a discipline in its own right, SKT is not precisely equivalent to MKT. Nonetheless, since statistics uses mathematics, categories of MKT subject matter knowledge and pedagogical content knowledge described by Hill, Ball, and Schilling (2008) are used as starting points to identify elements of SKT. The framework does, however, go beyond applying MKT category descriptions to statistics, since much of this work has been done elsewhere (Burgess, 2011; Groth, 2007; Noll, 2011; Wassong & Biehler, 2010). A unique aspect of the framework is that Simon's (2006) theoretical construct of "key developmental understandings" (KDUs) is used to identify landmarks in the development of SKT subject matter knowledge. Additionally, Silverman and Thompson's (2008) notion of "pedagogically powerful ideas" is used to identify landmarks and mechanisms in the development of SKT pedagogical content knowledge. The discussion of the framework is not exhaustive in terms of all possible KDUs and pedagogically

powerful ideas relevant to teaching statistics. Instead, the focus is on arguing that KDUs and pedagogically powerful ideas can be considered in combination with categories of subject matter knowledge and pedagogical content knowledge to form a unified theoretical framework. A summary of the main theoretical constructs in the exposition is provided in Table 1.

<INSERT TABLE 1 HERE>

Categories of Statistical Knowledge for Teaching

Hill, Ball, and Schilling's (2008) model of MKT subject matter knowledge and pedagogical content knowledge consists of six categories. They characterized subject matter knowledge as consisting of common content knowledge, specialized content knowledge, and horizon knowledge. Pedagogical content knowledge was thought to consist of knowledge of content and students, knowledge of content and teaching, and curriculum knowledge. Potential meanings of these categories within the context of statistics are considered next.

Common Content Knowledge, Specialized Content Knowledge, and Knowledge of Content and Students for Statistics

Groth's (2007) hypothetical SKT framework did not include all six MKT knowledge categories identified by Hill, Ball, and Schilling (2008). Instead, the framework focused on common content knowledge and specialized content knowledge, following Hill, Schilling, and Ball (2004). Groth defined these knowledge categories in the following manner:

> *Common knowledge* relates to competencies developed in conventional mathematics courses, such as computing accurately, making correct mathematical statements, and solving problems. *Specialized knowledge* is developed by carefully attending to mathematical issues and dilemmas that arise in teaching contexts. It relates to such tasks as providing understandable explanations, appraising students' unconventional methods

> for solving problems, and constructing and evaluating multiple representations for concepts (p. 428).

Using these definitions, Groth provided examples of common and specialized statistical knowledge. Common knowledge examples included accurately reading graphs, constructing survey questions, computing descriptive statistics, and choosing an appropriate descriptive statistic for a given context. Specialized knowledge examples included understanding challenges students may encounter reading different types of graphical displays, identifying properties of the arithmetic mean students may have difficulty comprehending, and realizing that students may compute a statistic without consideration of context.

Hill, Ball, and Schilling's (2008) description of common knowledge differs slightly from Groth's (2007) description. Hill, Ball, and Schilling described common content knowledge as "knowledge that is used in the work of teaching in ways *in common with* how it is used in many other professions or occupations that also use mathematics" (p. 377). Although this description differs from Groth's by emphasizing that common knowledge cuts across multiple professions, the examples of common knowledge he provided are still applicable. Accurately reading graphs, constructing survey questions, computing descriptive statistics, and choosing appropriate descriptive statistics for a given context are not tasks unique to the teaching profession. Therefore, the Hill, Ball, and Schilling characterization of common content knowledge seems to supplement Groth's description of common content knowledge rather than supplant it.

Although no substantive differences appear to exist between the Groth (2007) and Hill, Ball, and Schilling (2008) notions of common content knowledge, differences do exist between their characterizations of specialized content knowledge. For example, Groth categorized "understanding differences between how students read box plots and dot plots" (p. 430) as

specialized content knowledge. Hill, Ball, and Schilling, on the other hand, introduced a new category, knowledge of content and students: "content knowledge intertwined with knowledge of how students think about, know, or learn this particular content" (p. 375). Knowledge of content and students allows teachers to anticipate students' problem-solving strategies and to understand difficulties associated with learning concepts. Understanding differences between how students read box plots and dot plots, therefore, could be categorized as knowledge of content and students because children often have greater difficulty reading condensed displays such as box plots as opposed to displays that show each individual data value, such as dot plots (Zawojewski & Shaughnessy, 2000a) ("condensed" in this context means data are represented in a manner that reveals some distributional characteristics but not individual values). Other specialized knowledge examples given by Groth, such as "understanding students' strategies for measurement," (p. 430) can also be re-categorized as knowledge of content and students because they help teachers anticipate children's content-specific thinking.

Because Groth's (2007) examples of specialized knowledge can be re-categorized as knowledge of content and students, there is a need to re-consider what specialized content knowledge for teaching statistics might entail. Hill, Ball, and Schilling (2008) considered specialized knowledge to be "the mathematical knowledge that allows teachers to engage in particular *teaching* tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solutions to problems" (p. 377-378). In contrast to common knowledge, specialized knowledge is content knowledge of unique interest to teachers.

Examples to distinguish specialized knowledge from knowledge of content and students can be found in statistics education literature incorporating the two categories. Wassong and

Biehler (2010) discussed knowing how to represent the mean as a typical value, a fair share, a data-reducer, and a signal amid noise as potential examples of specialized knowledge. They then characterized knowing of students' difficulties conceiving of the mean as a signal amid noise and knowing that students have difficulty understanding the mean conceptually as examples of knowledge of content and students. Burgess (2011) discussed a teacher's appropriate analysis of a student's statistically naïve interpretation of survey results as indicative of specialized knowledge, and discussed another teacher's anticipation of students' difficulties sorting data as indicative of knowledge of content and students

Noll (2011) observed that it can be difficult to tease apart knowledge of content and students and specialized knowledge, and that distinguishing between specialized knowledge and common knowledge likewise is not always trivial. She wrote,

> When a teacher is grading a student solution to a homework problem, it may be difficult to decipher whether she is using specialized content knowledge, mathematically examining a non-standard student approach, or using her knowledge of content and students, recognizing a common student misconception or stage of development, or a combination. Likewise, there are some mathematical concepts that may fall on the boundary between common content knowledge and specialized content knowledge. For example, one might argue that some professions need knowledge of why the algorithm for multi-digit subtraction works, a financial consultant for instance, and, thus, this knowledge is not unique to teaching (p. 51).

Hence, throughout this paper, attention is given to identifying additional potential distinctions among specialized knowledge, common knowledge, and knowledge of content and students in the context of SKT.

Knowledge of Content and Teaching, Curriculum Knowledge, and Horizon Knowledge for Statistics

Three Hill, Ball, and Schilling (2008) categories remaining to be considered are knowledge of content and teaching, curriculum knowledge, and horizon knowledge. Hill, Ball, and Schilling considered knowledge of content and teaching and curriculum knowledge, along with knowledge of content and students, to be in the broader category of "pedagogical content knowledge." Knowledge of content and teaching provides teachers with content-specific teaching strategies. For instance, using the process of statistical investigation (Franklin et al., 2007) as a teaching strategy could be indicative of knowledge of content and teaching. Curriculum knowledge allows teachers to perform tasks such as appropriately sequencing the introduction of statistical ideas (Godino, Ortiz, Roa, & Wilhelmi, 2011). Hill, Ball, and Schilling grouped horizon knowledge with common content knowledge and specialized content knowledge under the broader category of "subject matter knowledge." Horizon knowledge entails knowing statistics beyond the prescribed curriculum. Such knowledge can help teachers guide students' investigations in productive directions and provide a foundation for learning in later grades (Ball, Thames, & Phelps, 2008).

Statistics education literature can be drawn on for specific examples of knowledge of content and teaching, curriculum knowledge, and horizon knowledge. Burgess (2011) considered a teacher's lack of strategies for remediating students' incorrect comparisons of two unequal-sized groups to be indicative of a gap in knowledge of content and teaching. Groth (in press a) discussed forming an opinion on when to introduce conventional statistical representations within a sequence of lessons as part of the development of curriculum knowledge. Godino et al. (2011) considered knowing of epistemological obstacles in the historical development of

probability to indicate horizon knowledge because it involves understanding broad disciplinary foundations to which specific ideas to be taught may connect. In a similar vein, Bakker and Gravemeijer (2006) demonstrated that knowledge of the historical development of mean and median can inform instructional decisions.

As with all of the knowledge categories discussed thus far, it should be noted that these three categories have a degree of overlap with others. For example, specialized knowledge and knowledge of content and teaching both involve making subject matter comprehensible to students. Additionally, knowledge of content and teaching and curriculum knowledge both involve using specific strategies to facilitate student learning. The preceding examples from statistics education literature provide some guidance in distinguishing among the knowledge categories, but undoubtedly still leave room for ambiguity. The theoretical framework discussed herein can further disentangle some ambiguities, though its primary purpose is to point toward relationships and variables to investigate (Johnson, 1980) rather than to resolve all questions.

Summary

In summary, two broad teacher knowledge categories can be identified: subject matter knowledge and pedagogical content knowledge. In the following, subject matter knowledge is assumed to consist of common content knowledge, specialized content knowledge, and horizon knowledge. Pedagogical content knowledge is assumed to consist of knowledge of content and students, knowledge of content and teaching, and curriculum knowledge. These assumptions require expanding the Groth (2007) SKT framework beyond common and specialized knowledge to include more recent work by Hill, Ball, and Schilling (2008) and re-categorizing Groth's specialized knowledge examples as knowledge of content and students. Nonetheless, the underlying premise of the earlier framework that MKT theory can advise the construction of

SKT theory remains. Since statistics and mathematics can be considered distinct disciplines, MKT and SKT are likely to overlap without being completely equivalent.

Key Developmental Understandings and SKT Subject Matter Knowledge

SKT knowledge categories, though important, seem unlikely to be wholly adequate for informing research and teacher education. Identifying cognitive landmarks in SKT development is also worthwhile because doing so can guide teacher education efforts beyond the level of merely naming categories of knowledge to be developed. If teacher educators are aware of such landmarks, they can focus attention on designing learning experiences to address them. The present framework proposes cognitive landmarks and mechanisms in the development of SKT in order to extend previous work on describing knowledge categories.

Simon (2006) posited "key developmental understandings" (KDUs) as a means for identifying cognitive landmarks in learning subject matter. In elaborating the construct, he stated, "A first characteristic is that KDUs involve a conceptual advance on the part of students. By conceptual advance, I mean a change in students' ability to think about and/or perceive particular mathematical relationships" (p. 362). To illustrate this characteristic, he described a task requiring students to form a square of a designated size on a geoboard and then put a red rubber band around one-half of the square. Most students formed two congruent rectangles, but some cut the original square along a diagonal. When Simon asked if the pieces formed by cutting along a diagonal were the same size as those formed by partitioning the square into congruent rectangles, students offered various opinions. Those who thought of one-half as an arrangement rather than a quantity believed that one of the types of pieces was larger or were not sure if the pieces were of equal size. Those who had developed the KDU of perceiving one-half as a

quantity thought it to be a trivial question, and believed the pieces formed by the different cutting strategies were the same size.

Simon (2006) ascribed another characteristic to KDUs, stating,

> A second characteristic of a KDU is that students without the knowledge do not tend to acquire it as the result of an explanation or demonstration. That is, the transition requires a building up of the understanding through students' activity and reflection and usually comes about over multiple experiences. This is not an empirical claim about KDUs; rather, it is an argument that a focus is needed on those understandings whose development tends to require more than an explanation or demonstration (p. 362).

Hence, KDUs are more than just discrete facts for students to learn. They involve significant shifts in students' thinking that occur through reflection on a series of conceptually similar tasks.

Simon (2006) observed that it is often difficult for those who have developed KDUs to anticipate learning obstacles for those who have not. His observation is corroborated by literature on the "expert blind spot" (Nathan & Petrosino, 2003) showing that those who have taken a number of advanced courses tend to lose sight of difficulties encountered by beginners. Hence, KDUs are usually not simple to identify a priori. Instead, researchers must make inferences from empirical data generated as students representing various levels of thinking engage with relevant content. To illustrate the use of KDUs in characterizing landmarks in SKT subject matter knowledge development, examples from empirical data relating to common content knowledge, specialized content knowledge, and horizon knowledge are considered next.

SKT Course Structure

The empirical examples discussed herein come from a semester-long SKT-focused undergraduate course for prospective Pre-K-8 teachers (hereafter referred to as the "SKT

course"). The SKT course was designed to address each of the Hill, Ball, and Schilling (2008) knowledge categories. As prospective teachers worked with the required text (Perkowski & Perkowski, 2007), they also completed statistical activities from Pre-K-8 curricula developed with funding from the National Science Foundation (NSF) (Senk & Thompson, 2003). This provided opportunities to learn of fundamental concepts of elementary curricula (common content knowledge), activities to foster children's statistical thinking (knowledge of content and teaching), and the structure of reform-oriented curricula (curriculum knowledge). Prospective teachers also read and wrote about articles from teacher-oriented journals to help familiarize them with common difficulties in statistical thinking (knowledge of content and students), unusual student approaches to problems (specialized knowledge), and statistical representations to help make content more understandable to children (specialized knowledge). The conceptual foundations of formal inference were introduced through simulation (Garfield & Ben-Zvi, 2008) to provide perspective on statistics beyond Pre-K-8 curricula (horizon knowledge). These teaching strategies are described in greater detail in Groth (in press a, in press b).

I taught the SKT for three consecutive semesters, and each time observed a range of responses among prospective teachers as they completed writing prompts associated with the assigned teacher-oriented journal articles about statistics. Writing was selected as an instructional strategy because it helps learners place organizational structures on their thinking (Vygotsky, 1987). The writing prompts required respondents to draw inferences from the articles, make statistical conjectures, solve statistical problems, analyze pedagogical positions advocated by authors, and provide opinions on different teaching strategies. Responses to the prompts are used as examples in the remainder of this article. (A more detailed description of the prompts, the associated articles, and the design and assessment process is provided in Groth, in press b). The

examples to be discussed are not meant to exhaust the possible KDUs in each subject matter knowledge category, or even all of the KDUs observed during the course. Instead, the examples are meant to illustrate the role of the KDU construct in mapping SKT development.

Common Content Knowledge: Experimental and Theoretical Probabilities

The distinction between theoretical and experimental probability is a key element in curriculum documents around the world (Jones, Langrall, & Mooney, 2007). It links statistics and probability as students analyze data generated by probability simulations to answer statistical questions. In the SKT course, I frequently asked prospective teachers to perform simulations and compare experimental results against theoretical. For example, on one occasion, they played a game called "Markers on a Line" (Burns, 2000), which prompted comparisons between data gathered from rolling a pair of dice several times and the theoretical probabilities of obtaining each possible sum from 2 to 12. On another occasion, they compared data gathered while playing several rounds of "rock, paper, scissors" (Nelson & Williams, 2009) against the theoretical probabilities associated with each player winning the game. They also read and responded to a teacher-oriented journal article (McMillen, 2008) describing activities that helped sixth-graders understand theoretical and experimental probabilities.

After reading the assigned journal article (McMillen, 2008), prospective teachers wrote responses to the following items: "Explain the difference between experimental and theoretical probability in your own words," and "Explain why experimental probabilities do not always match the theoretical probabilities." The two items were intended to prompt reflection on the article and activities from class. From these reflections, I expected to better understand prospective teachers' personal constructions of experimental and theoretical probability.

Prospective teachers' responses suggested that some misconstrued the intended relationship between experimental and theoretical probability. Some illustrative responses were:

> Sandy: Experimental probabilities do not always match the theoretical probabilities because sometimes when conducting an experiment, things do not always happen exactly how one would expect them to, therefore the preconceived ideas of what should happen can sometimes be misleading.
>
> Jessica: Theoretical probability is thinking about the probability of an outcome before actually performing the experiment to get the experimental probability. Experimental probability is when the probability is found by doing the actual experiment… They don't match the theoretical probability because theoretical probability is just a guess of a probability. A guess is rarely ever the same as the exact experiment's outcome.

SKT course activities portrayed theoretical probabilities as anchors for predicting long-term behavior over repeated trials rather than "misleading" measures or those that are "just a guess." In contrast, responses like the two above suggested a focus on predicting individual outcomes rather than long-term behavior. Theoretical probabilities usually do not predict an individual outcome precisely, but they form the basis for statistical inference by providing an anchor for predicting long-term behavior. The usefulness of theoretical probability lies within this capability rather than in predicting a single outcome. Given the qualitative shift in thinking needed to make the conceptual advance of focusing on long-term behavior, and the importance of the conceptual advance as a unifying curricular thread, I posit that conceiving of theoretical probability as an anchor for predicting long-term behavior is a statistical KDU.

Conceiving of theoretical probability as an anchor for predicting long-term behavior is a uniquely statistical, rather than purely mathematical, KDU. In discussing mathematical KDUs,

Simon (2006) asserted, "KDUs (and mathematical understandings more generally) are never the result of empirical learning processes" (p. 365-366). Simon defined "empirical learning processes" as follows:

> An empirical learning process is an inductive process through which students discover patterns. By inductive process, I mean multiple trials in which students make an input (or observe an input) and then observe an output. Students learn that the pattern exists. The phenomenon that generates the pattern may remain a black box to the students (p. 365).

Thus, for Simon, multiplying various whole numbers by 6 and observing that the product is always even does not constitute development of a mathematical KDU. One would only be developed when students understand why an even number is always produced. A mathematical KDU is thus grounded in deductive rather inductive reasoning. To distinguish between theoretical and experimental probability, however, inductive reasoning is foundational. Causes of *systematic* variability can be deduced by examining the data production context, but the existence and prevalence of *statistical* variability cannot be deduced a priori. Rather, its prevalence has motivated the development of mathematical tools for approximating it. For example, DeMoivre formulated the "empirical rule" (i.e., the "68-95-99.7 rule") through observation before the current mathematical model for normal distributions was devised (Bock, Velleman, & Deveaux, 2004). Pedagogically, it can also be advantageous to have students gather empirical data from trials in situations where there is no clear theoretical model (e.g., thumbtack tossing) (Konold et al., 2011). Given the foundational role of empirical observation in statistics, it seems likely that many other statistical KDUs also require empirical learning processes and inductive reasoning, distinguishing them from purely mathematical KDUs.

Horizon Knowledge: Standard Deviation and Mean Absolute Deviation

Developing techniques to measure spread was another goal of the SKT course. I introduced the idea of spread by asking prospective teachers to use their own strategies to describe the amount of spread in data sets. After discussing the strategies they employed in class, we discussed formal measures including range, interquartile range, and mean absolute deviation (MAD). In most states in the U.S., the mean absolute deviation (MAD) (see example in Figure 1) is to be introduced in sixth grade, and the standard deviation in ninth grade or later (National Governor's Association for Best Practices & Council of Chief State School Officers, 2010). Hence, for teachers in my SKT course, knowledge of standard deviation was horizon knowledge because the MAD is a precursor to the more complicated idea of standard deviation in later grades (Kader & Mamer, 2008). In the SKT course, I used activities described by Garfield and Ben-Zvi (2008) to introduce the MAD. In the activities, students used horizontal bars to represent the amount of deviation from the mean for each individual point in a data set (see Figure 1). The MAD was then thought of as the average bar length. In class, we compared this method for determining the MAD to a conventional formula for computing standard deviation. The intent of doing so was to draw attention to the idea that both the MAD and standard deviation employ deviation from the mean in describing the amount of spread in a data set.

<INSERT FIGURE 1 HERE>

As a follow-up to class activities about the MAD and standard deviation, I assigned a teacher-oriented journal article on describing data sets using measures of center and spread (including the MAD) (Kader & Mamer, 2008). The article also discussed a measure of spread that is useful primarily when comparing data sets of the same size: the sum of absolute deviations (SAD). As its name implies, the SAD is the sum of the absolute values of the

deviations from the mean. It can be misleading when comparing two unequal-size data sets because the SAD of a large data set can be larger than the SAD of a smaller data set even if the MAD is greater in the smaller data set. After reading the article, prospective teachers were to write responses to the following items:

(i) Construct two different sets of data that have the same mean. The data sets should have different numbers of values. Compute the SAD and MAD for each set of data. Show your work. Explain what the SAD and MAD tell you about the sets of data.

(ii) How is the MAD similar to the standard deviation? How is it different? How might understanding the MAD help students prepare to study the standard deviation?

A variety of qualitatively different responses were given to the items.

Prospective teachers' responses to the article-related items suggested that conceiving of the "typical" deviation as a measure of spread can be considered a KDU. Such a conception is needed to make sense of measures of spread introduced at various points in the curriculum. Several responses illustrated the difficulty individuals had with coordinating the ideas of average and deviation. In some cases, they simply calculated the averages of data sets they produced, thinking they were calculating the MAD. For example, Kimberly wrote the data sets {6, 5, 4} and {7, 5, 3} in response to the first article writing prompt, and then she calculated the MAD of each to be 5 by adding up the values in each data set and dividing by 3. She then went on to state,

> MAD is different from the standard deviation because standard deviation is squaring the average where MAD is the total number of values added, divided by the number of values shown. Understanding MAD might help students understand standard deviation by leading them up to the computations of standard deviation.

In such cases, the idea of deviation was not used as a tool for measuring spread. If deviation from the mean is not appreciated for its usefulness in measuring spread, it is not possible to understand how the MAD serves as a precursor to standard deviation.

In other cases, the idea of deviation was incorporated in responses to the article-related items, but the average of deviations was not. Some prospective teachers found the means for the data sets they invented and then calculated the deviations (i.e., distances from each data point to the mean). After doing so, however, some did not average the deviations. This type of thinking was apparent in a response from Angela. She used the invented data sets {2, 2, 2} and {1, 2, 3}. In regard to the former data set, Angela believed the MAD to be {0, 0, 0} and explained, "The MAD shows that all of the data points are the same distance from the mean." In regard to the latter, she calculated the MAD to be {1, 0, 1} and stated, "The MAD shows that two of the data points are one away from the mean and one point is the same as the mean." Although un-averaged deviations do provide a degree of information about spread, when deviations are not averaged, the potential of MAD as a precursor to standard deviation is not exploited to its fullest extent because both MAD and standard deviation summarize spread with a single number.

The idea that spread should be measured with more than one number also carried over to some prospective teachers' characterizations of standard deviation. For example, Amanda's response to the second of the two article-related items was, "MAD is similar to the standard deviation because it is the mean of all the standard deviations. However, it is different for the same reason because it is not the actual standard deviation, just the averages of them." Here, it seems the MAD was conceived of as one number, but the standard deviation was conceived of as a set of numbers obtained by calculating deviations from the mean. The idea that the standard deviation of a population is a set of numbers rather than a single number may partially be caused

by language conventionally used in statistics when standard deviation is used as a ruler. In such instances, it is common to discuss "how many standard deviations" an observation lies from the center of a distribution. This language-related influence was apparent in the thinking of Richard, a student who had previously taken an introductory college-level statistics course. He stated,

> The MAD and the standard deviation are similar because they both give an average away from the mean but the standard deviation can be more than one number. So you can be two standard deviations away from the mean but the MAD is only one number.

These types of responses pointed to a specific aspect of horizon knowledge in need of development: A key similarity between the MAD and standard deviation is that both summarize the spread of data with a single number. Conceiving of typical deviation from the mean as a measure of spread is a primarily mathematical KDU related to SKT. Structural similarities in the mathematical formulas for the two measures largely justify introducing the MAD in the earlier grades as a precursor to standard deviation in later grades.

Specialized Content Knowledge: Hat Plots

One of the advantages of an SKT-focused course is that it affords time to study representations that are of importance to teaching but not necessarily to other professions. One such representation I introduced in the SKT course was the hat plot (Watson, Fitzallen, Wilson, & Creed, 2008). Hat plots are generally not listed in curriculum documents as representations students are to learn. Box plots *are* often listed in curriculum documents, but children have difficulty interpreting them because they condense the data into quarters and do not display individual data points (Bakker, Biehler, & Konold, 2005). The value of the hat plot is that if shown with a dot plot (Figure 2) it provides an intermediate representation between data displays that show all individual values (e.g. dot plots) and those that condense the data (e.g., box plots).

Hat plots can be displayed above dot plots, as shown in Figure 2. In comparing the hat plot to the dot plot, children can begin to see how individual values can be condensed to a more compact display. In the hat plot shown in Figure 2, the median is not included, as it would be in a box plot. This allows students to focus initially on the intuitive idea of the "modal clump" of the data (which may or may not be the middle 50%) and partition the data set into the intuitive categories of "low," "medium," and "high" (Konold et al., 2002). Leaving the median out of the representation circumvents confusion that may occur as a result of the median being closer to one side of the box in a box plot than the other (Watson et al., 2008). When students understand the idea of condensing the data and partitioning into groups with a hat plot, adding the median to a hat plot in which the crown of the hat encompasses the middle 50% of the data can help complete the transition to box plots.

<INSERT FIGURE 2 HERE>

One of the assignments for the SKT course was to read an article (Watson et al., 2008) describing how children can generate hat plots to support their transition from data displays that show individual values to those that condense the data. After reading the article, prospective teachers were to write responses to the following questions: "How are hat plots similar to box-and-whisker plots? How are they different?" In the article, hat plots were shown placed above dot plots, as in Figure 2, so to be precise, the specialized knowledge representation under study was the special case of the two graphs shown in tandem (as opposed to just the hat plot itself). Their responses to the questions suggested that conceiving of hat plots (placed above dot plots) as transitional representations between uncondensed and condensed data displays is a KDU for specialized content knowledge.

Some of the prospective teachers who recognized the intermediary role hat plots play between uncondensed and condensed representations discussed hat plots in relation to box plots in the following ways:

> Rebecca: Hat plots are similar to box and whisker plots because they both have a box for the center 50% of the data and whiskers or lines that extend off of the brim/box to represent the highest and lowest data points. One difference is that box and whisker plots you show the median but the rest of the data is not shown whereas with a hat plot you see all of the data underneath the plot. I think showing the data underneath the plot is a great feature/idea because they children can understand what they are seeing.
>
> Stephanie: Hat plots are similar to box and whisker plots because they have a similar look. When you look at both graphs they have a brim or whiskers emanating from the box and it shows the ranges of the data from the lowest to highest. They both have a box shape that represents the data in a 50% range. The differences between the two are that with a box and whisker plot the median is shown by a line in the middle of the box. The hat plot does not show this. Also typically box and whisker plots do not show the data they are representing unlike a hat plot where it is shown below it in a dot sort of plot.

From the standpoint of specialized knowledge, recognizing that one representation shows individual data values while the other does not is significant because it provides perspective on why the uncondensed representation may be of educational use.

Other prospective teachers seemingly overlooked the intermediary representational role of hat plots, instead focusing on surface-level characteristics of each representation. Illustrative responses included:

> Veronica: Hat plots are similar to box and whisker plots because they both use a box and whiskers. They are different because the box plots whiskers come out of the box where the hat plots are underneath and the hat plot does not have a line determining the median of the whole data set.
>
> Sophia: Hat plots are similar to box-and-whisker plots, because they both show another representation of a certain graph. They both show the minimum and maximum values of a data set, and a middle 50% of the data.

These types of responses were not incorrect (though they may be considered cursory). However, from the specialized knowledge standpoint, they left much to be said. Because of the nature of the question, it cannot be said with certainty that these respondents did not recognize the intermediary role played by hat plots. Nonetheless, the responses may indicate fixation on the surface-level features of the two statistical representations. Specialized knowledge would entail conceiving of the hat plot as a precursor to the box plot rather than just another statistical representation with a few cosmetic features in common with the box plot.

Specialized Knowledge: Appraising Student-Invented Graphs

Specialized knowledge is also used to appraise the value of student-invented representations and strategies (Hill, Ball, & Schilling, 2008). On one occasion, I asked prospective teachers to write about the strengths and weaknesses of three student-invented representations in a teacher-oriented journal article about encouraging students to generate their own representations for a set of data showing how much time a group of seventh-grade students spent watching television each week (McClain, 1999). The data set is shown in Figure 3 along with two of the student-generated representations from the article. The first student-generated representation is similar to a conventional histogram, except that the intervals are not even and

the bars do not touch. The other student-generated representation shown in Figure 3 looks like a conventional histogram at first glance, but it was actually produced by rank ordering the data values, splitting them into five groups, computing the sum for each group, and then graphing the sums for each of the five groups. After reading the article, prospective teachers were asked to write about the strengths and weaknesses of each student-generated representation.

<INSERT FIGURE 3 HERE>

Prospective teachers' analyses of the student-generated representations shown in Figure 3 provided grounds for inferring another specialized knowledge KDU: recognizing unconventional modifications to conventional statistical representations. Some respondents recognized that frequencies were not displayed in the "hours of TV vs. number of students" representation. They noted that the heights of the bars represented sums rather than frequencies. On the other hand, some who did not notice this feature responded,

> Christy: [It] is a well-made histogram. Again, all information is clear and organized in equal increments allowing for easy understanding of the information given. You can see an obvious growth between the numbers of students and the number of hours watched.
>
> Stacey: [It] is the strongest of all the histograms. The group divided up the number of hours evenly and knew how to accurately display the data. The weakness in this graph is that it is hard to decipher the exact number of hours watched. This could have been spread out more to make it easier to see.
>
> Susan: [It] doesn't look much like a histogram it's more of a bar graph. They should have chosen intervals that would allow the bars to be right next to each other with no spaces in between. They also shouldn't have gone so high on the amount of hours, they counted to

120 but the graph stops at 100. On the positive side the graph was pretty organized and the students got the general idea of a histogram.

Although the prospective teachers had constructed and analyzed histograms during the SKT course, some did not identify differences between the student-invented histogram and a conventional one. It seems likely that the appraisal of children's novel representations was particularly difficult in this case because the "hours of TV vs. number of students" display appeared, on the surface, to be a conventional histogram rather than a one with student-invented modifications to common graphing conventions.

KDU Summary

In summary, four KDUs for SKT subject matter knowledge have been discussed. The first, related to common content knowledge, was conceiving of theoretical probability as an anchor for predicting long-term behavior. This KDU is related to common knowledge because the distinction between theoretical and experimental probability is part of many Pre-K-8 mathematics curricula and it is also relevant to professions other than teaching. The second, related to horizon knowledge, was conceiving of typical deviation as a measure of spread. This KDU relates to horizon knowledge because the idea of typical deviation, which is part of Pre-K-8 curricula, leads to standard deviation, which is often taught in later grades. Knowing the mathematics of this progression can influence the manner in which teachers introduce the idea of mean absolute deviation in the earlier grades. The third and fourth subject matter knowledge KDUs related to specialized knowledge, which entails the ability to select and interpret statistical representations relevant to student learning. Conceiving of hat plots as transitional representations between uncondensed and condensed data displays was proposed as a specialized knowledge KDU, as was recognizing unconventional modifications to conventional statistical

representations. Each of these understandings marks significant conceptual advances that appear to require more than just an explanation or demonstration to develop.

Pedagogically Powerful Ideas and Pedagogical Content Knowledge

Although the KDU construct is useful for identifying landmarks in the development of subject matter knowledge, it is not adequate for describing all aspects of MKT and SKT. Silverman and Thompson (2008) argued,

> Teachers who develop KDUs of particular mathematical ideas can do impressive mathematics with regard to those ideas, but it is not necessarily true that their understandings are powerful pedagogically; It is possible for a teacher to have a KDU and be unaware of its utility as a theme around which productive classroom conversations can be organized. Developing MKT, then, involves transforming these personal KDUs of a particular mathematical concept to an understanding of: (1) how this KDU could empower their students' learning of related ideas; (2) actions a teacher might take to support students' development of it and reasons why those actions might work (p. 502).

Silverman and Thompson used Piaget's notion of "decentering" to emphasize that teachers must learn to see things from children's perspectives in order to develop MKT. Decentering involves differentiating one's own viewpoint from that of another or attempting to view the world from another person's perspective. Through decentering, *personally powerful ideas* (i.e., KDUs) become *pedagogically powerful ideas*. Decentering allows teachers to understand potential student difficulties with content, providing a basis for developing and selecting strategies likely to support students' learning. In terms of the Hill, Ball, and Schilling (2008) framework, decentering would be required for development of the PCK categories of knowledge of content

and students, knowledge of content and teaching, and curriculum knowledge, since each category requires teachers to look beyond the KDUs they themselves have constructed.

Knowledge of Content and Students: Reading Data Displays

As noted in the earlier discussion of hat plots and box plots, the distinction between condensed and uncondensed data displays was studied in the SKT course. One of the articles assigned during the course (Kader & Mamer, 2008) described differences between reading condensed and uncondensed displays. It pointed out that children must transition from reading displays that show all data values to those that use groupings of data, and it provided examples of children's thinking with both types of displays. I assigned the following writing prompt to be completed in conjunction with the reading: "In your own words, and drawing upon the ideas in the article, explain why histograms and box plots are more challenging to use and interpret than line plots, dot plots, and picture graphs." Although this prompt deals with issues similar to those for the specialized knowledge prompt I assigned about hat plots and box plots, I discuss it in terms of knowledge of content and students to help differentiate specialized content knowledge from knowledge of content and students: The specialized knowledge prompt about hat plots and box plots asked for a comparison of two mathematical objects, but the knowledge of content and students prompt about histograms and box plots asked for identification of difficulties students may encounter in working with such objects. While it is true that responses to the specialized knowledge prompt could also include discussion of student difficulties, such discussion is only *required* in response to the knowledge of content and students prompt.

Many responses to the knowledge of content and students writing prompt discussed the specific issue of reading condensed displays vs. those showing individual values, but some did not. Some tended to write of potential student difficulties in fairly vague and general terms, as in

Angela's response: "With dot plots, picture graphs and line plots the information is much easier for students to read and understand, while histograms and box plots can be cluttered and quite confusing to interpret." Similarly, Sarah wrote,

> Histograms and box plots are more challenging to use and interpret than line plots, dot plots, and picture graphs because there is a more wide range of numbers. It is harder to see the difference when being compared to another set of data. When there are dot plots, line plots, and picture graphs it is easier to recognize overlap and draw a conclusion to that set of data. There is more variation in data, in terms of ranges, with dot plots, line plots, and picture graphs.

Others who did not address the issue of condensed versus uncondensed data displays attributed difficulties in reading condensed displays to students' previous experiences. For example,

> Anna: Histograms and box plots are more challenging to use and interpret because they are harder to read and understand than dot plots, line plots and picture graphs. I think this is because from a young age we are more exposed and more comfortable with dot plots, line plots and picture graphs.

Although such responses are not wholly inaccurate, and do actually seem to represent attempts to view ideas from children's perspectives, the types of interventions teachers with these kinds of superficial understandings of student difficulties might devise would be likely to leave key student difficulties unaddressed.

In terms of Silverman and Thompson's (2008) framework, understandings related to knowledge of content and students do not necessarily constitute pedagogically powerful ideas, because such understandings may not include specific actions teachers can take to help students build KDUs. However, according to Silverman and Thompson, a necessary part of constructing a

pedagogically powerful idea is that a teacher "has constructed models of the variety of ways students *may* understand the content (decentering)" (p. 508). Hence, teachers who recognize that the condensed nature of box plots and histograms pose a significant cognitive hurdle for students are further along the way to developing a pedagogically powerful idea than those who conceive of student difficulties in less specific terms. Once teachers comprehend students' thinking, they must devise or select classroom activities that are likely to build on it in productive ways or challenge it in order to fully develop a pedagogically powerful idea. The abilities to develop and select classroom activities that enhance children's thinking resonate with the Hill, Ball, and Schilling (2008) categories of knowledge of content and teaching and curriculum knowledge.

Knowledge of Content and Teaching: Mean and Median

Knowledge of content and teaching is evidenced by the ability to employ content-specific strategies to address student learning needs. One of the SKT course assignments was to analyze student results from statistics items on the National Assessment of Educational Progress (NAEP) as presented in a teacher-oriented journal article (Zawojewski & Shaughnessy, 2000b). Some of the NAEP results presented in the article indicated that when children are asked to select a measure of center to describe the typical value in a data set, they often choose mean over median without regard to the distribution of data. Prospective teachers wrote responses to the following item after reading the article, "Explain why some students believe the mean is always a better indicator of typical value than the median. How might you convince these students that the median is more appropriate in some cases?" The item was intended to elicit knowledge of content and teaching because it prompted respondents to describe teaching strategies likely to help develop a specific aspect of children's statistical thinking.

Some of the responses to the item indicated that prospective teachers were in the process of constructing KDUs related to the NAEP items while reading the article. For example,

> Lisa: To be completely honest, until reading the article I as well thought that the mean should have been used all of the time. It is very easy for children to get confused about this. When you hear average, you think of the total frequency. Children may believe this is the best because it is drawing from all of the info, not just a middle number. However, when the numbers are ranging for example 70 85 95, you should use the mean. If most of the numbers are in the same range you should use the median since it is drawing from that same group. I would tell them if most of the numbers are within 15 points of range to use the median, otherwise use the mean.

Although Lisa appeared to have set aside the idea that the mean is always superior to the median for describing data, some problematic elements needed resolution to allow her to develop a pedagogically powerful idea about teaching mean and median. One needed element was to shift attention from just the range of a data set to allow for examination of outliers as well, since the mean is sensitive to outliers but the median is not. Another needed element was to avoid embracing a deterministic rule for choosing between mean and median. This is largely a statistical concern, as the context for the data, and not just mathematical principles, are often taken into account when choosing between mean and median. Finally, the pedagogical choice to transmit the deterministic rule to students also blocked development of a pedagogically powerful idea, as transmission-oriented strategies are generally associated with lower levels of student learning (Campbell, Kyriakides, Muijs, & Robinson, 2004). Even some of those who attended to the issue of outliers, and not just the range, still favored transmission of deterministic rules (e.g., "I would advise the students to use the median whenever there is an outlier in the data").

For an example of a response showing evidence of further progress toward pedagogically powerful ideas for teaching mean and median, consider the following:

> Lori: I understand where those students who believe that mean is always a better indicator of typical value are coming from. The mean is the average of a data set and the term "average" sounds like it should be the correct indicator. When you think average you think common ground between all numbers in the data set. There is a number that will always affect the average and make it less representative of the numbers and they are known as outliers. Outliers will either bring your mean significantly up or significantly down. There are plenty of activities you could do in the classroom with students that would help them to see this more clearly. You could take something that relates to them such as their grades in school. Supply them with the example that they have five test grades for the term and they are 99, 97, 94, 92, 57. The 57 was a section they really had problems with and it was a very hard test. The median for this data set would be 94. The average for this data set is 87. You could then ask the students which would they rather tell their parents. An 87 is not bad at all but I think that most students would want to show their parents the 94 instead of the 87. The 94 looks better and even though the student never got any B's on test their average still comes out to a B which does not look as good compared to an A. This is one of the many examples you could use when explaining the affect [*sic*] of outliers to your students.

Although scholars in mathematics and statistics education could undoubtedly point to potential flaws in the pedagogical ideas described in Lori's repsonse, it has some desirable features not present in the responses described earlier. First, it avoids prescription of a deterministic rule for choosing between mean and median. Instead, it suggests letting students choose between the two

based on the context. The context for the proposed teaching scenario is also one with which students are likely to be familiar and have an emotional investment, as opposed to some responses that either lacked context or proposed contexts not as likely to engage students. Choosing a context for a statistical problem is a non-trivial matter, as engaging contexts make it more likely that children will bring the full power of their reasoning to a given task (Gal, 1998).

The prospective teachers' responses to the mean and median task suggest both similarities and differences between knowledge of content and teaching for MKT and SKT. One similarity between the two is that transmission-oriented strategies can block the development of pedagogically powerful ideas. Research generally does not support transmission-oriented teaching in statistics or mathematics classrooms (Campbell et al., 2004), but such teaching is prevalent in the U.S. (Jacobs et al., 2006), the site for the SKT course. One of the greatest barriers to the development of pedagogically powerful ideas in such a culture may be to have prospective teachers question transmission-oriented strategies.

A difference between MKT and SKT related to teaching mean and median is the role of determinism. The computations of mean and median require deterministic mathematical algorithms. There is no ambiguity about what the result should be when the algorithms are applied. Many mathematical activities have similar characteristics. Even in reform-oriented curricula that avoid transmission-oriented strategies for teaching mathematics, students are often led toward deterministic generalizations that apply across a range of problems. Arriving at such generalizations is part of the essence of doing mathematics. In statistics, on the other hand, arriving at reasoned judgments is a common activity. In the case of choosing between mean and median, there can be space for reasonable arguments that arrive at different conclusions. The space for different arguments is created by selection of distribution analysis techniques (e.g.,

deciding from among several available procedures for determining "outliers") and interpretations of the context in which the data are set (e.g., deciding if an observation is an outlier due to random variation, inaccurate measurement, data from a case with special characteristics, or some other cause). To develop SKT, prospective teachers must understand mathematics as a discipline where deterministic questions are often appropriate and desirable and statistics as a discipline where multiple viable conclusions based on reasoned arguments from context and other factors are the norm (Rossman, Chance, & Medina, 2006). Leading students to believe that a universal algorithm exists for deciding between mean and median contributes to a misleading portrayal of statistics as an essentially deterministic discipline.

Curriculum Knowledge: Introducing Conventional Graphs

Curriculum knowledge suggests the ability to understand the broad sweep of sequences of lessons rather than just knowledge of content-specific teaching strategies for a single lesson. To better understand prospective teachers' curriculum knowledge, I probed their thinking about a non-conventional curricular sequence for introducing statistical graphs. After reading McClain's (1999) teacher-oriented article describing how she encouraged students to invent graphs before introducing conventional ones, prospective teachers responded to the following item:

> On p. 374, the author asked, "Do students first need to know how to construct various types of graphs before they can engage in an analysis of data, or can they learn how to construct various types of graphs by engaging in data analysis?" Write a response to the author's question. Explain how your response compares to the position taken by the author of the article.

The item was carefully worded to elicit a justified opinion and compare it to the author's position rather than to require conformity with McClain's approach. Many respondents did, in fact, argue

that students should learn conventional graphical displays before analyzing data rather than adopting McClain's approach. The prospective teachers' preference for the traditional approach was not unexpected, given that the predominant mathematics classroom culture in the U.S. is one of "learning rules and practicing procedures" (Stigler & Hiebert, 1999), meaning that teachers generally demonstrate how to perform an exercise and then have students do it on their own.

Although positions advocated in the responses to the item were not surprising, the responses themselves revealed obstacles to developing pedagogically powerful ideas about data analysis curricula. In some cases, prospective teachers did not even realize their preference for a traditional approach conflicted with the curricular alternative proposed by McClain (1999). One such response was,

> Maggie: I believe students first need to know how to construct various graphs before they can engage in analysis of data. In the article since they didn't know how to correctly make a graph and distinguish why you need to make a certain type of graph they couldn't even make a correct graph in which you could analyze the data. My views coincide with the authors.

Maggie seemed to assume that McClain's primary goal was to have students produce "correct" graphs when the primary goal actually was to have students devise ways to make sense of data. The unrecognized conflict between instructional goals blocked serious consideration of McClain's proposed alternative to traditional sequencing of statistical ideas. The unrecognized conflict appears to be the primary barrier to development of curriculum knowledge rather than the decision not to adopt McClain's approach, although understanding and exploring the conflict in depth may well result in the adoption of at least some aspects of the approach.

Another factor preventing consideration of alternative curricular approaches was the underlying presence of conflicting theories of instruction. One prospective teacher wrote,

> Natasha: I think that it is more of how the students learn and how they represent the data in their own way. Some students may learn by first understanding what a graph is and do an analysis of the data, while others may do it in reverse. I am not really taking a position with or against the author, I just think what I think and from my knowledge of how children learn and their learning styles.

From this idiosyncratic "learning styles" theory position, McClain's question was dismissed as irrelevant, since students should not be forced into either a traditional curriculum or the alternative suggested in the article. A weakness of this dismissive stance is that it does not provide a specific, viable alternative curricular approach. Others harboring theories of learning that conflicted with those expressed in the article disagreed with McClain about the roles that struggle and confusion play in the learning process. For example, Angela wrote,

> I do believe students are capable to learn how to construct various types of graphs by engaging in data analysis, however I believe if it is required that way and teachers are not guiding them in how to construct graphs it will end up with a lot of frustration and probably more graphs that are unreadable in comparison to the data they have received. The author of this article would disagree with me, saying students need to explore the data and make their own graphs based on their knowledge of the data, but like I said, I believe that would end up with more confusion for the students than the few benefits.

In such responses, the productive roles that appropriate levels of cognitive struggle may play in learning were left unexplored. A key part of McClain's position was that such struggle is necessary and desirable as students attempt to derive meaning of data.

Some prospective teachers who considered struggle to be part of learning voiced agreement with McClain's position, as in the following response:

> Deanna: I think that students can learn how to construct various types of graphs by engaging in data analysis. I believe that people learn best through trial and error. If the teacher wants the students to graph a set of data, they should let the student decide how they think it should be graphed. If the student has incorrectly graphed the data, the teacher should ask them questions as to of why they graphed it the way they did and not another way, such as the correct way. The teacher should not come out and tell the student that they have incorrectly graphed the data, instead the child should be able to figure it out, based on the discussion with the teacher. I feel that the author of this article and I have the same views when it comes to children and figuring out what types of graphs represent different types of data.

Although this response overstated the extent of agreement between the respondent and McClain, it did allude to substantive grounds for analyzing McClain's approach. As in some of the other responses, there still was an unrecognized conflict between McClain's instructional goals and those of the prospective teacher – the primary goal of the former was to encourage students to make sense of data and the primary goal of the latter was to have them produce "correct" graphs. Nonetheless, the prospective teacher's willingness to allow "trial and error" suggests some recognition of the role cognitive struggle plays in learning.

The abilities to consider instructional goals and theories of learning that conflict with one's own appear to be essential to developing pedagogically powerful ideas. The notion of "decentering" is again relevant in this regard, although it is not just the type of decentering that allows teachers to view things from students' perspectives. Teachers must also develop the

ability to understand the perspectives of curriculum developers. Conflicts that exist between the instructional goals and learning theories of teachers and curriculum developers can contribute to low fidelity of implementation of innovative curriculum materials (Tarr et al., 2008). Teachers who wish to eliminate cognitive struggle and heavily emphasize the production of "correct" graphs as students begin to analyze data are not optimally positioned to implement curricula that encourage students to invent their own graphs before conventional ones are introduced. Hence, part of developing prospective teachers' curriculum knowledge must consist of helping them understand the goals and learning theories underlying innovative curricula.

Hypothesized SKT Framework

The SKT framework suggested by the preceding discussion integrates the work of Simon (2006), Silverman and Thompson (2008), and Hill, Ball, and Schilling (2008). Figure 4 summarizes the theoretical claims that have been made and hypothesizes relationships among them. The three rectangles at the top of the figure illustrate Silverman and Thompson's claim that KDUs must be transformed into pedagogically powerful ideas through the mechanism of decentering. The vertical branch on the far left side of the figure suggests that specific KDUs can be identified by using the Hill, Ball, and Schilling subject matter categories of common knowledge, specialized knowledge, and horizon knowledge. The arrow extending from the bottom center of the rectangle labeled "decentering" suggests that decentering is necessary for development of pedagogical content knowledge; as such knowledge entails making subject matter comprehensible to others. Hill, Ball, and Schilling's pedagogical content knowledge categories are shown in the figure to identify examples of this type of knowledge. Knowledge of content and teaching and curriculum knowledge are portrayed as categories of pedagogically powerful ideas, since such knowledge entails having content-specific strategies for helping

children develop KDUs. Knowledge of content and students is positioned as a potential basis for the formation of pedagogically powerful ideas.

<INSERT FIGURE 4 HERE>

The framework summarized in Figure 4 also refines and supplements the Groth (2007) SKT framework. Refinement is apparent in the re-categorization of Groth's specialized knowledge examples as knowledge of content and students. Specialized knowledge is framed in terms of knowledge of representations that facilitate students' learning and the ability to appraise students' novel strategies and representations. Supplements to Groth's SKT framework include the addition of the knowledge categories of horizon knowledge, knowledge of content and teaching, and curriculum knowledge. As with Groth's original SKT framework, some knowledge elements within each category serve to distinguish SKT from MKT while other elements suggest similarities. A final notable supplement to the Groth SKT framework is the identification of theoretical mechanisms suitable for characterizing and identifying cognitive landmarks in the development of subject matter knowledge (KDUs) and pedagogical content knowledge (pedagogically powerful ideas).

Theoretical frameworks of this nature are to guide researchers in asking questions, formulating hypotheses, and determining variables and relationships to investigate (Johnson, 1980). Research questions suggested by the framework summarized in Figure 4 include those related to identification of KDUs and pedagogically powerful ideas for SKT, further exploration of the relationship between SKT and MKT, and empirical tests of the hypothesized relationships among knowledge elements. For example, the framework motivates questions such as: What other KDUs and pedagogically powerful ideas are pertinent to SKT? In particular, what KDUs and pedagogically powerful ideas related to statistical ideas such as informal inference and

distribution remain to be described? What other traits are shared between MKT and SKT? What other traits differ? How strong are the proposed relationships among KDUs, pedagogically powerful ideas, and SKT knowledge categories? Answers to such questions can contribute to continued iterative refinement of the theoretical framework and its underlying hypotheses while also providing guidance for mathematics and statistics teacher education efforts.

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Table 1

Key Constructs Incorporated in SKT Theoretical Framework

Construct	Authors	Brief definition	Role within the framework
Statistical	Groth (2007)	Knowledge required for	Premise that statistics and

knowledge for teaching		teaching statistics	mathematics are not equivalent disciplines
Common content knowledge	Hill, Ball, & Schilling (2008)	Knowledge that teachers are responsible for teaching their students	Component of subject matter knowledge
Specialized content knowledge	Hill, Ball, & Schilling (2008)	Knowledge of various representations and unusual student strategies	Component of subject matter knowledge
Horizon knowledge	Hill, Ball, & Schilling (2008)	Knowledge of the broader content landscape in which curriculum is situated	Component of subject matter knowledge
Knowledge of content and students	Hill, Ball, & Schilling (2008)	Knowledge of common student difficulties with content and their thinking patterns	Component of pedagogical content knowledge
Knowledge of content and teaching	Hill, Ball, & Schilling (2008)	Knowledge of content-specific teaching strategies	Component of pedagogical content knowledge
Curriculum knowledge	Hill, Ball, & Schilling (2008)	Knowledge of how to arrange curricula to enhance student learning	Component of pedagogical content knowledge
Key developmental	Simon (2006)	Cognitive landmarks in the learning of fundamental	Mechanism to identify cognitive landmarks in subject

understandings (KDUs)		ideas needed to understand content	matter knowledge development
Pedagogically powerful ideas	Silverman & Thompson (2008)	Ideas that occur as the result of transforming KDUs (i.e., personally powerful ideas) into ideas that facilitate students' learning of the KDUs	Mechanism to identify cognitive landmarks and mechanisms in pedagogical content knowledge development

Figure Captions

Figure 1. Mean absolute deviation (MAD) example for a data set with weights of individuals

Figure 2. A hat plot representation for a data set with heights of individuals

Figure 3. Data set and student-invented representations from McClain (1999, pp. 375, 377, 379)

Figure 4. Hypothetical SKT elements and developmental structure

Figure 1

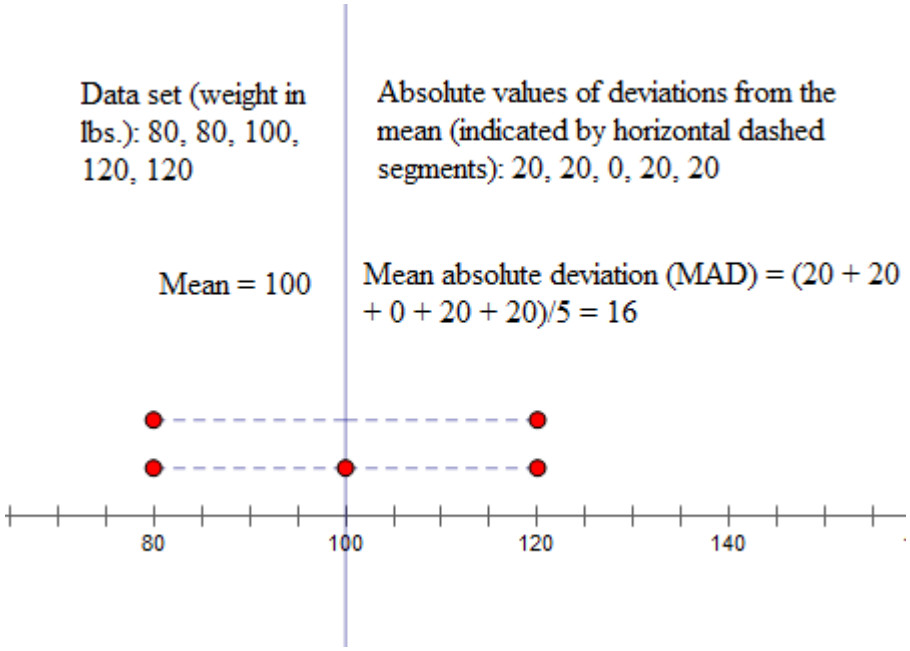

Data set (weight in lbs.): 80, 80, 100, 120, 120
Absolute values of deviations from the mean (indicated by horizontal dashed segments): 20, 20, 0, 20, 20
Mean = 100
Mean absolute deviation (MAD) = (20 + 20 + 0 + 20 + 20)/5 = 16
80
100
120
140

Figure 2

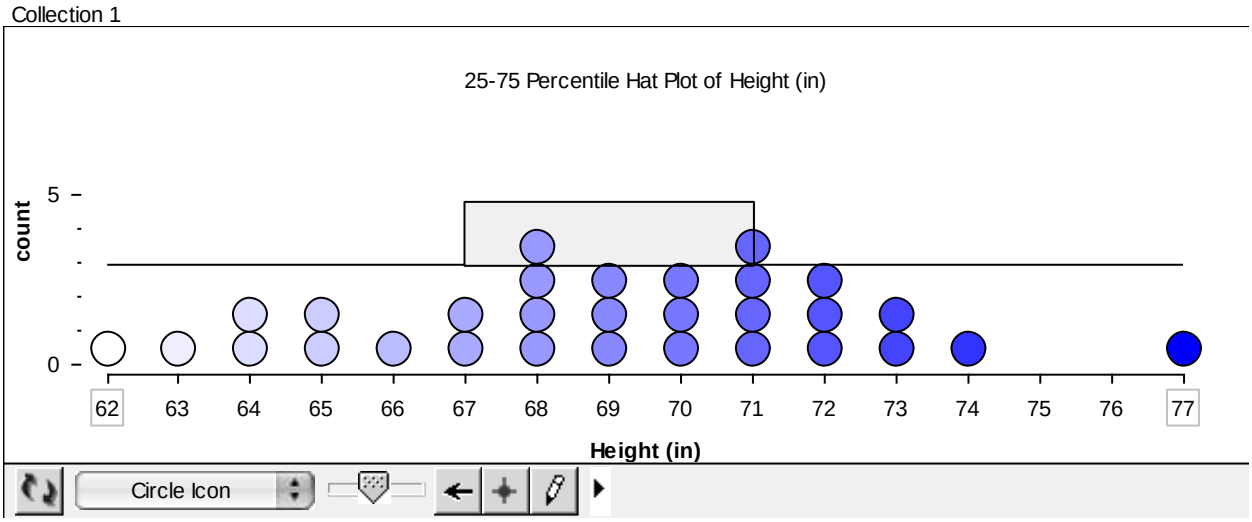
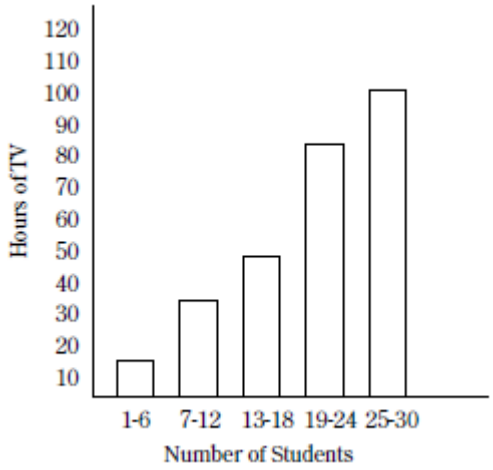

Collection 1
25-75 Percentile Hat Plot of Height (in)
count
5
0
62
63
64
65
66
67
68
69
70
71
72
73
74
75
76
77
Height (in)
Circle Icon

Figure 3

How Much Television?

Below are the results of a survey taken of 30 seventh graders to find out how many hours of television they watch in a week. The principal has asked you to summarize and represent these data in some form so that parents will be able to understand them quickly when they are posted on the bulletin board. The principal also asks you to write a short report for parents, explaining what the data show.

1.5	21	12.5
0	2.5	15
23	19	4
14	8	16
13.5	16.5	6
4.5	9	18
5	10.5	8.5
6	3	9
11.5	3.5	19.5
13	10	9



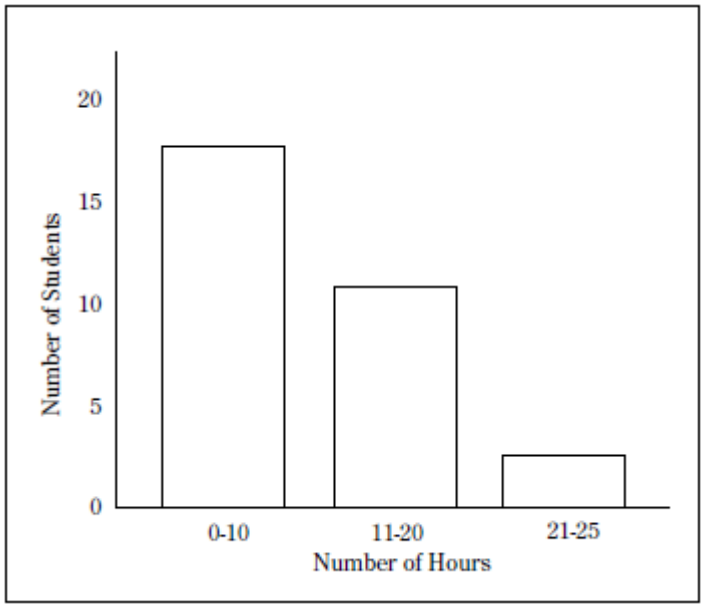
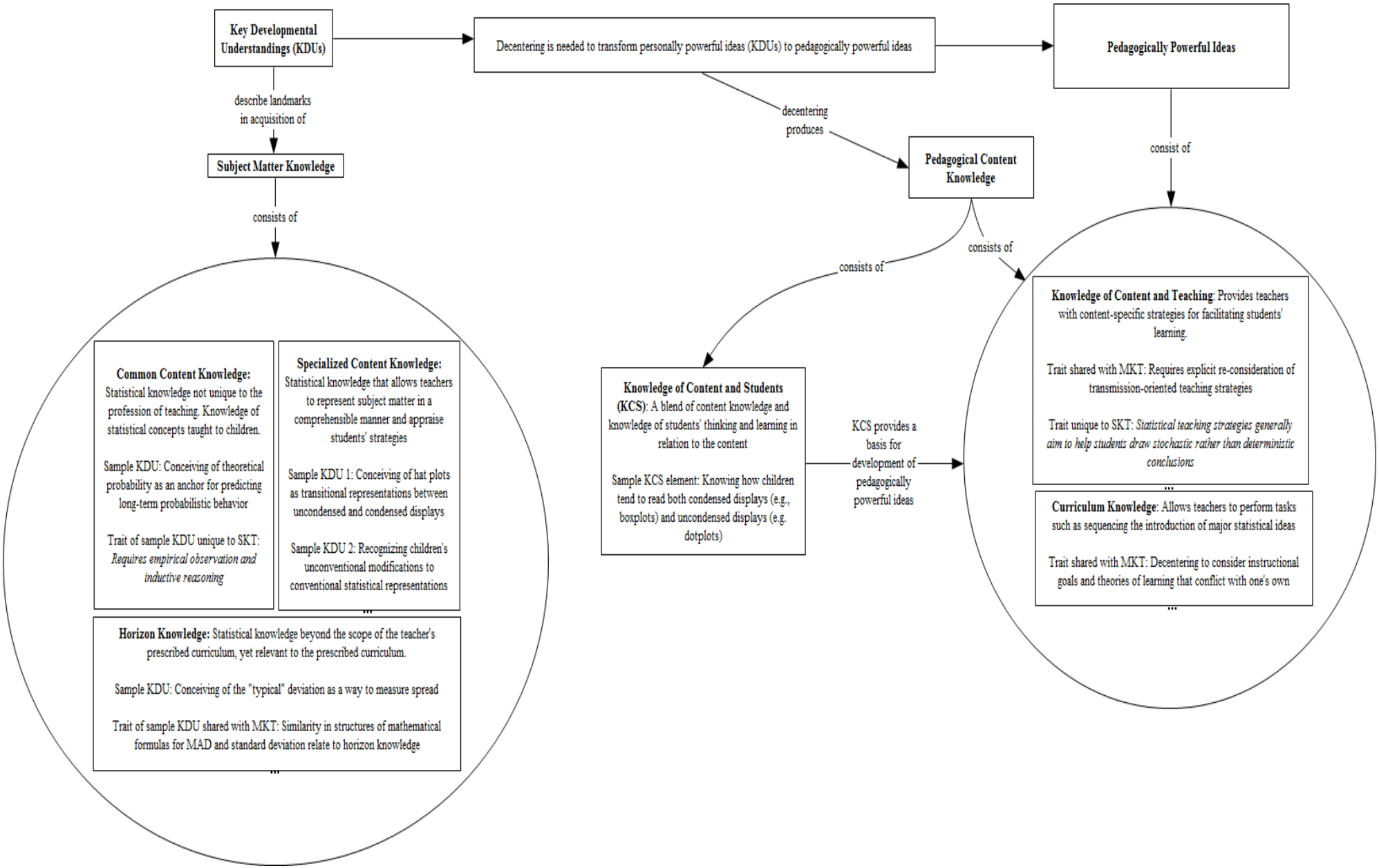


Figure 4


Key Developmental Understandings (KDUs)
Decentering is needed to transform personally powerful ideas (KDUs) to pedagogically powerful ideas
Pedagogically Powerful Ideas
describe landmarks in acquisition of
Subject Matter Knowledge
consists of
decentering produces
Pedagogical Content Knowledge
consist of
consists of
consists of
Common Content Knowledge: Statistical knowledge not unique to the profession of teaching. Knowledge of statistical concepts taught to children.
Sample KDU: Conceiving of theoretical probability as an anchor for predicting long-term probabilistic behavior
Trait of sample KDU unique to SKT: Requires empirical observation and inductive reasoning
Specialized Content Knowledge: Statistical knowledge that allows teachers to represent subject matter in a comprehensible manner and appraise students' strategies
Sample KDU 1: Conceiving of hat plots as transitional representations between uncondensed and condensed displays
Sample KDU 2: Recognizing children's unconventional modifications to conventional statistical representations
...
Horizon Knowledge: Statistical knowledge beyond the scope of the teacher's prescribed curriculum, yet relevant to the prescribed curriculum.
Sample KDU: Conceiving of the "typical" deviation as a way to measure spread
Trait of sample KDU shared with MKT: Similarity in structures of mathematical formulas for MAD and standard deviation relate to horizon knowledge
...
Knowledge of Content and Students (KCS): A blend of content knowledge and knowledge of students' thinking and learning in relation to the content
Sample KCS element: Knowing how children tend to read both condensed displays (e.g., boxplots) and uncondensed displays (e.g. dotplots)
KCS provides a basis for development of pedagogically powerful ideas
Knowledge of Content and Teaching: Provides teachers with content-specific strategies for facilitating students' learning.
Trait shared with MKT: Requires explicit re-consideration of transmission-oriented teaching strategies
Trait unique to SKT: Statistical teaching strategies generally aim to help students draw stochastic rather than deterministic conclusions
...
Curriculum Knowledge: Allows teachers to perform tasks such as sequencing the introduction of major statistical ideas
Trait shared with MKT: Decentering to consider instructional goals and theories of learning that conflict with one's own
...