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Title of Dissertation: Spectral sets and functions on Euclidean Jordan algebras

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Abstract

Title of dissertation: Spectral sets and functions
on Euclidean Jordan Algebras

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Dissertation directed by: Dr. Muddappa S. Gowda
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This thesis studies spectral and weakly spectral sets/functions on Euclidean Jordan algebras. These are generalizations of similar well-known concepts on $\mathcal{S}^n$ and $\mathcal{H}^n$, the algebras of $n \times n$ real symmetric and complex Hermitian matrices. Spectral sets and functions on a Euclidean Jordan algebra $\mathcal{V}$ are defined by:

$$E := \lambda^{-1}(Q) \quad \text{and} \quad F := f \circ \lambda,$$

where $Q \subseteq \mathcal{R}^n$ and $f : \mathcal{R}^n \to \mathcal{R}$ are permutation invariant and λ denotes the eigenvalue map that takes an element $x \in \mathcal{V}$ to its eigenvalue vector in $\mathcal{R}^n$ consisting of eigenvalues of x written in the decreasing order.

In this thesis, we study properties of such sets/functions and show how they are related to algebra automorphisms and majorization. We show they are indeed invariant under algebra automorphisms of $\mathcal{V}$, hence weakly spectral with converse holding when $\mathcal{V}$ is essentially simple.

For a spectral set K, we discuss the transfer principle and a related metaformula. When K is also a cone, we show that the dual of K is a spectral cone under certain conditions. We also discuss the dimension of K, and characterize the pointedness/solidness of K. Specializing, we study permutation invariant (proper) polyhedral cones in $\mathcal{R}^n$. We show that the Lyapunov rank of such a cone divides n.

Lastly, we study Schur-convexity of a spectral function F and describe some applications.

Spectral sets and functions on Euclidean Jordan algebras

by

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Dissertation submitted to the Faculty of the Graduate School of the University of Maryland, Baltimore County in partial fulfillment of the requirements for the degree of Doctor of Philosophy
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Dedication

To my family and my beloved one

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Juyoung Jeong

University of Maryland, Baltimore County

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Table of Contents

Notation

Symbol	Meaning
$\mathcal{R}$	The field of real numbers
$\mathcal{R}_+$	The set of nonnegative numbers
$\mathcal{R}^n$	The vector space of n-tuples with components in $\mathcal{R}$
$\mathcal{R}^n_+$	The nonnegative orthant in $\mathcal{R}^n$
$\mathcal{S}^n$	The space of all $n \times n$ real symmetric matrices
$\mathcal{H}^n$	The space of all $n \times n$ complex Hermitian matrices
$\mathcal{L}^n$	The Jordan spin algebra
$\mathcal{V}$	Euclidean Jordan algebra
$\mathcal{V}_+$	The symmetric cone of $\mathcal{V}$
Σ_n	The set of all $n \times n$ permutation matrices
$\mathbf{1}$	The vector of ones in an appropriate vector space $\mathcal{R}^d$
A^T	The transpose of a matrix A
$\langle x, y \rangle$	Inner product of x and y in $\mathcal{V}$
$\\|x\\|$	Norm of x in $\mathcal{V}$ defined by $\sqrt{\langle x, x \rangle}$
S°	The interior of a set S
$\bar{S}$	The closure of a set S
∂S	The boundary of a set S
S^*	The dual cone of a set S
$S^\perp$	The orthogonal complement of a set S
$\mathrm{conv}(S)$	The convex hull of a set S
$\mathrm{cone}(S)$	The conic hull of a set S
$\Sigma_n(S)$	The set given by $\{\sigma(u) \mid u \in S,\, \sigma \in \Sigma_n\}$
$\mathrm{Aut}(\mathcal{V})$	The set of all algebra automorphisms on $\mathcal{V}$
$\mathrm{Aut}(\mathcal{V}_+)$	The set of all cone automorphisms on $\mathcal{V}$
$\mathrm{Orth}(\mathcal{V})$	The set of all orthogonal transformations on $\mathcal{V}$
$\mathrm{DS}(\mathcal{V})$	The set of all doubly stochastic transformations on $\mathcal{V}$

Chapter 1

Introduction

This dissertation mainly deals with spectral sets and functions on Euclidean Jordan algebras.

In matrix algebra, a permutation matrix is obtained by permuting rows of the $n \times n$ identity matrix. The set of all permutation matrices is denoted by Σ_n. A set $Q \subseteq \mathcal{R}^n$ is permutation invariant if $\sigma(Q) = Q$ for all $\sigma \in \Sigma_n$. Likewise, a function $f : \mathcal{R}^n \to \mathcal{R}$ is said to be permutation invariant (also called symmetric) if $f(\sigma(u)) = f(u)$ for all $u \in \mathcal{R}^n$ and $\sigma \in \Sigma_n$, respectively. The permutation invariance of sets/functions has numerous applications in various fields.

Now, we generalize these concepts to Euclidean Jordan algebras.

A set E in a Euclidean Jordan algebra $\mathcal{V}$ is said to be a **spectral set** [1] if it is of the form

$$E = \lambda^{-1}(Q),$$

where Q is a permutation invariant set in $\mathcal{R}^n$ and $\lambda : \mathcal{V} \to \mathcal{R}^n$ is the eigenvalue map (which takes x to $\lambda(x)$, the vector of eigenvalues of x with entries written in the decreasing order). A function $F : \mathcal{V} \to \mathcal{R}$ is said to be a **spectral function** [1] if it is of the form

$$F = f \circ \lambda,$$

where $f : \mathcal{R}^n \to \mathcal{R}$ is a permutation invariant function.

The above concepts are generalizations of similar concepts that have been extensively studied in the setting of $\mathcal{R}^n$ and in $\mathcal{S}^n$ ($\mathcal{H}^n$), the space of all $n \times n$ real symmetric (respectively, complex Hermitian) matrices, see for example, [6], [8], [9], [20], [22], [27], [28], [30], [43], and the references therein. In the case of $\mathcal{S}^n$ ($\mathcal{H}^n$), spectral sets/functions are precisely those that are invariant under linear transformations of the form $X \to UXU^*$, where U is an orthogonal (respectively, unitary) matrix.

There are a few works that deal with spectrality on general Euclidean Jordan algebras. Baes [1] discusses some properties of Q which get transferred to E (such as closedness, openness, boundedness/compactness, and convexity) and properties of f which get transferred to F (such as convexity and differentiability). Sun and Sun [47] deal with the transferability of the semismoothness properties of f to F. Ramirez, Seeger, and Sossa [39] and Sossa [46] deal with a commutation principle and a number of applications.

For a Euclidean Jordan algebra $\mathcal{V}$, a linear transformation $\phi : \mathcal{V} \to \mathcal{V}$ is an algebra automorphism if it is invertible and preserves Jordan products:

$$\phi(x \circ y) = \phi(x) \circ \phi(y),$$

for all x, $y \in \mathcal{V}$. We say that a set $E \subseteq \mathcal{V}$ and a function $F : \mathcal{V} \to \mathcal{R}$ are **weakly spectral** [23] if $\phi(E) = E$ and $F(\phi(x)) = F(x)$ for all $x \in \mathcal{V}$ and all algebra automorphisms $\phi \in \mathrm{Aut}(\mathcal{V})$, respectively. Note that a permutation matrix $\sigma \in \Sigma_n$ is an algebra automorphism on $\mathcal{R}^n$ if we regard σ as a transformation from $\mathcal{R}^n$ to itself. Thus, we can naturally generalize the permutation invariance on $\mathcal{R}^n$ to any Euclidean Jordan algebra $\mathcal{V}$ by means of algebra automorphisms.

There are only a few works dealing with weak spectrality on Euclidean Jordan algebras, for instance, see [23], [24], [19]. In this thesis, we study some properties of permutation invariant sets/functions on $\mathcal{R}^n$ which get transferred to the corresponding spectral sets/functions on $\mathcal{V}$. Interconnections and relationships between spectrality and weak spectrality on $\mathcal{V}$ are also discussed.

1.1 Organization of the thesis

A brief outline of each chapter is as follows:

- In Chapter 2, we introduce basic definitions and properties of concepts in Euclidean Jordan algebra, convex analysis, and majorization which will be used in the thesis.
- Chapter 3 deals with spectral sets in Euclidean Jordan algebras. Specifically, we show that how spectral sets are related to weakly spectral sets. The transfer principle and a metaformula are also discussed in this chapter.
- Specializing results in Chapter 3, spectral cones in Euclidean Jordan algebras are studied in Chapter 4. Here, we study equivalent characterizations as well as dimensionality, pointedness, and solidness of spectral cones.
- Based on a result of Gowda and Tao [16], we investigate the Lyapunov rank of

permutation invariant proper polyhedral cone in Chapter 5.

- Chapter 6 focuses on spectral functions in Euclidean Jordan algebras. We relate spectrality and Schur-convexity and prove some majorization inequalities.
- In the final chapter, we make some concluding remarks as well as pose some open questions.

Chapter 2

Preliminaries

2.1 Euclidean Jordan algebras

2.1.1 Definitions and some properties

Definition 2.1.1 Let $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ be a finite dimensional real Hilbert space with a bilinear product $(x, y) \mapsto x \circ y : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ satisfying the following:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathcal{V}$.

(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, where $x^2 = x \circ x$.

(iii) $\langle x \circ y, z \rangle = \langle y, x \circ z \rangle$ for all $x, y, z \in \mathcal{V}$.

Then the triple $(\mathcal{V}, \circ, \langle \cdot, \cdot \rangle)$ is called a **Euclidean Jordan algebra** and the product $x \circ y$ is called the **Jordan product** of x and y.

In what follows, we assume that there exists a **unit element** $e \in \mathcal{V}$ such that $x \circ e = x$ for all $x \in \mathcal{V}$. The set of squares $\mathcal{V}_+ = \{x^2 \mid x \in \mathcal{V}\}$ is called the **symmetric cone** of $\mathcal{V}$. We say that x and y **operator commute** if $x \circ (y \circ z) = y \circ (x \circ z)$ for all $z \in \mathcal{V}$.

Example 2.1.2 Some basic examples of Euclidean Jordan algebras are:

(0) Euclidean Jordan algebra of n-dimensional vectors:

$$\mathcal{V} = \mathcal{R}^n, \quad \langle x,\, y\rangle = \sum_{i=1}^{n} x_i y_i, \quad x \circ y = x * y,$$

where $x * y$ denotes the component-wise product of vectors x and y. Here, the unit element is $e = (1,\, \ldots,\, 1) \in \mathcal{R}^n$ and the symmetric cone of $\mathcal{R}^n$ is the nonnegative orthant $\mathcal{R}^n$.

(1) Euclidean Jordan algebra of $n \times n$ symmetric matrices:

$$\mathcal{V} = \mathcal{S}^n, \quad \langle X,\, Y\rangle = \mathrm{tr}(XY), \quad X \circ Y = \frac{1}{2}(XY + YX).$$

Here, tr denotes the trace of a matrix. The identity matrix $I \in \mathcal{S}^n$ is the unit element of this algebra. Also, $\mathcal{S}^n_+$, the set of $n \times n$ positive semidefinite matrices, is the symmetric cone of $\mathcal{S}^n$.

(2) The Jordan spin algebra: Here, $\mathcal{V} = \mathcal{R}^n$ ($n \geq 2$) with the usual inner product. With the notation $z = \begin{bmatrix} z_1 \\ \bar{z} \end{bmatrix}$ where $z_1 \in \mathcal{R}$ and $\bar{z} \in \mathcal{R}^{n-1}$,

$$\left\langle \begin{bmatrix} x_1 \\ \bar{x} \end{bmatrix}, \begin{bmatrix} y_1 \\ \bar{y} \end{bmatrix} \right\rangle = x_1 y_1 + \langle \bar{x},\, \bar{y}\rangle, \quad \begin{bmatrix} x_1 \\ \bar{x} \end{bmatrix} \circ \begin{bmatrix} y_1 \\ \bar{y} \end{bmatrix} = \begin{bmatrix} x_1 y_1 + \langle \bar{x},\, \bar{y}\rangle \\ x_1 \bar{y} + y_1 \bar{y} \end{bmatrix}.$$

In this algebra, the unit element is $e = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$, where $\mathbf{0}$ is the zero vector in $\mathcal{R}^{n-1}$. The symmetric cone of $\mathcal{L}^n$ is given by

$$\mathcal{L}^n_+ = \left\{ \begin{bmatrix} x_1 \\ \bar{x} \end{bmatrix} \in \mathcal{R} \times \mathcal{R}^{n-1} \;\middle|\; x_1 \geq \|x\| \right\}.$$

This cone is called the second-order cone or the Lorentz cone.

Let $\mathcal{V}$ be a Euclidean Jordan algebra. A subspace $I \subseteq \mathcal{V}$ is an **ideal** of $\mathcal{V}$ provided

$x \in I$ and $y \in \mathcal{V}$ implies $x \circ y \in I$. A nonzero Euclidean Jordan algebra $\mathcal{V}$ is said to be **simple** if $\{0\}$ and $\mathcal{V}$ are the only ideals of $\mathcal{V}$. The classification theorem ([11], Theorem V.3.7 and Proposition III.4.4) says that there are, up to isomorphism, only five simple Euclidean Jordan algebras. Moreover, any nonzero Euclidean Jordan algebra is, in a unique way, a direct sum/product of simple Euclidean Jordan algebras. Note that the Item (0) in the examples above is not simple as, for example, $\mathcal{R}^{n-1} \times \{0\}$ is an ideal of $\mathcal{R}^n$. However, the Items (1) and (2) for $n \geq 3$ are simple Euclidean Jordan algebras. Other three simple Euclidean Jordan algebras are:

(3) the algebra $\mathcal{H}^n$ of $n \times n$ complex Hermitian matrices,

(4) the algebra $\mathcal{Q}^n$ of $n \times n$ quaternion Hermitian matrices,

(5) the algebra $\mathcal{O}^3$ of 3×3 octonian Hermitian matrices.

Definition 2.1.3 We say that a Euclidean Jordan algebra $\mathcal{V}$ is **essentially simple** if it is either simple or $\mathcal{R}^n$.

An element $c \in \mathcal{V}$ is an **idempotent** if $c^2 = c$ and is **primitive idempotent** if c cannot be written as the sum of two nonzero idempotents. Two idempotents c_1 and c_2 are **orthogonal** if $c_1 \circ c_2 = 0$. Let c_1, c_2 be idempotents. Then

$$c_1 \circ c_2 = 0 \implies \langle c_1, c_2 \rangle = \left\langle c_1^2, c_2 \right\rangle = \langle c_1 \circ c_1, c_2 \rangle = \langle c_1, c_1 \circ c_2 \rangle = \langle c_1, 0 \rangle = 0.$$

Definition 2.1.4 Let $\mathcal{V}$ be a Euclidean Jordan algebra. A set $\{e_1, e_2, \ldots, e_n\}$ of nonzero orthogonal primitive idempotents is called a **Jordan frame** of $\mathcal{V}$ if $e = e_1 + e_2 + \cdots + e_n$.

The **rank** of $\mathcal{V}$ is defined by $r = \max\{\deg(x) \mid x \in \mathcal{V}\}$, where $\deg(x)$ is the degree of $x \in \mathcal{V}$ given by $\deg(x) = \min\big\{k > 0 \;\big|\; \{e, x, x^2, \ldots, x^k\} \text{ is linearly dependent}\big\}$. *In what follows, we assume that* $\operatorname{rank}(\mathcal{V}) = n$ *unless explicitly mentioned.*

Example 2.1.5

(0) $\operatorname{rank}(\mathcal{R}^n) = n$.

(1) $\operatorname{rank}(\mathcal{S}^n) = n$, while $\dim(\mathcal{S}^n) = n(n+1)/2$.

(2) $\operatorname{rank}(\mathcal{L}^n) = 2$.

An $x \in \mathcal{V}$ is said to be **invertible** if there exists $y \in \operatorname{span}\{e, x, x^2, \ldots\}$ such that $x \circ y = e$. Such a y is unique and we write $y = x^{-1}$. Note that this is NOT the same as saying that there exists $y \in \mathcal{V}$ such that $x \circ y = e$.

2.1.2 Spectral and Peirce decompositions

Proposition 2.1.6 (Spectral decomposition theorem, [11], Theorem III.1.2) Suppose $\mathcal{V}$ is a Euclidean Jordan algebra of rank n. Then, for every $x \in \mathcal{V}$, there exists a Jordan frame $\{e_1, e_2, \ldots, e_n\}$ and real numbers $\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x)$ such that

$$x = \lambda_1(x)e_1 + \lambda_2(x)e_2 + \cdots + \lambda_n(x)e_n.$$

The numbers $\lambda_i(x)$'s are uniquely determined and are called the **eigenvalues** of x.

Note that, by renumbering the indices, we may assume that $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_n(x)$. *So from now on, without loss of generality, we assume that the eigenvalues in* *the spectral decomposition of x have decreasing order.*

It is easy to show that $x \in \mathcal{V}_+$ if and only if $\lambda_i(x) \geq 0$ for all i. Hence,

$$\mathcal{V}_+ = \{x \in \mathcal{V} \mid \lambda_i(x) \geq 0 \text{ for all } i\}\,.$$

Due to the uniqueness of the eigenvalues, we can respectively define the **trace** and the **determinant** of x as

$$\operatorname{tr}(x) := \sum_{i=1}^{n} \lambda_i(x) \quad \text{and} \quad \det(x) := \prod_{i=1}^{n} \lambda_i(x).$$

Note that $\operatorname{tr}(c) = 1$ for any primitive idempotent c in $\mathcal{V}$, and $\operatorname{tr}(e) = n$ and $\det(e) = 1$. It is known ([11], Proposition III.4.1) that, in any simple Euclidean Jordan algebra $\mathcal{V}$, there exists a $\theta > 0$ such that $\langle x,\, y\rangle = \theta \operatorname{tr}(x \circ y)$. Here, $\theta = \langle c,\, e\rangle = \|c\|^2$ for every primitive idempotent c in $\mathcal{V}$. In particular, we have $\|e_j\|^2 = \theta$ for every element of a Jordan frame $\{e_1,\, \ldots,\, e_n\}$.

Given any Euclidean Jordan algebra, we define an equivalent inner product called the **canonical inner product** by $\langle x,\, y\rangle = \operatorname{tr}(x \circ y)$. Various concepts, results, and decompositions remain the same when the given inner product is replaced by the canonical inner product. In particular, for an element of $\mathcal{V}$, the spectral decomposition, eigenvalues, and trace remain the same. We note that under the canonical inner product, the norm of any primitive element is one and $\operatorname{tr}(x) = \langle x,\, e\rangle$.

Proposition 2.1.7 $x \in \mathcal{V}$ is invertible if and only if all eigenvalues of x are nonzero.

In this case, we have

$$x = \sum_{i=1}^{n} \lambda_i(x) e_i \qquad \Longrightarrow \qquad x^{-1} = \sum_{i=1}^{n} \frac{1}{\lambda_i(x)} e_i.$$

Definition 2.1.8 The mapping $\lambda : \mathcal{V} \to \mathcal{R}^n$ defined by

$$\lambda(x) = \big(\lambda_1(x),\, \lambda_2(x),\, \ldots,\, \lambda_n(x)\big)^{\mathsf{T}} \in \mathcal{R}^n,$$

where $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_n(x)$, is called the **eigenvalue mapping**.

Example 2.1.9

(0) Take $\mathcal{V} = \mathcal{R}^n$. Let $e_j \in \mathcal{R}^n$ be a vector with 1 in j^{th} entry and 0s elsewhere. Then the set $\{e_1, e_2, \ldots, e_n\}$ is the only Jordan frame in $\mathcal{R}^n$. For any $x = (x_1, x_2, \ldots, x_n)^{\mathsf{T}} \in \mathcal{R}^n$, we have $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$.

(1) Let $\mathcal{V} = \mathcal{S}^n$ and $X \in \mathcal{V}$. As X is symmetric, the classical spectral decomposition theorem asserts that $X = U \Lambda U^{\mathsf{T}}$, where Λ is the diagonal matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of X in the diagonal and $U = [u_1, u_2, \ldots, u_n]$ is an orthogonal matrix. Then we have the spectral decomposition

$$X = \lambda_1 u_1 u_1^{\mathsf{T}} + \lambda_2 u_2 u_2^{\mathsf{T}} + \cdots + \lambda_n u_n u_n^{\mathsf{T}}.$$

One can verify that $\{u_1 u_1^{\mathsf{T}}, u_2 u_2^{\mathsf{T}}, \ldots, u_n u_n^{\mathsf{T}}\}$ forms a Jordan frame.

(2) For any $x = \begin{bmatrix} x_1 \\ \bar{x} \end{bmatrix} \in \mathcal{L}^n$ with $\bar{x} \neq 0$, we have

$$x = \underbrace{(x_1 + \|\bar{x}\|)}_{=\lambda_1} \underbrace{\frac{1}{2}\begin{bmatrix} 1 \\ \frac{\bar{x}}{\|\bar{x}\|} \end{bmatrix}}_{=e_1} + \underbrace{(x_1 - \|\bar{x}\|)}_{=\lambda_2} \underbrace{\frac{1}{2}\begin{bmatrix} 1 \\ \frac{-\bar{x}}{\|\bar{x}\|} \end{bmatrix}}_{=e_2} = \lambda_1 e_1 + \lambda_2 e_2.$$

It is easy to check that $e_1 \circ e_1 = e_1$, $e_2 \circ e_2 = e_2$, $e_1 \circ e_2 = 0$, and $e_1 + e_2 = e$. Then, $x = \lambda_1 e_1 + \lambda_2 e_2$ is a spectral decomposition of x.

We recall the following result from [18].

Proposition 2.1.10 Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two non-isomorphic simple Euclidean Jordan algebras. If $\phi \in \mathrm{Aut}(\mathcal{V}_1 \times \mathcal{V}_2)$, then ϕ is of the form (ϕ_1, ϕ_2) for some $\phi_i \in \mathrm{Aut}(\mathcal{V}_i)$, $i = 1, 2$, that is,

$$\phi(x) = \big(\phi_1(x_1), \phi_2(x_2)\big), \quad \forall\, x = (x_1, x_2) \in \mathcal{V}_1 \times \mathcal{V}_2.$$

Another important tool in Euclidean Jordan algebras is the Peirce decomposition.

Proposition 2.1.11 (Peirce decomposition theorem, [11], Theorem IV.2.1) Let $\mathcal{V}$ be a Euclidean Jordan algebra and $\{e_1, \ldots, e_n\}$ be a Jordan frame of $\mathcal{V}$. For $i, j \in \{1, \ldots, n\}$, we define the eigenspaces

$$\mathcal{V}_{ii} := \{x \in \mathcal{V} : x \circ e_i = x\} = \mathcal{R} e_i,$$

$$\mathcal{V}_{ij} := \{x \in \mathcal{V} : x \circ e_i = \tfrac{1}{2} x = x \circ e_j\} \quad (i \neq j).$$

Then, $\mathcal{V}$ is orthogonal direct sum of $\mathcal{V}_{ij}$s, i.e.,

$$\mathcal{V} = \sum_{i \leq j} \mathcal{V}_{ij} = \sum_{i=1}^{n} \mathcal{R} e_i + \sum_{i<j} \mathcal{V}_{ij}.$$

Furthermore, the following hold:

(1) $\mathcal{V}_{ij} \circ \mathcal{V}_{ij} \subseteq \mathcal{V}_{ii} + \mathcal{V}_{jj}$.

(2) $\mathcal{V}_{ij} \circ \mathcal{V}_{jk} \subseteq \mathcal{V}_{ik}$ if $i \neq k$.

(3) $\mathcal{V}_{ij} \circ \mathcal{V}_{kl} = \{0\}$ if $\{i, j\} \cap \{k, l\} = \emptyset$.

As a consequence, given any Jordan frame $\{e_1, \ldots, e_n\}$ in $\mathcal{V}$, we can write any element $x \in \mathcal{V}$ as

$$x = \sum_{1 \le i \le j \le n} x_{ij} = \sum_{i=1}^{n} x_i e_i + \sum_{1 \le i < j \le n} x_{ij},$$

where $x_i \in \mathcal{R}$ and $x_{ij} \in \mathcal{V}_{ij}$. This expression is called the **Peirce decomposition** of x associated with $\{e_1, \ldots, e_n\}$.

Example 2.1.12

(0) Let $\mathcal{V} = \mathcal{R}^n$ and $\{e_1, \ldots, e_n\}$ be the Jordan frame of $\mathcal{R}^n$. Then

$$\mathcal{V}_{ii} = \{a e_i : a \in \mathcal{R}\},\ i = 1, \ldots, n \quad \text{and} \quad \mathcal{V}_{ij} = \{0\},\ i \neq j.$$

Hence, any element $x \in \mathcal{R}^n$ can be written as $x = \sum_{i=1}^{n} x_i e_i$ and thus there is no difference between spectral and Peirce decomposition in $\mathcal{R}^n$.

(1) Let $\mathcal{V} = \mathcal{S}^n$ and define the set $\{E_1, \ldots, E_n\}$, where E_j is a diagonal matrix with 1 in the (j, j)-entry and 0's elsewhere. It can be verified that this set is a Jordan frame in $\mathcal{S}^n$. Associated with this Jordan frame, we have

$$\mathcal{V}_{ii} = \{a E_i : a \in \mathcal{R}\},\ i = 1, \ldots, n \quad \text{and} \quad \mathcal{V}_{ij} = \{b E_{ij} : b \in \mathcal{R}\},\ i \neq j,$$

where E_{ij} is a matrix with 1 in the (i, j) and (j, i)-entries and 0's elsewhere. Thus, $X \in \mathcal{S}^n$ has the Peirce decomposition with respect to $\{E_1, \ldots, E_n\}$ by

$$X = \sum_{i=1}^{n} x_i E_i + \sum_{1 \le i < j \le n} x_{ij} E_{ij}.$$

(2) Let $\mathcal{V} = \mathcal{L}^n$ and $\{e_1, e_2\}$ defined by $e_1 = (\frac{1}{2}, \frac{1}{2}, 0_{n-2})$ and $e_1 = (\frac{1}{2}, -\frac{1}{2}, 0_{n-2})$,

where 0_{n-2} is a vector of zeros in $\mathbb{R}^{n-2}$. Clearly, this set is a Jordan frame of $\mathcal{L}^n$. It is easy to show that

$$V_{ii} = \{ae_i : a \in \mathcal{R}\},\ i = 1, 2 \quad \text{and} \quad V_{12} = \{x \in \mathcal{R}^n : x_1 = x_2 = 0\}.$$

Thus, given an $x \in \mathcal{L}^n$, we can write

$$x = (x_1 + x_2)e_1 + (x_1 - x_2)e_2 + (0,\, 0,\, x_3,\, \ldots,\, x_n),$$

which is the Peirce decomposition of x associated with $\{e_1,\, e_2\}$.

2.1.3 Some special linear transformations

In any Euclidean Jordan algebra $\mathcal{V}$, one can define automorphism groups in the following way:

Definition 2.1.13 Let $\phi : \mathcal{V} \to \mathcal{V}$ be a linear transformation. Then,

(i) ϕ is called an **algebra automorphism** of $\mathcal{V}$ if it is invertible and $\phi(x \circ y) = \phi(x) \circ \phi(y)$ for all $x,\, y \in V$. The set of all algebra automorphisms of $\mathcal{V}$ is denoted by $\mathrm{Aut}(\mathcal{V})$.

(ii) ϕ is a **(symmetric) cone automorphism** if $\phi(\mathcal{V}_+) = \mathcal{V}_+$. The set of all cone automorphisms of $\mathcal{V}$ is denoted by $\mathrm{Aut}(\mathcal{V}_+)$.

(iii) ϕ is said to be **doubly stochastic** if ϕ is *positive* (i.e., $\phi(\mathcal{V}_+) \subseteq \mathcal{V}_+$), *unital* (i.e., $\phi(e) = e$), and *trace preserving* (i.e., $\mathrm{tr}(\phi(x)) = \mathrm{tr}(x)$ for all $x \in \mathcal{V}$). We denote the set of all doubly stochastic linear transformations by $\mathrm{DS}(\mathcal{V})$.

(iv) ϕ is said to be **orthogonal** if $\langle \phi(x),\, \phi(y) \rangle = \langle x,\, y \rangle$ for all $x,\, y \in \mathcal{V}$. The set of all orthogonal linear transformations is denoted by $\mathrm{Orth}(\mathcal{V})$.

Example 2.1.14

(0) For $\mathcal{R}^n$, we easily see that $\mathrm{Aut}(\mathcal{R}^n)$ consists of permutation matrices, and any element in $\mathrm{Aut}(\mathcal{R}^n_+)$ has a form DP, where P is a permutation matrix and D is a diagonal matrix with positive diagonal entries. Take any $A = [a_{ij}] \in \mathrm{DS}(\mathcal{R}^n)$. Then one can show that all entries of A are nonnegative and $A\mathbf{1} = 1 = A^{\mathsf{T}}\mathbf{1}$, where $\mathbf{1} \in \mathcal{R}^n$ denotes the vector of ones. This is precisely the definition of doubly stochastic matrix on $\mathcal{R}^n$.

(1) In $\mathcal{S}^n$, it is known [12] that, corresponding to any $\phi \in \mathrm{Aut}(\mathcal{S}^n)$, there exists an orthogonal matrix $U \in \mathcal{R}^{n\times n}$ such that $\phi(X) = UXU^{\mathsf{T}}$ for all $X \in \mathcal{S}^n$. Also, for $\psi \in \mathrm{Aut}(\mathcal{S}^n_+)$, there exists an invertible matrix $Q \in \mathcal{R}^{n\times n}$ such that $\psi(X) = QXQ^{\mathsf{T}}$ for all $X \in \mathcal{S}^n$.

(2) For $\mathcal{L}^n$, if $\phi \in \mathrm{Aut}(\mathcal{L}^n)$, then (because of $\phi(e) = e$) it can be written as $\phi = \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix}$, where U is an $(n-1)\times(n-1)$ orthogonal matrix. Although the explicit description of $\mathrm{Aut}(\mathcal{L}^n_+)$ is not verified, it is known [31] that $\psi \in \mathrm{Aut}(\mathcal{L}^n_+)$ if and only if there exists $\mu > 0$ such that $\psi^{\mathsf{T}} J \psi = \mu J$, where $J = \mathrm{diag}(1, -1, \ldots, -1) \in \mathcal{R}^{n\times n}$.

The following result will be used in many of our theorems, particularly, in the converse statements. Here and elsewhere, we implicitly assume that a Jordan frame, in addition to being a set, is also an ordered listing of its objects.

Proposition 2.1.15 Algebra automorphisms map Jordan frames to Jordan frames.

Thus, eigenvalues of an element remain the same under the action of an automorphism. In particular,

$$\lambda(\phi(x)) = \lambda(x), \quad \forall\, x \in \mathcal{V},\, \phi \in \mathrm{Aut}(\mathcal{V}).$$

Furthermore, if $\mathcal{V}$ is essentially simple and $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ are any two Jordan frames in $\mathcal{V}$, then there exists $\phi \in \mathrm{Aut}(\mathcal{V})$ such that $\phi(e_i) = e'_i$ for all $i = 1, \ldots, n$. In particular, if $\lambda(x) = \lambda(y)$ in $\mathcal{V}$, then there exists $\phi \in \mathrm{Aut}(\mathcal{V})$ such that $x = \phi(y)$.

We list below some more properties of linear transformations.

Proposition 2.1.16 For a Euclidean Jordan algebra $\mathcal{V}$ with the canonical inner product, the following hold:

(a) The positivity of $\phi : \mathcal{V} \to \mathcal{V}$ is equivalent to that of $\phi^{\mathsf{T}} : \mathcal{V} \to \mathcal{V}$, see [18].

(b) The trace preserving (unital) property of ϕ is equivalent to the unital (trace preserving) property of its transpose. In particular, ϕ is doubly stochastic if and only if ϕ^{T} is doubly stochastic, see [18].

(c) It is known ([11], p.57) that $\mathrm{Aut}(\mathcal{V}) = \mathrm{Aut}(\mathcal{V}_+) \cap \mathrm{Orth}(\mathcal{V})$. Gowda [18] showed that, if $\mathcal{V}$ is simple, we further have $\mathrm{Aut}(\mathcal{V}) = \mathrm{Aut}(\mathcal{V}_+) \cap \mathrm{DS}(\mathcal{V})$.

2.2 Convex cones

We explain some definitions and properties of closed convex cones which will be used throughout the thesis. Further concepts and results of closed convex cones can be found in [2].

Definition 2.2.1 For a nonempty set K in an inner product space V, we say that

(1) K is **convex** if $(1-t)x + ty \in K$ for all $x, y \in K$ and $0 \le t \le 1$.

(2) K is a **cone** if $tx \in K$ for all $x \in K$ and $t \ge 0$.

(3) K is **closed** if it is closed in the topology of V.

Note that if K is a convex cone in V, then $K - K = \{x - y \mid x,\, y \in K\}$ is the minimal subspace of V containing K, and $K \cap (-K)$ is the maximal subspace of V contained in K.

Definition 2.2.2 Let K be a convex cone in an inner product space V. We say that

(1) K is **pointed** if $K \cap (-K) = \{0\}$.

(2) K is **solid** if $K^{\circ} \neq \emptyset$, and **reproducing** if $K - K = V$.

(3) K is said to be **proper** if it is closed, pointed, and solid.

For any nonempty set S in V, the **dual cone** of S is

$$S^* := \{x \in V \mid \langle x,\, y\rangle \ge 0 \text{ for every } y \in S\}\,.$$

It is known that the dual cone of any set S is a closed convex cone.

Proposition 2.2.3 ([2], Propositions 1.17, 1.18) For a closed convex cone K in V,

(1) K is solid if and only if it is reproducing.

(2) K is pointed if and only if K^* is solid.

Given a nonempty set S in V, the **convex hull** of S is

$$\operatorname{conv}(S) = \left\{ \sum_{i=1}^{k} \alpha_i x_i \;\middle|\; k \in \mathbb{N},\, x_i \in S,\, \alpha_i \geq 0,\, \sum_{i=1}^{k} \alpha_i = 1 \right\}$$

and the **conic hull** of S is

$$\operatorname{cone}(S) = \left\{ \sum_{i=1}^{k} \alpha_i x_i \;\middle|\; k \in \mathbb{N},\, x_i \in S,\, \alpha_i \geq 0 \right\}.$$

Note that $\operatorname{conv}(S)$ is always a convex set and $\operatorname{cone}(S)$ is always a convex cone for any nonempty S.

Definition 2.2.4 A convex cone K is a **polyhedral cone** if it is finitely generated, that is, $K = \operatorname{cone}(S)$ for some finite set S.

Definition 2.2.5 Let K be a convex cone. A nonzero vector $x \in K$ is called an **extreme vector** of K if $x = y + z$, where $y, z \in K$, implies that y and z are both nonnegative scalar multiples of x. We say that two extreme vectors x_1 and x_2 are equivalent (and hence consider them to be the 'same') if they are nonnegative scalar multiples of each other. Define

$$\operatorname{ext}(K) = \{x \in K \mid x \text{ is an extreme vector of } K\}.$$

Proposition 2.2.6 ([2], Proposition 1.19, Theorem 1.38) Let K be a convex cone in V. We have the following:

(1) If K is a polyhedral cone, then it is necessarily closed.

(2) K is a polyhedral cone if and only if K has a finite number of extreme vectors.

(3) K is a polyhedral cone if and only if it is the finite intersection of closed half-spaces.

Theorem 2.2.7 (Caratheodory's Theorem, [41]) Let S be a nonempty set in an inner product space V of dimension d. Then any $x \in \text{conv}(S)$ is a convex combination of at most $d+1$ elements of S. Similarly, any $x \in \text{cone}(S)$ can be expressed as a nonnegative combination of at most d elements of S.

Definition 2.2.8 Let S be a nonempty set in V. The set

$$S^* := \{y \in V \mid \langle y, x \rangle \geq 0 \text{ for every } x \in S\}$$

is called the **dual** of S.

2.3 Permutation matrices and majorization

2.3.1 Permutation matrices

Vectors in $\mathcal{R}^n$ are considered as column vectors and $\mathcal{R}^n$ carries the usual inner product. An $n \times n$ **permutation matrix** is a matrix obtained by permuting the rows of an $n \times n$ identity matrix. The set of all $n \times n$ permutation matrices is denoted by Σ_n. For notational convenience, an element $\sigma \in \Sigma_n$ can be regarded as either a permutation matrix P or a permutation σ of indices $\{1, 2, \ldots, n\}$.

For any $u = (u_1, u_2, \ldots, u_n)^{\mathsf{T}}$ in $\mathcal{R}^n$ consider

$$\Sigma_n(u) = \{\sigma(u) : \sigma \in \Sigma_n\},$$

the set of all possible permutations of u. If we look (only) at the first components of vectors in this collection, we see u_i (for $i = 1, 2, \ldots, n$) appearing exactly $(n-1)!$-times. Hence, adding all these first components, we get the sum $(n-1)!\ \mathrm{tr}(u)$, where $\mathrm{tr}(u) = u_1 + u_2 + \cdots + u_n$. The same sum is obtained when other components are considered. Thus,

$$\sum_{\sigma \in \Sigma_n} \sigma(u) = (n-1)!\ \mathrm{tr}(u)\,\mathbf{1}, \tag{2.1}$$

where $\mathbf{1}$ denotes the vector in $\mathcal{R}^n$ with all entries 1.

2.3.2 Majorization in $\mathcal{R}^n$

For any vector $u \in \mathcal{R}^n$, we write $u^{\downarrow}$ for its decreasing rearrangement.

Definition 2.3.1 Given two vectors u and v in $\mathcal{R}^n$ with their decreasing rearrangements $u^{\downarrow}$ and $v^{\downarrow}$, we say that u is **majorized** by v and write $u \prec v$ if

(i) $\displaystyle\sum_{i=1}^{k} u_i^{\downarrow} \le \sum_{i=1}^{k} v_i^{\downarrow}$ for all $1 \le k \le n-1$, and

(ii) $\displaystyle\sum_{i=1}^{n} u_i^{\downarrow} = \sum_{i=1}^{n} v_i^{\downarrow}$.

If $u \prec v$, then we have, by setting $k = 1$ and $k = n-1$,

$$\max_i u_i \le \max_i v_i \quad \text{and} \quad \min_i u_i \ge \min_i v_i. \tag{2.2}$$

We start by recalling two classical results in matrix theory. The first one is due to

Hardy, Littlewood, and Pólya and the second one is due to Birkhoff.

Proposition 2.3.2 ([3], Theorem II.1.10) Let $u,\ v \in \mathcal{R}^n$. A necessary and sufficient condition for $u \prec v$ is that there exists a doubly stochastic matrix A such that $u = A(v)$.

Proposition 2.3.3 ([3], Theorem II.2.3) The set of all $n \times n$ doubly stochastic matrices is a compact convex set whose extreme points are permutation matrices. In particular, every doubly stochastic matrix is a convex combination of permutation matrices.

Hence, combining two propositions above, $u \prec v$ if and only if

$$u = \sum_{i=1}^{N} \alpha_i P_i(v) \tag{2.3}$$

where $\alpha_i \geq 0$ with $\sum_{i=1}^{N} \alpha_i = 1$, and $P_i \in \Sigma_n$.

The next proposition, which is essential for Sections 4 and 6, is somewhat classical and well known. It easily follows from the above two propositions.

Proposition 2.3.4

(1) If Q is convex and permutation invariant in $\mathcal{R}^n$, then

$$u \prec v,\ v \in Q \quad \Longrightarrow \quad u \in Q.$$

(2) If $f : \mathcal{R}^n \to \mathcal{R}$ is convex and permutation invariant, then f is Schur-convex, that is,

$$u \prec v \quad \Longrightarrow \quad f(u) \leq f(v).$$

Proof.

(1) As $u \prec v$, there exists $\alpha_i \geq 0$ with $\sum_{i=1}^{N} \alpha_i = 1$, and $P_i \in \Sigma_n$ such that (2.3) holds. Since Q is permutation invariant, each $P_i(v) \in Q$. Since Q is also convex, u, a convex combination of elements in Q, is in Q.

(2) Let $u \prec v$. From (2.3) and the fact that f is convex and permutation invariant, it is easy to see that

$$f(u) = f\left(\sum_{i=1}^{N} \alpha_i P_i(v)\right) \leq \sum_{i=1}^{N} \alpha_i f(P_i(v)) = \sum_{i=1}^{N} \alpha_i f(v) = f(v).$$

Hence, $f(u) \leq f(v)$ completing the proof. $\square$

The following elementary proposition will be useful.

Proposition 2.3.5 Suppose $u\,(\neq 0)$ and v be vectors in $\mathcal{R}^n$ with decreasing entries and $\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} v_i = 0$. Then,

(1) $\sum_{i=1}^{k} u_i > 0$ for all k with $1 \leq k \leq n-1$, and

(2) $v \prec \alpha\, u$ for some positive number α.

Proof.

(1) Suppose $\sum_{i=1}^{k} u_i \leq 0$ for some k with $1 \leq k \leq n-1$. As $\sum_{i=1}^{n} u_i = 0$, we have $\sum_{i=k+1}^{n} u_i \geq 0$. Since the entries of u are decreasing, we must have $u_{k+1} \geq 0$. This implies that $u_1 \geq u_2 \geq \cdots \geq u_k \geq 0$. But then, $\sum_{i=1}^{k} u_i \leq 0$ implies that $u_1 = u_2 = \cdots = u_k = 0$. From this we get $0 \geq u_{k+1} \geq u_{k+2} \geq \cdots \geq u_n$. As these inequalities imply $0 \geq \sum_{i=k+1}^{n} u_i$, we see that $0 = \sum_{i=k+1}^{n} u_i$ from which we get $0 = u_{k+1} = u_{k+2} = \cdots = u_n$. Thus, $u = 0$, leading to a contradiction.

Hence we have (i).

(2) Now, because of (i), we can find a positive α such that $\sum_{i=1}^{k} v_i \leq \alpha(\sum_{i=1}^{k} u_i)$ for all k with $1 \leq k \leq n-1$. Since $\alpha(\sum_{i=1}^{n} u_i) = \sum_{i=1}^{n} v_i = 0$, we see that $v \prec \alpha\, u$. This gives (ii). $\square$

2.3.3 Majorization in Euclidean Jordan algebras

Definition 2.3.6 Let x, y be elements in a Euclidean Jordan algebra $\mathcal{V}$. We say x is **majorized** by y in $\mathcal{V}$ and write $x \prec y$ if $\lambda(x) \prec \lambda(y)$ in $\mathcal{R}^n$.

Recall that $\mathcal{V}$ is essentially simple if it is either simple or $\mathcal{R}^n$. The result below describes a connection between majorization, automorphisms, and doubly stochastic transformations in the setting of Euclidean Jordan algebras.

Proposition 2.3.7 (Gowda [18]) For x, $y \in \mathcal{V}$, consider the following statements:

(a) $x = \Phi(y)$, where Φ is a convex combination of automorphisms of $\mathcal{V}$.

(b) $x = \Psi(y)$, where Ψ is doubly stochastic on $\mathcal{V}$.

(c) $x \prec y$.

Then, $(a) \Rightarrow (b) \Rightarrow (c)$. Furthermore, when $\mathcal{V}$ is essentially simple, reverse implications hold.

When $\mathcal{V}$ is a simple Euclidean Jordan algebra, it is known [37] that $\lambda(x+y) \prec \lambda(x) + \lambda(y)$ for all x, $y \in \mathcal{V}$. Here is a generalization of this result.

Proposition 2.3.8 Let $\mathcal{V}$ be a Euclidean Jordan algebra of rank n. Then for $x,\, y \in \mathcal{V}$, there exist $n \times n$ doubly stochastic matrices A and B such that

$$\lambda(x+y) = A\lambda(x) + B\lambda(y). \tag{2.4}$$

When $\mathcal{V}$ is simple, we can take $A = B$.

Proof. As the result for a simple algebra is known, we assume that $\mathcal{V}$ is non-simple, that is, $\mathcal{V}$ is a product of simple algebras. For simplicity, we assume $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$, where $\mathcal{V}_1,\, \mathcal{V}_2$ are simple algebras of rank $n_1,\, n_2$, respectively. Now, let $x = (x_1,\, x_2)$, $y = (y_1,\, y_2)$, where $x_i,\, y_i \in \mathcal{V}_i$ for $i = 1,\, 2$. Define $u_i = \lambda(x_i) \in \mathcal{R}^{n_i}$ for $i = 1,\, 2$ and $u = (u_1,\, u_2) \in \mathcal{R}^{n_1+n_2} = \mathcal{R}^n$. As eigenvalues of x come from the eigenvalues of x_1 and x_2, we have $\lambda(x) = \sigma_1(u)$ for some $\sigma_1 \in \Sigma_n$. Similarly, there exist $\sigma_2,\, \sigma_3 \in \Sigma_n$ such that

$$\begin{matrix} \lambda(y) = \sigma_2(v) \\ \lambda(x+y) = \sigma_3(w) \end{matrix} \quad \text{where} \quad \begin{matrix} v = (v_1,\, v_2),\ v_i = \lambda(y_i) \\ w = (w_1,\, w_2),\ w_i = \lambda(x_i + y_i) \end{matrix} \quad \text{for } i = 1,\, 2.$$

Since $\mathcal{V}_1,\, \mathcal{V}_2$ are simple,

$$w_i = \lambda(x_i + y_i) \prec \lambda(x_i) + \lambda(y_i) = u_i + v_i \qquad \text{for} \quad i = 1,\, 2.$$

Thus, there are doubly stochastic matrices C_i on R^{n_i} such that $w_i = C_i(u_i + v_i)$ for $i = 1,\, 2$. So,

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}.$$

Letting C denote the the block diagonal matrix that appears above, one can easily verify that C is doubly stochastic on $\mathcal{R}^n$. Moreover, we have $w = C(u+v) = C(u) + C(v)$. Now, define matrices A and B by

$$A = \sigma_3 C \sigma_1^{-1} \qquad \text{and} \qquad B = \sigma_3 C \sigma_2^{-1}.$$

As products of doubly stochastic matrices are doubly stochastic, A and B are doubly stochastic. Finally, we have $A\lambda(x) + B\lambda(y) = \lambda(x+y)$. $\square$

Chapter 3

Spectral Sets in Euclidean Jordan Algebras

3.1 Introduction

This chapter focuses on interconnections between spectral sets in $\mathcal{V}$ and the corresponding permutation invariant sets in $\mathcal{R}^n$. We first note that permutations are (algebra) automorphisms of $\mathcal{R}^n$. A set in $\mathcal{V}$ is said to be **weakly spectral** if it is invariant under algebra automorphisms of $\mathcal{V}$. In this manner, we see that weakly spectral sets are direct generalization of permutation invariant sets. We also define a **spectral set** $E = \lambda^{-1}(Q)$ in $\mathcal{V}$, where Q is a permutation invariant set in $\mathcal{R}^n$.

When $\mathcal{V} = \mathcal{S}^n$, it is known that a set E is spectral if and only if

$$X \in E \quad \Longrightarrow \quad UXU^{\mathsf{T}} \in E, \quad \forall\, U \in \mathcal{O}^n,$$

where $\mathcal{O}^n$ denotes the set of all $n \times n$ orthogonal matrices. Since a map $X \mapsto UXU^T$ is an algebra automorphism of $\mathcal{S}^n$, spectral sets and weak spectral sets coincide when $\mathcal{V} = \mathcal{S}^n$. Indeed, it is shown that every spectral set is weakly spectral, and the converse holds when $\mathcal{V}$ is essentially simple. The behavior of the inverse of the eigenvalue mapping is also explored.

We conclude the chapter with a discussion on the *Transfer Principle* which asserts that numerous properties of a permutation invariant set Q, such as closedness, convexity, connectedness, etc., get transferred to a spectral set $E = \lambda^{-1}(Q)$. We also prove a metaformula

$$\# \diamondsuit = \diamondsuit \#,$$

where # is a set operation (such as the closure, interior, boundary, convex hull, etc.) and $\diamondsuit$ is the set operation defined by

$$Q^{\diamondsuit} = \lambda^{-1}(Q) \ \text{ for } \ Q \subseteq \mathcal{R}^n \quad \text{and} \quad E^{\diamondsuit} = \Sigma_n(\lambda(E)) \ \text{ for } \ E \subseteq \mathcal{V},$$

with Σ_n denoting the set of all $n \times n$ permutation matrices.

The organization of this chapter is as follows:

- Section 2 deals with spectral and weakly spectral sets in $\mathcal{V}$. Especially, a characterization of a spectral set and a relation between spectral sets and weakly spectral sets are discussed.
- The transfer principle and a metaformula are presented in Section 3.

3.2 Spectral and weakly spectral sets

Throughout this paper, $\mathcal{V}$ is assumed to be a Euclidean Jordan algebra of rank n.

Definition 3.2.1 A set E in $\mathcal{V}$ is **spectral** if there exists a permutation invariant set Q in $\mathcal{R}^n$ such that $E = \lambda^{-1}(Q)$. Also, E is said to be **weakly spectral** if it is invariant under algebra automorphisms of $\mathcal{V}$.

Note that the symmetric cone $\mathcal{V}_+$ can be written as

$$\mathcal{V}_+ = \{x \in \mathcal{V} \mid \lambda(x) \geq 0\} = \lambda^{-1}(\mathcal{R}^n_+).$$

This shows that $\mathcal{V}_+$ is spectral. As automorphisms preserve eigenvalues, it is easily seen that every spectral set is weakly spectral.

The following result can be readily obtained by using the definition, thus we state it without proof.

Proposition 3.2.2 Let E_1 and E_2 be spectral/weakly spectral sets. Then union, intersection, and/or complements of E_1 and E_2 are also spectral/weakly spectral.

We first characterize spectral sets via spectral equivalence and a 'diamond' operation. This operation is motivated by the question of recovering Q from E. We note that in the case of $\mathcal{V} = \mathcal{S}^n$, if E is a spectral set, then the corresponding Q is given by $Q = \{u \in \mathcal{R}^n : \mathrm{Diag}(u) \in E\}$, where $\mathrm{Diag}(u)$ is a diagonal matrix with u as the diagonal [22].

Definition 3.2.3 For a set Q in $\mathcal{R}^n$, we define the set $Q^\diamondsuit$ in $\mathcal{V}$ by

$$Q^\diamondsuit := \lambda^{-1}(Q) = \{x \in \mathcal{V} : \lambda(x) \in Q\}. \tag{3.1}$$

For a set E in $\mathcal{V}$, we let $E^\diamondsuit$ in $\mathcal{R}^n$ be

$$E^\diamondsuit := \Sigma_n(\lambda(E)) = \left\{u \in \mathcal{R}^n : \ u^\downarrow = \lambda(x) \text{ for some } x \in E\right\}. \tag{3.2}$$

For simplicity, we write $Q^{\diamondsuit\diamondsuit}$ in place of $(Q^\diamondsuit)^\diamondsuit$, etc.

From now on, for elements $x,\, y \in \mathcal{V}$, we write $x \sim y$ if and only if $\lambda(x) = \lambda(y)$.

Theorem 3.2.4 The following statements hold:

(i) For any Q that is permutation invariant in $\mathcal{R}^n$, $Q^\diamond$ is a spectral set in $\mathcal{V}$ and $Q^{\diamond\diamond} = Q$.

(ii) For any set E in $\mathcal{V}$, $E^\diamond$ is permutation invariant in $\mathcal{R}^n$.

(iii) $E^{\diamond\diamond} = \{x \in \mathcal{V} : \; x \sim y \text{ for some } y \in E\}$.

Proof. For any set Q in $\mathcal{R}^n$, we define the *core of* Q by

$$Q^{\downarrow} := \{u^{\downarrow} : u \in Q\}.$$

We immediately note the following when Q is permutation invariant:

$$Q^{\downarrow} \subseteq Q, \quad Q = \Sigma_n(Q^{\downarrow}), \quad \text{and} \quad \lambda(\lambda^{-1}(Q)) = Q^{\downarrow}.$$

(i) Let Q be permutation invariant. Then $E := Q^\diamond$ is a spectral set by the definition. We also have

$$(Q^\diamond)^\diamond = E^\diamond = \Sigma_n(\lambda(E)) = \Sigma_n(\lambda(\lambda^{-1}(Q)) = \Sigma_n(Q^{\downarrow}) = Q.$$

(ii) Since Σ_n is a group, $\Sigma_n(E^\diamond) = \Sigma_n(\Sigma_n(\lambda(E)) = \Sigma_n(\lambda(E)) = E^\diamond$. Thus, $E^\diamond$ is permutation invariant.

(iii) This follows from the fact that

$$\begin{aligned} E^{\diamond\diamond} = (E^\diamond)^\diamond &= \left\{x \in \mathcal{V} \;\middle|\; \lambda(x) \in E^\diamond\right\} \\ &= \{x \in \mathcal{V} \mid \lambda(x) = \lambda(y) \text{ for some } y \in E\}. \end{aligned}$$ $\square$

Example 3.2.5 This example shows that in Item (i) above, one needs the permutation invariance property of Q to get the equality $Q^{\diamondsuit\diamondsuit} = Q$. Let $\mathcal{V} = \mathcal{S}^2$ and $Q_1 = \{u \in \mathcal{R}^2_+ : u_1 \geq u_2\}$. Then, it is easy to see that $Q_1^{\diamondsuit} = \mathcal{S}^2_+$ and so $Q_1^{\diamondsuit\diamondsuit} = \mathcal{R}^2_+$. Thus, we have $Q_1 \subsetneq Q_1^{\diamondsuit\diamondsuit}$. On the other hand, consider $Q_2 = \{u \in \mathcal{R}^2_+ : u_1 \leq u_2\}$. Then, $Q_2^{\diamondsuit} = \{aI : a \geq 0\}$; therefore $Q_2^{\diamondsuit\diamondsuit} = \{u \in \mathcal{R}^2_+ : u_1 = u_2\}$. This means $Q_2^{\diamondsuit\diamondsuit} \subsetneq Q_2$.

We now characterize spectral sets.

Theorem 3.2.6 The following are equivalent for any set E in $\mathcal{V}$.

(a) $E = \lambda^{-1}(Q)$, where Q is permutation invariant in $\mathcal{R}^n$.

(b) If $x \sim y$ and $y \in E$, then $x \in E$.

(c) $E^{\diamondsuit\diamondsuit} = E$.

Proof. $(a) \Rightarrow (b)$: Let $E = \lambda^{-1}(Q)$, where Q is permutation invariant. If $x \sim y$ with $y \in E$, then $\lambda(x) = \lambda(y) \in Q$. Hence, $x \in \lambda^{-1}(Q) = E$.

$(b) \Rightarrow (c)$: By Item (iii) in Theorem 3.2.4, we have

$$E^{\diamondsuit\diamondsuit} = \{x \in \mathcal{V} : \ x \sim y \text{ for some } y \in E\}.$$

Then $E \subseteq E^{\diamondsuit\diamondsuit}$ is now clear from this observation. To see the reverse implication, take $x \in E^{\diamondsuit\diamondsuit}$, then there exists $y \in E$ with $x \sim y$. This implies $x \in E$ from (b). Hence, $E^{\diamondsuit\diamondsuit} \subseteq E$.

$(c) \Rightarrow (a)$: When $E^{\diamondsuit\diamondsuit} = E$, we let $Q := E^{\diamondsuit}$. Since Σ_n forms a group,

$$\Sigma_n(E^{\diamondsuit}) = \Sigma_n(\Sigma_n(\lambda(E)) = \Sigma_n(\lambda(E)) = E^{\diamondsuit}.$$

Thus, $Q = E^{\diamondsuit}$ is permutation invariant; hence $E = Q^{\diamondsuit} = \lambda^{-1}(Q)$ is spectral. $\square$

Remarks.

(1) The above result shows that $Q \mapsto Q^{\diamondsuit}$ sets up is a one-to-one correspondence between permutation invariant sets in $\mathcal{R}^n$ and spectral sets in $\mathcal{V}$.

(2) For any spectral set E in $\mathcal{V}$, we can obtain the corresponding permutation invariant set Q in $\mathcal{R}^n$ by taking $Q = E^{\diamondsuit}$.

We now describe the relation between spectral and weakly spectral sets. This result explains why in $\mathcal{S}^n$ or $\mathcal{H}^n$, spectral sets are completely characterized by automorphism invariance.

Theorem 3.2.7 Every spectral set in $\mathcal{V}$ is weakly spectral. Converse holds when $\mathcal{V}$ is essentially simple.

Proof. Suppose E is a spectral set, $x \in E$, and $\phi \in \mathrm{Aut}(\mathcal{V})$. As eigenvalues remain the same under the action of automorphisms, we see that $\phi(x) \sim x$. By Item (b) in Theorem 3.2.6, $\phi(x) \in E$. This proves that E is weakly spectral.

To see the converse, assume that E is weakly spectral and $\mathcal{V}$ is essentially simple. We verify Item (b) in Theorem 3.2.6 to show that E is spectral. To this end, let $x \sim y$, $y \in E$. Then, $\lambda(x) = \lambda(y)$. By Proposition 2.1.15, there exists $\phi \in \mathrm{Aut}(\mathcal{V})$ such that $x = \phi(y)$. As E is invariant under automorphisms, we must have $x \in E$. This concludes the proof. $\square$

Example 3.2.8 In the theorem above, the converse may not hold for general algebras. To see this, consider $\mathcal{V} = \mathcal{R} \times \mathcal{S}^2$ and $E = \mathcal{R} \times \mathcal{S}^2_+$. Then, by Proposition 2.1.10, every automorphism ϕ on $\mathcal{V}$ is of the form $\phi = (\phi_1, \phi_2)$, where $\phi_1 \in \mathrm{Aut}(\mathcal{R})$, $\phi_2 \in \mathrm{Aut}(\mathcal{S}^2)$. It is easy to check that E is invariant under automorphisms. Now consider two elements,

$$x = \left(0, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) \notin E, \quad \text{and} \quad y = \left(-1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \in E.$$

As $\lambda(x) = \lambda(y) = (1, 0, -1)^{\mathsf{T}}$, we have $x \sim y$. Thus, E violates condition (b) in Theorem 3.2.6. Hence, E is not a spectral set. It is easy to verify that $x \in E^{\diamond\diamond}$. Thus, $E \neq E^{\diamond\diamond}$ even though E is invariant under automorphisms.

The following result shows that on permutation invariant (convex) sets, λ^{-1} has linear behavior.

Theorem 3.2.9 The following statements hold:

(i) Let Q be a permutation invariant set in $\mathcal{R}^n$ and $\alpha \geq 0$ in $\mathcal{R}$. Then

$$\lambda^{-1}(-Q) = -[\lambda^{-1}(Q)] \quad \text{and} \quad \lambda^{-1}(\alpha\, Q) = \alpha\, \lambda^{-1}(Q).$$

(ii) Let Q_1 and Q_2 be permutation invariant convex sets in $\mathcal{R}^n$. Then,

$$\lambda^{-1}(Q_1 + Q_2) = \lambda^{-1}(Q_1) + \lambda^{-1}(Q_2).$$

Proof.

(i) We first observe that $-Q$ is permutation invariant. Let $x \in \lambda^{-1}(Q)$ with its spectral decomposition $x = \sum_1^n \lambda_i(x) e_i$. From $-x = \sum_1^n [-\lambda_i(x)] e_i$ we get $\lambda(-x) = [-\lambda(x)]^{\downarrow} = -[\lambda(x)]^{\uparrow}$, where $u^{\uparrow}$ denotes the increasing rearrangement of u. Since Q is permutation invariant and $\lambda(x) \in Q$, we have $\lambda(x)^{\uparrow} \in Q$ and $-[\lambda(x)]^{\uparrow} \in -Q$. Thus, $\lambda(-x) \in -Q$ and $-x \in \lambda^{-1}(-Q)$. Hence, $-[\lambda^{-1}(Q)] \subseteq \lambda^{-1}(-Q)$. Now, let $y \in \lambda^{-1}(-Q)$. Then, by the previous inclusion applied to $-Q$, $-y \in \lambda^{-1}[-(-Q)] = \lambda^{-1}(Q)$, or equivalently, $y \in -[\lambda^{-1}(Q)]$. This proves the inclusion $\lambda^{-1}(-Q) \subseteq -[\lambda^{-1}(Q)]$. Thus we have the first part of statement (i). The second part, for $\alpha = 0$ is obvious; the case $\alpha > 0$ follows easily from the positive homogeneity of λ.

(ii) Now suppose that Q_1 and Q_2 are permutation invariant convex sets in $\mathcal{R}^n$. Then $Q_1 + Q_2$ is also permutation invariant and convex. Let $x \in \lambda^{-1}(Q_1)$ and $y \in \lambda^{-1}(Q_2)$ so that $\lambda(x) \in Q_1$ and $\lambda(y) \in Q_2$. Then, by Proposition 2.3.8,

$$\lambda(x+y) = A\,\lambda(x) + B\,\lambda(y),$$

where A and B are doubly stochastic matrices on $\mathcal{R}^n$. By a well-known theorem of Birkhoff ([3], Theorem II.2.3) A and B are convex combinations of permutation matrices. As Q_1 and Q_2 are permutation invariant convex sets, it follows that $A\lambda(x) \in Q_1$ and $B\lambda(y) \in Q_2$. Thus, $\lambda(x+y) \in Q_1 + Q_2$. This implies that $x + y \in \lambda^{-1}(Q_1 + Q_2)$. Hence,

$$\lambda^{-1}(Q_1) + \lambda^{-1}(Q_2) \subseteq \lambda^{-1}(Q_1 + Q_2).$$

To see the reverse inclusion, let $z \in \lambda^{-1}(Q_1 + Q_2)$ with spectral decomposition $z = \sum_1^n \lambda_i(z) e_i$. Then, $\lambda(z) \in Q_1 + Q_2$. Let $\lambda(z) = u + v$, where $u \in Q_1$ and $v \in Q_2$. Define $x = \sum_1^n u_i e_i$ and $y = \sum_1^n v_i e_i$ so that $z = x + y$. As $\lambda(x) = u^{\downarrow} \in Q_1$ and $\lambda(y) = v^{\downarrow} \in Q_2$, we see that $x \in \lambda^{-1}(Q_1)$, $y \in \lambda^{-1}(Q_2)$.

Thus, $z = x+y \in \lambda^{-1}(Q_1)+\lambda^{-1}(Q_2)$. Hence, $\lambda^{-1}(Q_1+Q_2) \subseteq \lambda^{-1}(Q_1)+\lambda^{-1}(Q_2)$.

Thus we have (ii).

$\square$

The following is an easy consequence.

Corollary 3.2.10 Let Q_1 and Q_2 be permutation invariant convex sets in $\mathcal{R}^n$ and $\alpha_1, \alpha_2 \in \mathcal{R}$. Then,

$$\lambda^{-1}(\alpha_1\, Q_1 + \alpha_2\, Q_2) = \alpha_1\, \lambda^{-1}(Q_1) + \alpha_2\, \lambda^{-1}(Q_2).$$

The following theorem describes how the convexity gets transferred between spectral sets and the corresponding permutation invariant sets.

Theorem 3.2.11 Let $Q \in \mathcal{R}^n$ be a permutation invariant set and $E = \lambda^{-1}(Q)$ be a spectral set in $\mathcal{V}$. Then Q is convex if and only if E is convex.

Proof. Let $E = \lambda^{-1}(Q)$, where Q is permutation invariant and convex in $\mathcal{R}^n$. The convexity of E has already been proved in Theorem 27, [1]. For completeness, we provide a (slightly different) proof. Let $x, y \in E$ and $t \in [0, 1]$. By Proposition 2.3.8, $\lambda(tx + (1-t)y)) = tu + (1-t)v$, where $u = A(\lambda(x))$, $v = B(\lambda(y))$ for some doubly stochastic matrices A and B. By Proposition 2.3.4, we see that $u, v \in Q$. As Q is convex, $\lambda(tx + (1-t)y) \in Q$. Thus, $tx + (1-t)y \in E$. This proves the convexity of E.

Now, let $E = \lambda^{-1}(Q)$ be convex in $\mathcal{V}$. To show that Q is convex, let u, $v \in Q$ and

$t \in [0,1]$. Then there exist x, $y \in E$ such that $x := \sum_1^n u_i e_i$ and $y := \sum_1^n v_i e_i'$ for some Jordan frames $\{e_1, e_2, \dots, e_n\}$ and $\{e_1', e_2', \dots, e_n'\}$. Now, define $\bar{x} = \sum_1^n u_i e_i'$. Then $\lambda(\bar{x}) = \lambda(x)$, hence $\bar{x} \in E$ by Item (b) in Theorem 3.2.6. As $\bar{x}$, $y \in E$ and E is convex, we get

$$\begin{aligned}\sum_{i=1}^{n}(tu_i + (1-t)v_i)e_i' &= t\sum_{i=1}^{n} u_i e_i' + (1-t)\sum_{i=1}^{n} v_i e_i' \\ &= t\bar{x} + (1-t)y \in E.\end{aligned}$$

This proves that $tu + (1-t)v \in Q$. Thus, Q is convex. □

3.3 The transfer principle and metaformulas

In the context of spectral sets $E = \lambda^{-1}(Q)$ and spectral functions $F = f \circ \lambda$, the *Transfer Principle* asserts that (many) properties of Q (of f) get transferred to E (respectively, to F). Theorem 3.2.11, where convexity gets transferred, illustrates this principle. In addition to this, Baes ([1], Theorem 27) shows that closedness, openness, boundedness and compactness properties of Q are carried over to E. Sun and Sun [47] show that semismoothness property of f gets transferred to F in the setting of a Euclidean Jordan algebras. Numerous specialized results exist in the setting of $\mathcal{S}^n$ and $\mathcal{H}^n$; see the recent article [9] and the references therein for a discussion on the transferability of C^∞-manifold property of Q and various types differentiability properties of f (e.g., prox-regularity, Clarke-regularity, and smoothness). Related to this, in the context of normal decomposition systems, Lewis [27] has observed that the 'metaformula'

$$\#(\lambda^{-1}(Q)) = \lambda^{-1}(\#(Q))$$

holds for sets Q that are invariant under a group of orthogonal transformations and certain set operations $\#$. In particular, it was shown in Theorem 5.4, [27], that the formula holds when $\#$ is the closure/interior/boundary operation. Consequently, because of a result in [35], it is also valid in any (essentially) simple Euclidean Jordan algebra; see [20], [22] for results of this type for spectral cones in $\mathcal{S}^n$. Motivated by these, we present the following 'metaformula' in the setting of general Euclidean Jordan algebras.

In what follows, for a set S (either in $\mathcal{R}^n$ or in $\mathcal{V}$), we consider closure, interior, boundary, convex hull, and conic hull operations, which are respectively denoted by $\overline{S}$, S°, ∂S, $\mathrm{conv}(S)$, and $\mathrm{cone}(S)$. Recall that for a set Q in $\mathcal{R}^n$ and a set E in $\mathcal{V}$,

$$Q^\diamondsuit := \lambda^{-1}(Q) \quad \text{and} \quad E^\diamondsuit := \Sigma_n(\lambda(E)).$$

(We remark that when $\mathcal{V} = \mathcal{R}^n$ and $E = Q$, these two definitions coincide.)

Theorem 3.3.1 Let $\#$ denote one of the operations of closure, interior, boundary, convex hull, or conic hull. Then, over permutation invariant sets in $\mathcal{R}^n$ and spectral sets in $\mathcal{V}$, the operations $\#$ and $\diamondsuit$ commute; symbolically,

$$\# \diamondsuit = \diamondsuit \#.$$

In preparation for the proof, we first state and prove the following result:

Proposition 3.3.2 Let Q be a permutation invariant set in $\mathcal{R}^n$ and $E = \lambda^{-1}(Q)$ in $\mathcal{V}$. Then,

(i) $\overline{E} = \lambda^{-1}(\overline{Q})$.

(ii) $E^\circ = \lambda^{-1}(Q^\circ)$.

(iii) $\partial E = \lambda^{-1}(\partial Q)$.

(iv) $\text{conv}(\lambda^{-1}(Q)) = \lambda^{-1}(\text{conv}(Q))$.

(v) $\text{cone}(\lambda^{-1}(Q)) = \lambda^{-1}(\text{cone}(Q))$.

Consequently, the closure/interior/boundary/convex hull/conic hull of a spectral set is spectral.

Proof.

(i) By continuity of λ (see [1], Corollary 24), $\overline{E} = \overline{\lambda^{-1}(Q)} \subseteq \lambda^{-1}(\overline{Q})$. To see the reverse inclusion, let $x \in \lambda^{-1}(\overline{Q})$ so that $\lambda(x) \in \overline{Q}$. Let $\lambda(x) = \lim q_k$, where $q_k \in Q$. We consider the spectral decomposition $x = \sum_1^n \lambda_i(x) e_i$ and define $x_k := \sum_1^n (q_k)_i e_i$, for $k = 1, 2, \ldots$. Then, $\lambda(x_k) = q_k^\downarrow \in Q$ (recall that Q is permutation invariant). Thus, $x_k \in \lambda^{-1}(Q)$ for all k. As $x_k \to x$, we see that $x \in \overline{E}$. Hence, $\overline{E} = \lambda^{-1}(\overline{Q})$.

(ii) As Q^c is permutation invariant and λ^{-1} preserves complements, we see, by (i), that $\overline{E^c} = \lambda^{-1}(\overline{Q^c})$. Taking complements and using the identity $E^\circ = (\overline{E^c})^c$, we get $E^\circ = \lambda^{-1}(Q^\circ)$.

(iii) This comes from the previous items using the definition $\partial E = \overline{E} \backslash E^\circ$ and the fact that λ^{-1} preserves set differences.

(iv) We first prove that $\mathrm{conv}(\lambda^{-1}(Q)) \subseteq \lambda^{-1}(\mathrm{conv}(Q))$ in $\mathcal{V}$. It is clear that $\mathrm{conv}(Q)$ is convex and permutation invariant. Hence, by Theorem 3.2.11, $\lambda^{-1}(\mathrm{conv}(Q))$ is convex. As $\lambda^{-1}(Q) \subseteq \lambda^{-1}(\mathrm{conv}(Q))$, we see that $\mathrm{conv}(\lambda^{-1}(Q)) \subseteq \lambda^{-1}(\mathrm{conv}(Q))$. To prove the reverse implication, let $x \in \lambda^{-1}(\mathrm{conv}(Q))$. Then, $\lambda(x) = \sum_{k=1}^{N} \alpha_k \, q_k$, where α_ks are positive with sum one and $q_k \in Q$ for all k. Writing the spectral decomposition of $x = \sum_{i=1}^{n} \lambda_i(x) e_i$, we see that

$$x = \sum_{i=1}^{n} \left(\sum_{k=1}^{N} \alpha_k \, (q_k)_i \right) e_i = \sum_{k=1}^{N} \alpha_k \left(\sum_{1}^{n} (q_k)_i e_i \right).$$

Letting $x_k := \sum_1^n (q_k)_i e_i$, we get $\lambda(x_k) = q_k^{\downarrow} \in Q$. Hence, $x_k \in \lambda^{-1}(Q)$ and x is now a convex combination of elements of $\lambda^{-1}(Q)$. Thus, $x \in \mathrm{conv}(\lambda^{-1}(Q))$ proving the required reverse inclusion.

(v) Using (iv) and the positive homogeneity of λ, it is easy to verify that $\lambda^{-1}(\mathrm{cone}(Q))$ is a convex cone. Together with this fact, $\lambda^{-1}(Q) \subseteq \lambda^{-1}(\mathrm{cone}(Q))$ implies $\mathrm{cone}(\lambda^{-1}(Q)) \subseteq \lambda^{-1}(\mathrm{cone}(Q))$. For the reverse implication, we repeat the proof of (iv) except q_ks are now not required to have the sum one.

Finally, the last statement follows from Items (i) - (v) together with the observation that the closure/interior/boundary/convex hull/conic hull of a permutation invariant set in $\mathcal{R}^n$ is permutation invariant. $\square$

Proof of the Theorem. Suppose Q is any permutation invariant set in $\mathcal{R}^n$. Then from Proposition 3.3.2 we have $\lambda^{-1}(\#(Q)) = \#(\lambda^{-1}(Q))$, that is,

$$[\#(Q)]^{\diamondsuit} = \#(Q^{\diamondsuit}).$$

This means that on permutation invariant sets in $\mathcal{R}^n$, the operations $\#$ and $\diamondsuit$ commute.

Now suppose that E is any spectral set in $\mathcal{V}$. Then, $Q := E^{\diamondsuit}$ is permutation invariant and $Q^{\diamondsuit} = E^{\diamondsuit\diamondsuit} = E$ (by Theorem 3.2.6). Hence, $\#(E) = \#(Q^{\diamondsuit}) = [\#(Q)]^{\diamondsuit}$. As $\#(Q)$ is permutation invariant, from Theorem 3.2.4 (i), $[\#(Q)]^{\diamondsuit\diamondsuit} = \#(Q)$. Thus,

$$[\#(E)]^{\diamondsuit} = [\#(Q)]^{\diamondsuit\diamondsuit} = \#(Q) = \#(E^{\diamondsuit}).$$

This proves that the operations $\#$ and $\diamondsuit$ commute on spectral sets in $\mathcal{V}$. $\square$

It will be shown below that the equality $\overline{E^{\diamondsuit}} = \overline{E}^{\diamondsuit}$ holds for any set $E \subseteq \mathcal{V}$. We now provide examples to show that in the above theorem, permutation invariance of Q and/or spectrality of E is needed to get the remaining results.

Example 3.3.3

(a) Let $\mathcal{V} = \mathcal{S}^n$ and $Q = \{u \in \mathcal{R}^n_+ : u_1 < u_2 < \cdots < u_n\}$. Clearly, Q is not permutation invariant. As $Q^{\diamondsuit} = \lambda^{-1}(Q) = \emptyset$, we have $\overline{Q^{\diamondsuit}} = \emptyset$. On the other hand, we have $\overline{Q} = \{u \in \mathcal{R}^n_+ : u_1 \leq u_2 \leq \cdots \leq u_n\}$ and $\overline{Q}^{\diamondsuit} = \lambda^{-1}(\overline{Q})$ is the (nonempty) set of all nonnegative multiples of the identity matrix. Thus, $\overline{Q}^{\diamondsuit} \neq \overline{Q^{\diamondsuit}}$.

(b) One consequence of Theorem 3.2.11 is that *when E is spectral and convex, $E^{\diamondsuit}$ is convex.* This may fail if E is not spectral: In $\mathcal{V} = \mathcal{R}^2$, let e_1 and e_2 denote the standard coordinate vectors. Then, $E = \{e_1\}$ is convex, while $E^{\diamondsuit} = \{e_1, e_2\}$ is not. In particular, $[\text{conv}(E)]^{\diamondsuit} \neq \text{conv}(E^{\diamondsuit})$.

(c) Let $\mathcal{V} = \mathcal{S}^2$ and let E be the set of all nonnegative diagonal matrices. Then, one can easily verify that $E^{\diamondsuit} = \mathcal{R}^2_+$ and so $(E^{\diamondsuit})^\circ = \mathcal{R}^2_{++}$, where $\mathcal{R}^2_{++}$ denotes the set of all positive vectors in $\mathcal{R}^2$. However, we have $E^\circ = \emptyset$ and $(E^\circ)^{\diamondsuit} = \emptyset$. Hence, $(E^{\diamondsuit})^\circ \neq (E^\circ)^{\diamondsuit}$.

The preservation of certain properties is an important feature of the 'diamond' operation.

Theorem 3.3.4 If a set E is closed/open/bounded/compact in $\mathcal{V}$, then so is $E^{\diamondsuit}$ in $\mathcal{R}^n$.

Proof. We assume, without loss of generality, that E is nonempty.

- Suppose that E is closed. Let $\{u_k\}$ be a sequence in $E^{\diamondsuit}$ such that $u_k \to u$ for some $u \in \mathcal{V}$. For each $u_k \in E^{\diamondsuit}$, we can find $x_k \in E$ with $\lambda(x_k) = u_k^{\downarrow}$. Let $x_k = \sum_i \lambda_i(x_k) e_i^{(k)}$ be the spectral decomposition of x_k for each k. As the set of all primitive idempotents in $\mathcal{V}$ forms a compact set (see [11], page 78), there exists a sequence k_m such that $e_i^{(k_m)} \to e_i$ for all $i = 1, 2, \ldots, n$. Then $\{e_1, e_2, \ldots, e_n\}$ forms a Jordan frame and

$$x_{k_m} = \sum_{i=1}^{n} \lambda_i(x_{k_m}) e_i^{(k_m)} \to \sum_{i=1}^{n} u_i^{\downarrow} e_i.$$

Since E is closed, we have $x := \lim x_{k_m} \in E$, which implies that $\lambda(x) = u^{\downarrow}$. Hence, $u \in E^{\diamondsuit}$ proving the closedness of $E^{\diamondsuit}$.

- For any (primitive) idempotent c in $\mathcal{V}$ we observe that $||c||^2 = \langle c, c\rangle = \langle c \circ c, e\rangle = \langle c, e\rangle \leq ||c||\,||e||$ and hence $||c|| \leq ||e||$. Then, for any Jordan frame

$\{e_1, e_2, \ldots, e_n\}$ in $\mathcal{V}$ and $u \in \mathcal{R}^n$, we have

$$\left\|\sum_{i=1}^{n} u_i e_i\right\| \leq \left(\sum_{i=1}^{n} |u_i|\right) ||e|| \leq \sqrt{n}\, ||u||\, ||e|| \tag{3.3}$$

where $||u||$ denotes the 2-norm of u.

Now assume that E is open in $\mathcal{V}$. To show that $E^\diamond$ open, let $u \in E^\diamond$. Then there exists $x \in E$ such that $\lambda(x) = u^\downarrow$. As E is open, there exists $\epsilon > 0$ such that $B(x, \epsilon) := \{y \in \mathcal{V} : \|y - x\| < \epsilon\} \subseteq E$. Putting $\delta := \frac{\epsilon}{\sqrt{n}||e||}$, we show that the ball $B(u, \delta) := \{v \in \mathcal{R}^n : \|u - v\| < \delta\}$ is contained in $E^\diamond$. To this end, let $x = \sum_i u_i^\downarrow e_i$ be the spectral decomposition of x. Then, there exists a permutation matrix $\sigma \in \Sigma_n$ such that $x = \sum_i u_i e_{\sigma(i)}$. Now, for any $v \in B(u, \delta)$, define $y := \sum_i v_i e_{\sigma(i)} \in \mathcal{V}$. Then, from (3.3),

$$\|x - y\| \leq \sqrt{n}\, ||e||\, \|u - v\| < \epsilon.$$

This proves that $y \in E$. As $\lambda(y) = v^\downarrow$, we see that $v \in E^\diamond$. This shows that $B(u, \delta) \subseteq E^\diamond$; hence $E^\diamond$ is open.

- Now let E be compact. Then $\lambda(E)$ is compact, by the continuity of λ. As Σ_n is compact, we see that $E^\diamond = \Sigma_n(\lambda(E))$ is also compact.

- Finally, if E is bounded, then $\overline{E}$ is compact. Hence, $(\overline{E})^\diamond$ is compact. Clearly, $E^\diamond$ is bounded as it is a subset of $(\overline{E})^\diamond$. □

Corollary 3.3.5 For any set E in $\mathcal{V}$, $\overline{E^\diamond} = \overline{E}^\diamond$.

Proof. Without loss of generality, we assume that E is nonempty.

Since $E^\diamond \subseteq \overline{E}^\diamond$ and $\overline{E}^\diamond$ is closed by the above Theorem, we have $\overline{E^\diamond} \subseteq \overline{E}^\diamond$.

To see the reverse inclusion, let $u \in \overline{E}^\diamond$. Then there exists $x \in \overline{E}$ such that $\lambda(x) = u^\downarrow$. As $u^\downarrow$ is some permutation of u, there exists a permutation $\sigma \in \Sigma_n$ such that $\sigma^{-1}(u) = u^\downarrow = \lambda(x)$. As $x \in \overline{E}$, there exists a sequence $\{x_k\}$ of E converging to x. Then, by the continuity of λ, we get $\lambda(x) = \lim_{k\to\infty} \lambda(x_k)$. This implies

$$u = \sigma(\lambda(x)) = \sigma\left(\lim_{k\to\infty} \lambda(x_k)\right) = \lim_{k\to\infty} \sigma(\lambda(x_k)).$$

Letting $u_k := \sigma(\lambda(x_k))$, we get $u_k^\downarrow = \lambda(x_k)$; thus $u_k \in E^\diamond$ for all k. As $u_k \to u$, we have $u \in \overline{E^\diamond}$ proving $\overline{E}^\diamond \subseteq \overline{E^\diamond}$. This completes the proof. $\square$

Our final result deals with the 'double diamond' operation. For a set E in $\mathcal{V}$, we call $E^{\diamond\diamond}$, the *spectral hull* of E in $\mathcal{V}$. Here are some properties of the spectral hull.

Proposition 3.3.6 For any set $E \subseteq \mathcal{V}$, we have

(i) $E \subseteq E^{\diamond\diamond}$.

(ii) If $E_1 \subseteq E_2$, then $E_1^{\diamond\diamond} \subseteq E_2^{\diamond\diamond}$.

(iii) $E^{\diamond\diamond}$ is the smallest spectral set containing E.

(iv) E is spectral if and only if $E = E^{\diamond\diamond}$.

(v) $(E_1 \cap E_2)^{\diamond\diamond} = E_1^{\diamond\diamond} \cap E_2^{\diamond\diamond}$, $(E_1 \cup E_2)^{\diamond\diamond} = E_1^{\diamond\diamond} \cup E_2^{\diamond\diamond}$.

(vi) $\overline{E^{\diamond\diamond}} = (\overline{E})^{\diamond\diamond}$. In particular, if E is closed, then so is $E^{\diamond\diamond}$.

(vii) If E is open, then $E^{\diamond\diamond}$ is open.

(viii) If E is compact, then $E^{\diamond\diamond}$ is compact.

Proof. Items (i)-(v) follow from Item (iii), Theorem 3.2.4.

(vi) As $E^{\diamondsuit}(= Q)$ is permutation invariant, this follows from Proposition 3.3.2 and Corollary 3.3.5.

(vii) As $E^{\diamondsuit}$ is permutation invariant, it is clear from Theorem 3.3.4 and Proposition 3.3.2 that $E^{\diamondsuit\diamondsuit}$ is open whenever E is open.

(viii) When E is compact, by Theorem 3.3.4, $Q := E^{\diamondsuit}$ is compact and permutation invariant. By Theorem 27 in [1], $E^{\diamondsuit\diamondsuit} = \lambda^{-1}(Q)$ is also compact. $\square$

Chapter 4

Spectral Cones in Euclidean Jordan Algebras

4.1 Introduction

Continuing the study of spectral sets in $\mathcal{V}$, in this chapter, we study a special case of spectral sets, namely spectral cones. A convex cone K in a Euclidean Jordan algebra is **spectral** if $K = \lambda^{-1}(Q)$ for some permutation invariant convex cone in $\mathcal{R}^n$. The symmetric cone in a Euclidean Jordan algebra is an important example of a spectral cone as it comes from $Q = \mathcal{R}^n_+$ (the nonnegative orthant in $\mathcal{R}^n$). Because any spectral cone is a spectral set, all properties/theorems of spectral sets can be applied to spectral cones.

Moreover, due to rich properties of cones, we may expect additional properties of a spectral cone K can get transferred from the corresponding Q. To be specific, when $\mathcal{V}$ is simple or carries the canonical inner product, we show that the dual of spectral cone is also spectral and be written as $K^* = \lambda^{-1}(Q^*)$. The pointedness and solidness of a spectral cone are characterized for any $\mathcal{V}$. We also show that for any spectral cone K in $\mathcal{V}$, $\dim(K) \in \{0, 1, d-1, d\}$, where $\dim(K)$ denotes the dimension of of K and d is the dimension of $\mathcal{V}$.

The organization of this chapter is as follows:

- We relate, in Section 3, spectral cones with majorization, doubly stochastic maps, and algebra automorphisms.
- In Section 4, dimensionality of a given spectral cone is discussed.
- Characterizations of pointedness and solidness of spectral cones are presented in Section 5.

4.2 Equivalence formulations of spectral cones

Definition 4.2.1 A convex cone K in $\mathcal{V}$ is called a **spectral cone** if $K = \lambda^{-1}(Q)$ for some permutation invariant convex cone Q in $\mathcal{R}^n$.

It is easy to verify, by the positive homogeneity of λ, that K is indeed a cone and, by Proposition 3.2.11, convex. Thus, every spectral cone is a convex cone. The metaformula (Theorem 3.3.1) shows that the closure/interior/convex hull of a spectral cone is again a spectral cone. For a list of spectral cones in $\mathcal{S}^n$, see [22].

The following result characterizes spectral cones among convex cones in $\mathcal{V}$.

Theorem 4.2.2 For a convex cone K in $\mathcal{V}$, consider the following statements:

(a) K is a spectral cone.

(b) If $x \prec y$ and $y \in K$, then $x \in K$.

(c) If $x \sim y$ and $y \in K$, then $x \in K$.

(d) For every doubly stochastic map Ψ on $\mathcal{V}$, $\Psi(K) \subseteq K$.

(e) For every algebra automorphism ϕ on $\mathcal{V}$, $\phi(K) \subseteq K$.

Then, $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ and $(b) \Rightarrow (d) \Rightarrow (e)$. Moreover, when $\mathcal{V}$ is essentially simple, all the above statements are equivalent.

Proof. $(a) \Rightarrow (b)$: Suppose $x \prec y$ and $y \in K$. Then, $\lambda(x) \prec \lambda(y)$ and $\lambda(y) \in Q$. By Proposition 2.3.4, $\lambda(x) \in Q$ and hence $x \in K$.

$(b) \Rightarrow (c)$: This is obvious as $x \sim y$ implies $x \prec y$.

$(c) \Rightarrow (a)$: When (c) holds, by Theorem 3.2.6, K is spectral. Now, let $K = \lambda^{-1}(Q)$, where Q is permutation invariant. To show that Q is convex, let u, $v \in Q$ and $t \in [0, 1]$. Then there exist x, $y \in K$ such that $x := \sum_{i=1}^{n} u_i e_i$ and $y := \sum_{i=1}^{n} v_i e_i'$ for some Jordan frames $\{e_1, e_2, \ldots, e_n\}$ and $\{e_1', e_2', \ldots, e_n'\}$. Now, define $\bar{x} = \sum_{i=1}^{n} u_i e_i'$. Then $\lambda(\bar{x}) = \lambda(x)$, hence $\bar{x} \in K$ by (c). As $\bar{x}$, $y \in K$ and K is convex, we get

$$\begin{aligned} \sum_{i=1}^{n} (tu_i + (1-t)v_i)e_i' &= t \sum_{i=1}^{n} u_i e_i' + (1-t) \sum_{i=1}^{n} v_i e_i' \\ &= t\bar{x} + (1-t)y \in K. \end{aligned}$$

This proves that $tu + (1-t)v \in Q$. Thus, Q is convex. Since K is given to be a cone, we see that Q (which equals $\Sigma_n(\lambda(K))$, see (3.2)) is a cone. Thus, K is a spectral cone.

$(b) \Rightarrow (d)$: Assume that K is spectral cone and let Ψ be a doubly stochastic map on $\mathcal{V}$. For $y \in K$, let $x = \Psi(y)$. Then by Proposition 2.3.7, $x \prec y$. It follows that $x \in K$.

$(d) \Rightarrow (e)$: This is obvious, as every algebra automorphism is doubly stochastic.

Now we prove $(e) \Rightarrow (b)$ assuming $\mathcal{V}$ is essentially simple. Let $x \prec y$ in $\mathcal{V}$. Then, Proposition 2.3.7 shows that $x = \Phi(y)$ where Φ is a convex combination of algebra automorphisms. Using the convexity of K and (e), it is easy to see that $x \in K$, which proves (b). $\square$

Example 4.2.3 We can use Example 3.2.8 to show that in a general $\mathcal{V}$, (d) may not imply (a), (b), or (c) in the above theorem.

Remark. When $\mathcal{V}$ is not essentially simple, the reverse implications in the above theorem may not hold. For instance, consider $\mathcal{V} = \mathcal{R} \times \mathcal{S}^2$, $K = \mathcal{R}_+ \times \mathcal{S}^2$, and

$$x = \left(-1, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right), \; y = \left(1, \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\right).$$

From Proposition 2.1.10, we see that K is invariant under automorphisms of $\mathcal{V}$. However, K is not a spectral set as $x \sim y$, $y \in K$, and $x \notin K$. Thus, (e) holds, but not (a).

The above example shows that the *Cartesian product of two spectral cones need not be spectral.* However, as we see below, certain projections of a spectral cone are spectral.

Corollary 4.2.4 Suppose $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_N$ be the product of simple Euclidean Jordan algebras. Let $\Pi_j : \mathcal{V} \to \mathcal{V}_j$ denote the projection map and K be a spectral cone in $\mathcal{V}$. Then $\Pi_j(K)$ is a spectral cone in $\mathcal{V}_j$ for all $j = 1, 2 \ldots, N$.

Proof. By the linearity of Π_j, $\Pi_j(K)$ is a convex cone in $\mathcal{V}_j$. To show that $\Pi_j(K)$ is a spectral cone in the simple algebra $\mathcal{V}_j$, we show that it is invariant under automorphisms of $\mathcal{V}_j$ and apply Theorem 4.2.2. Now, without loss of generality, let

$j = 1$, $x_1 \in \Pi_1(K)$ and $\phi_1 \in \mathrm{Aut}(\mathcal{V}_1)$. Then, there exists $x_i \in \mathcal{V}_i$, $i = 2, 3, \ldots, N$ such that $x := (x_1, x_2, \ldots, x_N) \in K$. Let ϕ_i be the identity transformation on $\mathcal{V}_i$, $i = 2, \ldots, N$. Then, $\phi := (\phi_1, \phi_2, \phi_3, \ldots, \phi_N) \in \mathrm{Aut}(\mathcal{V})$, where the action of ϕ on $\mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_N$ is given by $\phi(z_1, z_2, \ldots, z_n) := (\phi(z_1), \phi(z_2), \ldots, \phi(z_n))$. Since $\phi(x) \in K$ by Theorem 4.2.2, we see that $\phi_1(x_1) \in \Pi_1(K)$. This completes the proof. $\square$

Remarks.

(1) The automorphisms of $\mathcal{R}^n$ are just permutation matrices. Since $\mathcal{R}^n$ is also essentially simple, spectral cones in $\mathcal{R}^n$ are just permutation invariant convex cones.

(2) Let $\mathcal{V}$ be essentially simple. For any $z \in \mathcal{V}$, consider

$$K = \mathrm{cone}\{\phi(z) : \phi \in \mathrm{Aut}(\mathcal{V})\},$$

the convex cone generated by $\{\phi(z) : \phi \in \mathrm{Aut}(\mathcal{V})\}$. Then, by Theorem 4.2.2, K is a spectral cone.

Theorem 4.2.5 Suppose $\mathcal{V}$ is either simple or carries the canonical inner product. Then, for any permutation invariant set Q in $\mathcal{R}^n$,

$$\left[\lambda^{-1}(Q)\right]^* = \lambda^{-1}(Q^*).$$

Proof. By assumption, all primitive idempotents in $\mathcal{V}$ have the same norm. Let $\omega := ||c||^2$ for any primitive idempotent c in $\mathcal{V}$. Now, let $x \in [\lambda^{-1}(Q)]^*$ and $x = \sum_i \lambda_i(x) e_i$ be its spectral decomposition. For any $q \in Q$, set $y = \sum_i q_i e_i$. As

$\lambda(y) = q^{\downarrow} \in Q$, we have $y \in \lambda^{-1}(Q)$ and so $\langle x,\, y\rangle \geq 0$. Thus, we have

$$
\begin{aligned}
0 &\leq \langle x,\, y\rangle \\
&= \left\langle \sum_{i=1}^{n} \lambda_i(x) e_i,\, \sum_{i=1}^{n} q_i e_i \right\rangle \\
&= \sum_{i=1}^{n} \lambda_i(x) q_i \left\| e_i \right\|^2 \\
&= \omega \, \langle \lambda(x),\, q\rangle ,
\end{aligned}
$$

where $\langle \lambda(x),\, q\rangle$ denotes the usual inner product between the vectors $\lambda(x)$ and q in $\mathcal{R}^n$. Since $\omega > 0$, $\langle \lambda(x),\, q\rangle \geq 0$. As q is arbitrary in Q, we have $\lambda(x) \in Q^*$ and so $x \in \lambda^{-1}(Q^*)$. This shows $[\lambda^{-1}(Q)]^* \subseteq \lambda^{-1}(Q^*)$.

To prove the reverse implication, let $x \in \lambda^{-1}(Q^*)$ so that $\lambda(x) \in Q^*$. Now, let $y \in \lambda^{-1}(Q)$ with its spectral decomposition $y = \sum_i \lambda_i(y) e_i$. We write the Peirce decomposition of x with respect to the Jordan frame $\{e_1,\, \ldots,\, e_n\}$ as

$$
x = \sum_{i=1}^{n} x_i e_i + \sum_{i<j} x_{ij},
$$

and define $\mathrm{diag}(x) := \sum_{i=1}^{n} x_i e_i$. Let $u := (x_1, x_2, \ldots, x_n)^{\mathsf{T}}$ in $\mathcal{R}^n$. Because of the orthogonality of the Peirce spaces, we have

$$
\begin{aligned}
\langle x,\, y\rangle &= \left\langle \sum_{i=1}^{n} x_i e_i,\, \sum_{i=1}^{n} \lambda_i(y) e_i \right\rangle \\
&= \sum_{i=1}^{n} x_i \lambda_i(y) \left\| e_i \right\|^2 \qquad (4.1) \\
&= \omega \, \langle u,\, \lambda(y)\rangle .
\end{aligned}
$$

Now, it is known that $\mathrm{diag}(x) \prec x$ (see [18], Example 7); hence, $u \prec \lambda(x)$. By

Proposition 2.3.2, $u = A\lambda(x)$ for some doubly stochastic matrix A; furthermore, by Proposition 2.3.3, this A is a convex combination of permutation matrices. Thus,

$$u = \left(\sum_{k=1}^{N} \alpha_k \sigma_k\right)\lambda(x), \quad \text{where } \alpha_k > 0 \text{ with } \sum_{k=1}^{N} \alpha_k = 1, \text{ and } \sigma_k \in \Sigma_n.$$

Since Q^* is permutation invariant and convex, $u \in Q^*$.

Since $y \in \lambda^{-1}(Q) \Rightarrow \lambda(y) \in Q$, we must have $\langle u, \lambda(y)\rangle \geq 0$. Hence, from (4.1), we get $\langle x,\, y\rangle \geq 0$. As y is arbitrary in $\lambda^{-1}(Q)$, this proves that $x \in [\lambda^{-1}(Q)]^*$. Thus, $\lambda^{-1}(Q^*) \subseteq [\lambda^{-1}(Q)]^*$, completing the proof. □

We now summarize the following:

Corollary 4.2.6 Let K_1, K_2, and K be spectral cones in $\mathcal{V}$ and $\alpha_1, \alpha_2 \in \mathcal{R}$. Then, the following hold:

(a) $\alpha_1 K_1 + \alpha_2 K_2$, $\overline{K}$, and K° are spectral cones.

(b) When $\mathcal{V}$ is simple or carries the canonical inner product, K^* and $K^\perp$ are spectral cones.

Proof.

(a) Let $K_i = \lambda^{-1}(Q_i)$, where Q_i is a permutation invariant convex cone in $\mathcal{R}^n$ for $i = 1, 2$. Then, $\alpha_1 Q_1 + \alpha_2 Q_2$ is a permutation invariant convex cone in $\mathcal{R}^n$ and from Corollary 3.2.10, $\alpha_1 K_1 + \alpha_2 K_2 = \lambda^{-1}(\alpha_1 Q_1 + \alpha_2 Q_2)$. Thus, $\alpha_1 K_1 + \alpha_2 K_2$ is a spectral cone. Let $K = \lambda^{-1}(Q)$, where Q is a permutation invariant convex cone in $\mathcal{R}^n$. It is easy to see that $\overline{Q}$, Q°, Q^* and $Q^\perp$ (defined with respect to the usual inner product in $\mathcal{R}^n$) are permutation invariant convex cones in $\mathcal{R}^n$.

(Note that $Q^\perp = Q^* \cap -Q^*$.) That $\overline{K}$ and K° are spectral cones follow by Theorem 3.3.1. Thus we have (a).

(b) Suppose that $\mathcal{V}$ is simple or carries the canonical inner product. Then, Theorem 4.2.5 shows that K^* is a spectral cone. Finally, the equality

$$K^\perp = K^* \cap -K^* = \lambda^{-1}(Q^*) \cap \lambda^{-1}(-Q^*) = \lambda^{-1}(Q^* \cap -Q^*) = \lambda^{-1}(Q^\perp) \quad (4.2)$$

shows that $K^\perp$ is a spectral cone. □

Example 4.2.7 Consider the algebra $\mathcal{V} = \mathcal{R}^2$, where for $x = (x_1, x_2)$ and $y = (y_1, y_2)$, the Jordan and inner products are defined, respectively, by

$$x * y := (x_1 y_1, x_2 y_2) \qquad \text{and} \qquad \langle x, y \rangle := 2x_1 y_1 + 3x_2 y_2.$$

Since the set (subspace) $Q := \{(x_1, x_2) \mid x_1 = x_2\}$ in $\mathcal{R}^2$ is permutation invariant, $K := \lambda^{-1}(Q) = \mathcal{R}\, e$ is a spectral set in our algebra $\mathcal{V}$. With respect to the above inner product,

$$K^* = K^\perp = \{(x_1, x_2) \mid 2x_1 + 3x_2 = 0\}.$$

Now, $(2, -3) \sim (-3, 2)$ with $(-3, 2) \in K^*$ and $(2, -3) \notin K^*$. Hence, by Proposition 3.2.6, K^* is not spectral.

Remark. In [22], Lemma 3, it is shown that the dual of a spectral cone in $\mathcal{S}^n$ is a spectral cone. Item (b) in the above theorem is a generalization. However, it does not hold in a general algebra: The set K in Example 4.2.7 is a spectral cone (actually, a subspace) while K^* (which is $K^\perp$) is not even a spectral set.

4.3 A dimensionality result

Inspired by a result of Lahtonen and its proof in [26], we present the following dimensionality result for spectral cones. Recall that the dimension of a convex cone K is the dimension of the subspace $K - K$. Throughout this section, for the Euclidean Jordan algebra $\mathcal{V}$, we let

$$d := \dim(\mathcal{V}).$$

Theorem 4.3.1 For any spectral cone K in $\mathcal{V}$,

$$\dim(K) \in \{0,\, 1,\, d-1,\, d\}.$$

Proof. We assume without loss of generality that K is nonempty and that $\mathcal{V}$ carries the canonical inner product so that the norm of any primitive idempotent is one. (See the paragraph right before Proposition 2.1.7). Now we show that either every element of K is a scalar multiple of e or every element of $K^\perp$ is a scalar multiple of e. Assuming the contrary, let $x \in K$ and $y \in K^\perp$ have at least two distinct eigenvalues. Writing their spectral decompositions,

$$x = x_1e_1 + x_2e_2 + \cdots + x_ne_n \quad \text{and} \quad y = y_1e_1' + y_2e_2' + \cdots + y_ne_n',$$

we assume, without loss of generality, $x_1 \neq x_2$ and $y_1 \neq y_2$. Now, define $\bar{x}$, $\bar{\bar{x}}$ by

$$\bar{x} := x_1e_1' + x_2e_2' + x_3e_3' + \cdots + x_ne_n',$$

$$\bar{\bar{x}} := x_2e_1' + x_1e_2' + x_3e_3' + \cdots + x_ne_n'.$$

We note that $\bar{x} \sim x$ and $\bar{\bar{x}} \sim x$. Since K is a spectral set and $x \in K$, by Theorem 4.2.2, $\bar{x}$, $\bar{\bar{x}} \in K$; hence, $\bar{x}$, $\bar{\bar{x}} \perp y$. Then,

$$\begin{aligned} 0 &= \langle \bar{x}, y \rangle - \left\langle \bar{\bar{x}}, y \right\rangle = \left\langle \bar{x} - \bar{\bar{x}}, y \right\rangle \\ &= \langle (x_1 - x_2)(e'_1 - e'_2), y_1 e'_1 + y_2 e'_2 + \cdots + y_n e'_n \rangle \\ &= (x_1 - x_2) y_1 - (x_1 - x_2) y_2 \\ &= (x_1 - x_2)(y_1 - y_2) \\ &\neq 0. \end{aligned}$$

We reach a contradiction. This shows that either K or $K^{\perp}$ must contain just multiples of e. Thus, the dimension of K or $K^{\perp}$ is at most 1. As $\dim(K) + \dim(K^{\perp}) = \dim(\mathcal{V}) = d$, we deduce that the possible values for $\dim(K)$ are $0,\ 1,\ d-1$, and d. $\square$

It is easy to see that a nonempty (spectral) cone has dimension zero if and only if it is $\{0\}$. Also, it is of dimension d if and only if it has nonempty interior. We now describe nonempty spectral cones in $\mathcal{V}$ with dimensions 1 and $d-1$.

Theorem 4.3.2 For a nonempty spectral cone K in $\mathcal{V}$, the following statements hold.

(a) $\dim(K) = 1$ if and only if K is a nonzero cone contained in $\mathcal{R}\, e$.

(b) $\dim(K) = d - 1$ if and only if $K = \{x \in \mathcal{V} \mid \operatorname{tr}(x) = 0\}$.

Proof.

(a) As the 'if' part is obvious, We prove the 'only if' part. Suppose $\dim(K) = 1$. Let $K = \lambda^{-1}(Q)$, where Q is a permutation invariant convex cone in $\mathcal{R}^n$. We claim that Q has dimension one. If, on the contrary, u, v are linearly independent vectors in Q, then for any Jordan frame $\{e_1, e_2, \ldots, e_n\}$, $x := \sum_1^n u_i e_i$ and $y := \sum_1^n v_i e_i$ are two linearly independent elements in K contradicting $\dim(K) = 1$. Thus, Q has dimension one; let Q be contained in the span of a nonzero vector w. As Q is permutation invariant, this w must be a multiple of $\mathbf{1}$. Then $K = \lambda^{-1}(Q)$ is contained in $\mathcal{R}\, e$, as $\{e\} = \lambda^{-1}(\{\mathbf{1}\})$.

(b) If $K = \{x \in \mathcal{V} : \operatorname{tr}(x) = 0\}$, then it has dimension $d - 1$ as $x \mapsto \operatorname{tr}(x)$ from $\mathcal{V}$ to $\mathcal{R}$ is linear with null space K.

Now suppose K has dimension $d - 1$. Letting $K = \lambda^{-1}(Q)$, we see, from (4.2), $K^\perp = \lambda^{-1}(Q^\perp)$ has dimension one. As in Item (a), we can show that $Q^\perp$ is spanned by the vector $\mathbf{1}$ in $\mathcal{R}^n$; hence, $\operatorname{tr}(v) = \langle v, \mathbf{1} \rangle = 0$ for all $v \in Q$. This means that $Q \subseteq M$, where $M = \{v \in \mathcal{R}^n \mid \operatorname{tr}(v) = 0\}$. Now, take any vector $v \in M$ and a nonzero $u \in Q$. As M and Q are permutation invariant, we may assume that the entries of u and v are decreasing. Note that $\sum_{i=1}^n v_i = \sum_{i=1}^n u_i = 0$. Now, from Proposition 2.3.5, $v \prec \alpha\, u$ for some $\alpha > 0$. Since Q is a permutation invariant convex cone, $\alpha u \in Q$ and by Theorem 4.2.2, $v \in Q$. This proves that $Q = M$. From this, we get $K = \lambda^{-1}(Q) = \{x \in \mathcal{V} \mid \operatorname{tr}(x) = 0\}$. This proves the result. $\square$

Remark. We note that in $\mathcal{V}$, there are only five nonzero cones contained in $\mathcal{R}\, e$, namely, $\mathcal{R}_+ e$, $\mathcal{R}_{++} e$, $-\mathcal{R}_+ e$, $-\mathcal{R}_{++} e$, and $\mathcal{R} e$, where $\mathcal{R}_{++}$ is the set of positive numbers in $\mathcal{R}$.

Corollary 4.3.3 Suppose K is a spectral cone in $\mathcal{V}$ such that

$$\{0\} \subseteq K \subseteq \{x \in \mathcal{V} \mid \operatorname{tr}(x) = 0\}.$$

Then, either $K = \{0\}$ or $K = \{x \in \mathcal{V} \mid \operatorname{tr}(x) = 0\}$.

Proof. For the specified K, let $K = \lambda^{-1}(Q)$, where Q is a permutation invariant convex cone in $\mathcal{R}^n$. Then,

$$\{0\} \subseteq \lambda^{-1}(Q) \subseteq \lambda^{-1}(M),$$

where $M = \{v \in \mathcal{R}^n : \operatorname{tr}(v) = 0\}$. Now, from Theorem 3.2.4, $P = \Sigma_n(\lambda(\lambda^{-1}(P)))$ for any permutation invariant set P in $\mathcal{R}^n$; thus,

$$\{0\} \subseteq Q \subseteq M.$$

The proof of Item (b) in the above theorem shows that either $Q = \{0\}$ or $Q = M$. Then, K, which is $\lambda^{-1}(Q)$, is either $\{0\}$ or $\{x \in \mathcal{V} : \operatorname{tr}(x) = 0\}$. $\square$

Remark. Suppose K is a nonempty spectral cone which is different from $\{0\}$ and $\{x \in \mathcal{V} \mid \operatorname{tr}(x) = 0\}$. Then, either $e \in K$ or $-e \in K$. This is seen as follows. Let $K = \lambda^{-1}(Q)$, where Q is a permutation invariant convex cone in $\mathcal{R}^n$. Then, Q is nonempty and different from $\{0\}$ and $\{u \in \mathcal{R}^n : \operatorname{tr}(u) = 0\}$. Let u be a nonzero element of Q with $\mathrm{tr}(u) \neq 0$. As Q is permutation invariant, from (2.1),

$$(n-1)!\ \mathrm{tr}(u)\,\mathbf{1} = \sum_{\sigma \in \Sigma_n} \sigma(u) \in Q.$$

Since Q is a cone, we see that either $\mathbf{1} \in Q$ or $-\mathbf{1} \in Q$. From this, we see that either $e \in K$ or $-e \in K$.

4.4 Pointed/Solid spectral cones

The following result is a generalization of a similar result stated in the setting of $\mathcal{V} = \mathcal{S}^n$ (see [20], Theorem 3.3).

Theorem 4.4.1 Let $K = \lambda^{-1}(Q)$, where Q is a permutation invariant convex cone in $\mathcal{R}^n$. Then

(i) K is pointed if and only if Q is pointed,

(ii) K is solid if and only if Q is solid.

Proof. (i) From Theorem 3.2.9, $K \cap -K = \lambda^{-1}(Q \cap -Q)$. Thus, $K \cap -K \subseteq \{0\}$ if and only if $Q \cap -Q \subseteq \{0\}$. This proves (i).

(ii) This follows from the fact that $K^\circ = \lambda^{-1}(Q^\circ)$, see Proposition 3.3.1. $\square$

Lemma 4.4.2 Let Q be a nonempty nonzero permutation invariant convex cone in $\mathcal{R}^n$. Then Q is pointed if and only if exactly one of the following conditions holds:

(i) $\mathrm{tr}(u) > 0$ for all nonzero $u \in Q$ (and $\mathbf{1} \in Q$).

(ii) $\mathrm{tr}(u) < 0$ for all nonzero $u \in Q$ (and $-\mathbf{1} \in Q$).

Proof. Suppose Q is pointed. We first show that $\mathrm{tr}(u) \neq 0$ for every nonzero $u \in Q$. Suppose, on the contrary, there is a nonzero vector $u \in Q$ such that $\mathrm{tr}(u) = 0$. Then by (2.1), $0 = (n-1)!\,\mathrm{tr}(u)\,\mathbf{1} = \sum_{\sigma \in \Sigma_n} \sigma(u)$. This implies

$$v := \sum_{\sigma \in \Sigma_n,\, \sigma \neq I} \sigma(u) = \left[\sum_{\sigma \in \Sigma_n} \sigma(u)\right] - I(u) = -I(u) = -u,$$

where I is the identity matrix. However, as both u and v are in Q, we reach a contradiction to the pointedness of Q. Now, if there exist nonzero $u, v \in Q$ such that $\mathrm{tr}(u) > 0$ and $\mathrm{tr}(v) < 0$, then for a suitable convex combination w of u and v, we have $\mathrm{tr}(w) = 0$. As Q is convex, $w \in Q$. From what has been proved earlier, $w = 0$. This contradicts the pointedness of the convex cone Q. Hence, either $\mathrm{tr}(u) > 0$ for all nonzero $u \in Q$ or $\mathrm{tr}(u) < 0$ for all nonzero $u \in Q$.

Now, in the first case, by (2.1),

$$(n-1)!\,\mathrm{tr}(u)\,\mathbf{1} = \sum_{\sigma \in \Sigma_n} \sigma(u) \in Q.$$

It follows (by scaling) that $\mathbf{1} \in Q$. This is Item (i). Similarly, when $\mathrm{tr}(u) < 0$ for all nonzero $u \in Q$, we get Item (ii). By the pointedness of Q, both (i) and (ii) cannot hold simultaneously.

To see the reverse implication, suppose without loss of generality (i) holds so that $\mathrm{tr}(u) > 0$ for every nonzero $u \in Q$. By the linearity of the trace function, $-u$ cannot be in Q for any nonzero $u \in Q$. Thus, Q is pointed. $\square$

Theorem 4.4.3 Let K be a nonempty, nonzero spectral cone in $\mathcal{V}$. Then K is pointed if and only if exactly one of the following conditions holds:

(i) $\mathrm{tr}(x) > 0$ for all nonzero $x \in K$ (and $e \in K$).

(ii) $\mathrm{tr}(x) < 0$ for all nonzero $x \in K$ (and $-e \in K$).

Proof. Let $K = \lambda^{-1}(Q)$ where Q is a permutation invariant convex cone. Then, by an earlier result, K is pointed if and only if Q is pointed. Thus, by the above lemma, K is pointed if and only if either $\mathrm{tr}(u) > 0$ for all $u \in Q$ (and $\mathbf{1} \in Q$) or $\mathrm{tr}(u) < 0$ for all $u \in Q$ (and $-\mathbf{1} \in Q$). As $\mathrm{tr}(\lambda(x)) = \mathrm{tr}(x)$, and $\pm\mathbf{1} \in Q$ if and only if $\pm e \in K$, we get the stated results from $K = \lambda^{-1}(Q)$. $\square$

We say that a vector in $\mathcal{R}^n$ is a **nonconstant vector** if it is not a multiple of $\mathbf{1}$.

Lemma 4.4.4 Let Q be a permutation invariant convex cone in $\mathcal{R}^n$. Then the following are equivalent:

(a) Q is solid.

(b) $\mathbf{1} \in Q^\circ$ or $-\mathbf{1} \in Q^\circ$.

(c) When $n \geq 2$, Q has a nonconstant vector, and either $\mathbf{1} \in Q$ or $-\mathbf{1} \in Q$.

Proof. $(a) \Rightarrow (b)$: Suppose Q is solid, that is, $Q^\circ \neq \emptyset$. As Q° cannot be contained in the subspace $\{u \in \mathcal{R}^n : \mathrm{tr}(u) = 0\}$, there exists a nonzero vector $u \in Q^\circ$ with $\mathrm{tr}(u) \neq 0$. As Q° is permutation invariant, $\sigma(u) \in Q^\circ$ for every $\sigma \in \Sigma_n$. Now, as Q° is a convex cone, by (2.1),

$$(n-1)!\ \mathrm{tr}(u)\,\mathbf{1} = \sum_{\sigma \in \Sigma_n} \sigma(u) \in Q^\circ.$$

Since $\mathrm{tr}(u) \neq 0$, we must have, $\pm\mathbf{1} \in Q^\circ$.

$(b) \Rightarrow (c)$: This is obvious.

$(c) \Rightarrow (a)$: Let $n \geq 2$. Assume that there is a nonconstant vector $u = (u_1, u_2, \ldots, u_n)^{\mathsf{T}}$ in Q and (without loss of generality) $\mathbf{1} \in Q$. Suppose, if possible, Q is not solid. Then $Q - Q \neq \mathcal{R}^n$ and so there exists a nonzero $v \in (Q-Q)^{\perp} = Q^{\perp}$. As $\mathbf{1} \in Q$, we must have $\sum_i v_i = \langle \mathbf{1}, v \rangle = 0$, where $v = (v_1, v_2, \ldots, v_n)^{\mathsf{T}}$. Now, as u has at least two distinct components, we may assume (by permuting the coordinates) that $u_1 \neq u_2$. Let $\bar{u}$ be the vector in Q obtained from u by interchanging u_1 and u_2. Since $u, \bar{u} \perp v$, we get

$$\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \sum_{i=3}^{n} u_i v_i = 0, \text{ and}$$
$$\langle \bar{u}, v \rangle = u_2 v_1 + u_1 v_2 + \sum_{i=3}^{n} u_i v_i = 0.$$

From these, we get $0 = \langle u - \bar{u}, v \rangle = (u_1 - u_2)(v_1 - v_2)$. As $u_1 - u_2 \neq 0$, we must have $v_1 = v_2$. Now, we can permute u so that the resulting vector has different entries in i and j slots, $i \neq j$. Then, the above argument can be repeated to get $v_i = v_j$. Hence, v is a multiple of $\mathbf{1}$. However, as $\sum_i v_i = 0$, we must have $v = 0$, a contradiction. Thus (a) holds. $\square$

Theorem 4.4.5 Let K be a spectral cone in $\mathcal{V}$. Then the following are equivalent:

(a) K is solid.

(b) $e \in K^{\circ}$ or $-e \in K^{\circ}$.

(c) When $n \geq 2$, K has a vector which is not a multiple of e, and either $e \in K$ or $-e \in K$.

Proof. Let $K = \lambda^{-1}(Q)$ where Q is a permutation invariant convex cone. As $K^{\circ} = \lambda^{-1}(Q^{\circ})$ and $\lambda(e) = \mathbf{1}$, we see that the items listed above are equivalent to the similar ones in the previous lemma. This completes the proof. $\square$

Chapter 5

The Lyapunov Rank of Permutation Invariant Proper Polyhedral Cones

5.1 Introduction

In optimization theory, a complementarity problem corresponding to a mapping $f : \mathcal{V} \to \mathcal{V}$ and a closed convex cone K in $\mathcal{V}$ is to find $x \in \mathcal{V}$ such that

$$x \in K,\ s = f(x) \in K^*,\ \langle x,\, s\rangle = 0, \tag{5.1}$$

where K^* denotes the dual of K.. There are various strategies for solving complementarity problems, see [4]. Notice that, when $\dim(\mathcal{V}) = n$, there are $2n$ variables x_i, s_i for $i = 1, 2, \ldots, n$, while there are $n+1$ equations, namely, $s_i = f_i(x)$ for $i = 1, 2, \ldots, n$ and $\langle x,\, s\rangle = 0$. Thus, we may want to rewrite the last bilinear relation $\langle x,\, s\rangle = 0$ by an equivalent system of n linearly independent bilinear relations in order to get a square system. If $\mathcal{V} = \mathcal{R}^n$ and $K = \mathcal{R}^n_+$, then $\langle x,\, s\rangle = 0$ can be replaced by $x_i s_i = 0$ for $i = 1, 2, \ldots, n$. Although this situation does not always happen, it will be useful to identify cones where this is possible.

Now, for a closed convex cone K in $\mathcal{V}$, a linear transformation $L : \mathcal{V} \to \mathcal{V}$ is said to be a *Lyapunov-like transformation* on K provided

$$x \in K,\ s \in K^*,\ \langle x,\, s\rangle = 0 \quad \Longrightarrow \quad \langle L(x),\, s\rangle = 0. \tag{5.2}$$

This concept was first introduced in [14] as a generalization of the Lyapunov transformation $X \mapsto AX + XA^{\mathsf{T}}$ that appears in the linear dynamical system theory, and has been a subject of several recent works. Some examples and properties related to the Lyapunov-like transformation can be found in [14], [16], and [17].

Recently, G. Rudolf et al [42] and Gowda and Tao [16] introduced the following: For a proper cone $K \in \mathcal{V}$, the *Lyapunov rank* of K is defined by

$$\beta(K) = \dim \mathrm{LL}(K) = \dim \mathrm{Lie}(\mathrm{Aut}(K)),$$

where $\mathrm{LL}(K)$ represents the set of all Lyapunov-like transformations on K. Thus, $\beta(K)$ measures the number of independent Lyapunov-like transformations on K. In the case of $\beta(K) < n$, the complementarity problem can never be written as a square system with Lyapunov-like transformations alone. When $\beta(K) > n$, one needs to carefully choose n linearly independent Lyapunov-like transformations to get a square system. Hence, the problem is desirable when $\beta(K) = n$.

The below is a summary of results on the Lyapunov rank: Gowda et al ([16], [17], and [38]) have shown that

(1) For any proper cone $K \subseteq \mathcal{R}^n$, $\beta(K) \leq (n-1)^2$.

(2) For any proper polyhedral cone in $K \subseteq \mathcal{R}^n$, $1 \leq \beta(K) \leq n$, $\beta(K) \neq n-1$.

(3) $\beta(\mathcal{S}^n_+) = n^2$ and $\beta(\mathcal{CP}_n) = n$ where $\mathcal{CP}_n$ denotes the set of all $n \times n$ completely positive matrices.

(4) In $\mathcal{R}^n$, for $n \geq 3$, the l_p-cone, defined by $l_{p,+} := \{u = (u_0, \bar{u}) \in \mathcal{R} \times \mathcal{R}^{n-1} : u_0 \geq ||u||_p\}$, $\beta(l_{p,+}) = 1$ where $1 \leq p \leq \infty$, $p \neq 2$. When $p = 2$, $\beta(l_{2,+}) =$

$(n^2 - n + 2)/2$.

Inspired by Item 2, we prove that the Lyapunov rank of a permutation invariant proper polyhedral cone divides n. The organization of this chapter is as follows:

- In Section 2, a special type of spectral cone in $\mathcal{R}^n$, namely a rearrangement cone is studied. The Lyapunov rank this cone is provided.
- In Section 3, we compute the Lyapunov rank of a permutation invariant proper polyhedral cone.

5.2 Rearrangement cones

In this section, we describe some permutation invariant convex cones in $\mathcal{R}^n$. Given any nonempty set S in $\mathcal{R}^n$,

$$Q := \text{cone}\,(\Sigma_n(S)),$$

the convex cone generated by $\Sigma_n(S) := \{\sigma(u) \mid u \in S, \sigma \in \Sigma_n\}$, is a permutation invariant convex cone. A rearrangement cone Q_p^n is a particular example of a permutation invariant cone which is proper and polyhedral. Recall that, for any $u \in \mathcal{R}^n$, $u^\uparrow$ denotes the vector obtained by rearranging the components of u in increasing order. In other words, $u^\uparrow$ is the rearrangement of u satisfying the following inequalities

$$u_1^\uparrow \leq u_2^\uparrow \leq u_3^\uparrow \leq \cdots \leq u_n^\uparrow.$$

Given $1 \leq p \leq n$, define the **rearrangement cone** [20]

$$Q_p^n = \left\{ u \in \mathcal{R}^n \;\middle|\; u_1^\uparrow + u_2^\uparrow + \cdots + u_p^\uparrow \geq 0 \right\}.$$

Note that

$$\begin{aligned} Q_1^n &= \left\{ u \in \mathcal{R}^n \;\middle|\; u_1^\uparrow \geq 0 \right\} \\ &= \{ u \in \mathcal{R}^n \mid u_i \geq 0 \text{ for all } i = 1,\, 2,\, \ldots,\, n \}\,, \\ Q_n^n &= \left\{ u \in \mathcal{R}^n \;\middle|\; u_1^\uparrow + u_2^\uparrow + \cdots + u_n^\uparrow \geq 0 \right\} \\ &= \{ u \in \mathcal{R}^n \mid u_1 + u_2 + \cdots + u_n \geq 0 \}\,. \end{aligned}$$

Hence, $Q_1^n = \mathcal{R}_+^n$ and Q_n^n is a half-space of $\mathcal{R}^n$. Thus, we are interested in Q_p^n when $2 \leq p \leq n-1$. We start with some basic properties of Q_p^n.

Proposition 5.2.1 The following statements hold:

(a) Q_p^n is a permutation invariant polyhedral (closed convex) solid cone; it is pointed when $1 \leq p \leq n-1$.

(b) $Q_1^n \subseteq Q_2^n \subseteq \cdots \subseteq Q_n^n$.

(c) Q_p^n is isomorphic to Q_{n-p}^n for $1 \leq p \leq n-1$.

Proof.

(a) For any nonempty index set $I \subseteq \{1,\, 2,\, \ldots,\, n\}$, let $|I|$ denote the cardinality of I and e_I be the vector with $(e_I)_i = 1$ for $i \in I$ and $(e_I)_i = 0$ otherwise. Then,

$$u_1^\uparrow + u_2^\uparrow + \cdots + u_p^\uparrow = \min_{|I|=p} \langle u,\, e_I \rangle\,,$$

hence, we get

$$\begin{aligned} Q_p^n &= \{ u \in \mathcal{R}^n :\; u_1^\uparrow + u_2^\uparrow + \cdots + u_p^\uparrow \geq 0 \} \\ &= \left\{ u \in \mathcal{R}^n \;\middle|\; \min_{|I|=p} \langle u,\, e_i \rangle \geq 0 \right\} \end{aligned}$$

$$= \{u \in \mathcal{R}^n \mid \langle u,\, e_I\rangle \geq 0 \text{ for every } e_I \text{ with } |I| = p\}\,.$$

Since Q_p^n is now the intersection of a finite number of closed half-spaces, it is a polyhedral closed convex cone. Since $\sigma(u)^\uparrow = u^\uparrow$ for any permutation matrix $\sigma \in \Sigma_n$, we see that Q_p^n is permutation invariant. That Q_p^n has nonempty interior follows from the inclusion $\mathcal{R}_+^n \subseteq Q_p^n$. Now let $1 \leq p \leq n-1$. To see the pointedness, suppose both u and $-u$ are in Q_p^n. Then, the inequalities $\min_{|I|=p} \langle u,\, e_I\rangle \geq 0$ and $\min_{|I|=p} \langle -u,\, e_I\rangle \geq 0$ imply that $\langle u,\, e_I\rangle = 0$ whenever $|I| = p$. This means that sum of any p entries of u is zero. Hence, as $p \leq n-1$, all entries of u are equal, and so, $u = 0$. Thus, Q_p^n is pointed when $1 \leq p \leq n-1$.

(b) Let $1 \leq p \leq n-1$. If $u \in Q_p^n$, then $u_1^\uparrow + u_2^\uparrow + \cdots + u_p^\uparrow \geq 0$. Since $0 \leq u_p^\uparrow \leq u_{p+1}^\uparrow$, we get $u_1^\uparrow + u_2^\uparrow + \cdots + u_p^\uparrow + u_{p+1}^\uparrow \geq 0$. Thus, $u \in Q_{p+1}^n$.

(c) We let $1 \leq p \leq n-1$. Let $\mathbf{1}$ denote the vector in $\mathcal{R}^n$ with all entries one, $E := \mathbf{1}\,\mathbf{1}^\mathsf{T}$ (the $n \times n$ matrix with all entries one), and I denote the identity matrix. Defining

$$M_p := \frac{1}{p}E - I, \tag{5.3}$$

we easily verify that $M_p\,M_{n-p} = I$; hence M_p is invertible. Now, let $u \in Q_{n-p}^n$; this means that the sum of $n-p$ smallest entries of u is nonnegative, that is, $\sum_{p+1}^n u_i^\downarrow \geq 0$. Then, from $M_p u = \dfrac{\mathrm{tr}(u)}{p}\,\mathbf{1} - u$ and $(-u)^\uparrow = -u^\downarrow$ we have $(M_p u)^\uparrow = \dfrac{\mathrm{tr}(u)}{p}\,\mathbf{1} - u^\downarrow$. Hence,

$$\sum_{i=1}^p (M_p u)_i^\uparrow = \mathrm{tr}(u) - \sum_1^p u_i^\downarrow = \sum_{p+1}^n u_i^\downarrow \geq 0.$$

Thus $M_p u \in Q_p^n$, and so $M_p(Q_{n-p}^n) \subseteq Q_p^n$. Similarly, $M_{n-p}(Q_p^n) \subseteq Q_{n-p}^n$. Hence, $M_p(Q_{n-p}^n) = Q_p^n$. □

5.2.1 The Lyapunov rank of a rearrangement cone

Now, fix $p \in \{2, 3, \ldots, n-1\}$ and consider the rearrangement cone Q_p^n. The main part of this section is to find the set of extreme vectors and compute the Lyapunov rank of Q_p^n. Since Q_p^n is a proper polyhedral cone, it contains finitely many extreme vectors and by using these vectors, we may compute the Lyapunov rank of Q_p^n. In order to find the set of all extreme vectors of Q_p^n, we start with a lemma.

Lemma 5.2.2 Let $u = (u_1, u_2, \ldots, u_n)^\mathsf{T}$ be a vector in $\mathcal{R}^n$. If there exist $p \in \{1, 2, \ldots, n-1\}$ and $\alpha \in \mathcal{R}$ such that $\langle u, e_I \rangle = \alpha$ for all I satisfying $|I| = p$, then $u_1 = u_2 = \cdots = u_n = \alpha/p$.

Proof. In the case of $|I| = 1$, we clearly have $u_1 = \cdots = u_n = \alpha$. Thus, suppose $p \in \{2, \ldots, n-1\}$ and consider the sums

$$\begin{aligned} u_1 + u_2 + \cdots + u_p \qquad &= \alpha \\ u_2 + \cdots + u_p + u_k &= \alpha, \quad \text{for } k = p+1, \ldots, n. \end{aligned}$$

Thus, we get $u_1 = u_k$ for all $k = p+1, \ldots, n$. Similarly, we can show that $u_l = u_k$ for all $k = 1, \ldots, p$. This implies that $u_1 = u_2 = \cdots = u_n = \alpha/p$. □

Theorem 5.2.3 For a proper cone Q_p^n, $p = 2, 3, \ldots, n-1$, let $d_{p_j} \in \mathcal{R}^n$, for $j = 1, \ldots, n$, denote a vector in $\mathcal{R}^n$ such that its j^{th} entry is $1 - p$ and other entries are 1's. Then, for $2 \leq p \leq n-2$,

$$\operatorname{ext}(Q_p^n) = \{e_1,\, e_2,\, \ldots,\, e_n,\, d_{p_1},\, d_{p_2},\, \ldots,\, d_{p_n}\}.$$

When $p = 1$ or $p = n-1$, we respectively have

$$\operatorname{ext}(Q_1^n) = \{e_1,\, e_2,\, \ldots,\, e_n\},$$

$$\operatorname{ext}(Q_{n-1}^n) = \{d_{(n-1)_1},\, d_{(n-1)_2},\, \ldots,\, d_{(n-1)_n}\}.$$

Proof. Since $Q_1^n = \mathcal{R}_+^n$, it is clear that $\operatorname{ext}(Q_1^n) = \{e_1,\, e_2,\, \ldots,\, e_n\}$. Note that Q_1^n is isomorphic to Q_{n-1}^n with an isomorphism M_{n-1} given in (5.3). Since

$$(n-1)M_{n-1}e_j = [E - (n-1)I]e_j = \mathbf{1} - (n-1)e_j = d_{(n-1)_j},$$

for all j, it follows that $\operatorname{ext}(Q_{n-1}^n) = \{d_{(n-1)_1},\, d_{(n-1)_2},\, \ldots,\, d_{(n-1)_n}\}$.

Now, fix $2 \leq p \leq n-2$. First, we show $\{d_{p_1}, d_{p_2}, \ldots, d_{p_n}\} \subseteq \operatorname{ext}(Q_p^n)$. Consider the case of d_{p_1} only as proofs of the other cases will be similar. Suppose we have vectors $u = (u_1,\, u_2,\, \ldots,\, u_n)^\mathsf{T}$ and $v = (v_1,\, v_2,\, \ldots,\, v_n)^\mathsf{T}$ in Q_p^n such that $d_{p_1} = u + v$. We show that u and v are nonnegative scalar multiples of d_{p-1}. Write $u = (u_1,\, \bar{u})^\mathsf{T}$, $v = (v_1,\, \bar{v})^\mathsf{T}$, where $\bar{u},\, \bar{v} \in \mathcal{R}^{n-1}$. Note that $u_1 + v_1 = 1 - p$ and $\bar{u} + \bar{v} = \mathbf{1}$, where $\mathbf{1} \in \mathcal{R}^{n-1}$. Since $u,\, v \in Q_p^n$, we have $u_1 + \langle e_I,\, \bar{u}\rangle \geq 0$ and $v_1 + \langle e_I,\, \bar{v}\rangle \geq 0$ for all $e_I \in \mathcal{R}^{n-1}$ satisfying $|I| = p-1$. Hence we get,

$$0 \leq v_1 + \langle e_I,\, \bar{v}\rangle = (1 - p - u_1) + \langle e_I,\, \mathbf{1} - \bar{u}\rangle = -u_1 - \langle e_I,\, \bar{u}\rangle\,.$$

This implies $u_1 + \langle e_I,\, \bar{u}\rangle \leq 0$, and hence $u_1 + \langle e_I,\, \bar{u}\rangle = 0$ or $\langle e_I,\, \bar{u}\rangle = -u_1$. Applying Lemma 5.2.2, we get $u_2 = \cdots = u_n = u_1/(1-p)$. Hence, $u = \alpha d_{p_1}$ for some α and

$d_{p_1} \in \mathrm{ext}(Q_p^n)$.

Now, we show $\{e_1, e_2, \ldots, e_n\} \subseteq \mathrm{ext}(Q_p^n)$. Suppose that vectors $u = (u_1, u_2, \ldots, u_n)^\mathsf{T}$ and $v = (v_1, v_2, \ldots, v_n)^\mathsf{T}$ in Q_p^n with $e_1 = u + v$. We show that $u = \alpha e_1$ for some $\alpha \geq 0$ or, equivalently, $\bar{u} = 0$ where $u = (u_1, \bar{u})^\mathsf{T}$. Since $\bar{u} + \bar{v} = 0$ and $u, v \in Q_p^n$, we have $\langle e_I, \bar{u}\rangle \geq 0$ and $\langle e_I, \bar{v}\rangle \geq 0$, where $e_I \in \mathbb{R}^{n-1}$ with $|I| = p - 1$. Hence,

$$0 \leq \langle e_I, \bar{v}\rangle = \langle e_I, -\bar{u}\rangle = -\langle e_I, \bar{u}\rangle .$$

Thus, we get $\langle e_I, \bar{u}\rangle \leq 0$, and hence $\langle e_I, \bar{u}\rangle = 0$. This implies $\bar{u} = 0$ by Lemma 5.2.2. We can similarly prove that $e_i \in \mathrm{ext}(Q_p^n)$ for all i.

We now show $\mathrm{cone}\{e_1, e_2, \ldots, e_n, d_{p_1}, d_{p_2}, \ldots, d_{p_n}\} = Q_p^n$. Pick a vector $u = (u_1, u_2, \ldots, u_n)^\mathsf{T} \in Q_p^n$. Without loss of generality, assume $u = u^\uparrow$. If u has non-negative entries only, then u can be represented as a nonnegative linear combination of $\{e_1, e_2, \ldots, e_n\}$. Otherwise, if $u_i < 0$ for $1 \leq i \leq p$ and $u_i \geq 0$ for $p+1 \leq i \leq n$, define

$$u^d = (u_1, u_2, \ldots, u_p, u_p, \ldots, u_p)^\mathsf{T}, \quad u^e = u - u^d.$$

Note that $u = u^d + u^e$ and u^e has nonnegative entries only. Then u^e can be represented as a nonnegative linear combination of $\{e_1, e_2, \ldots, e_n\}$.

We now show u^d can be represented as a $\{d_{p_1}, d_{p_2}, \ldots, d_{p_n}\}$. Consider an $n \times n$ matrix M whose i^{th} column is d_{p_i}. Note that $M = pM_p$, so the inverse of M exists and $M^{-1} = \frac{1}{p}M_{n-p}$. Then the problem is converted to showing $u := M^{-1}u^d = \frac{1}{p}M_{n-p}u^d \geq 0$. Indeed, we can show $u_j \geq 0$ for all $j = 1, 2, \cdots, n$ by direct calculation. Finally, the given vector $u \in Q_p^n$ can be represented a nonnegative linear combination,

$$u = u^d + u^e = \alpha_1 d_{p_1} + \cdots + \alpha_n d_{p_n} + \beta_1 e_1 + \cdots + \beta_n e_n.$$

Hence, $\text{cone}\{e_1,\, e_2,\, \ldots,\, e_n,\, d_{p_1},\, d_{p_2},\, \ldots,\, d_{p_n}\} = Q_p^n$. $\square$

Finally, we now compute the Lyapunov rank of cone Q_p^n as a corollary.

Corollary 5.2.4 $\beta(Q_p^n) = 1$ when $p = 2, 3, \ldots, n-2$ and $\beta(Q_p^n) = n$ when $p = 1, n-1$.

Proof. When $p = 1$, we have $Q_1^n = \mathcal{R}_+^n$, in which case, $\beta(Q_1^n) = 1$. When $p = n-1$, Q_{n-1}^n is isomorphic to Q_1^n by Proposition 5.2.1; hence $\beta(Q_{n-1}^n) = n$.

We now suppose $p \in \{2,\, 3,\, \ldots,\, n-2\}$ and let L be a Lyapunov-like transformation on Q_p^n. Since Q_p^n is a proper polyhedral cone, every extreme vector of Q_p^n is an eigenvector of a Lyapunov-like transformation L, see [16]. As Q_p^n has extreme vectors $\{e_1,\, e_2,\, \ldots,\, e_n\}$, L is a diagonal matrix, say $L = \text{diag}(a_{11},\, a_{22},\, \ldots,\, a_{nn})$. Now, assume that λ is an eigenvalue of L corresponding to d_{p_1}. Then we get,

$$(1-p)a_{11} = \lambda(1-p),\ a_{22} = \lambda,\ a_{33} = \lambda,\ \ldots,\ a_{nn} = \lambda.$$

Hence, $L = \text{diag}(a_{11},\, a_{22},\, \ldots,\, a_{nn}) = \lambda I$. Thus every Lyapunov-like transformation is a multiple of I. Hence, $\beta(Q_p^n) = 1$ completing the proof. $\square$

5.3 The Lyapunov rank of PIPP cone

It has been shown that the Lyapunov rank of a proper polyhedral cone in $\mathcal{R}^n$ can only take a value between 1 to n except $n-1$, see [16]. Motivated by this, we deal with the Lyapunov rank of permutation invariant proper polyhedral cone. (In short, a PIPP cone)

Lemma 5.3.1 Suppose a proper cone Q is reducible, so $Q = Q_1 \oplus Q_2$ where Q_1, Q_2 are non-empty and non-zero cones and $\operatorname{span}(Q_1) \cap \operatorname{span}(Q_2) = \{0\}$. Then, $\operatorname{ext}(Q)$ is the union of disjoint set $\operatorname{ext}(Q_1)$ and $\operatorname{ext}(Q_2)$. In other words,

$$\operatorname{ext}(Q_1) \cup \operatorname{ext}(Q_2) = \operatorname{ext}(Q) \quad \text{and} \quad \operatorname{ext}(Q_1) \cap \operatorname{ext}(Q_2) = \emptyset.$$

Proof. First, we show $\operatorname{ext}(Q) \subseteq \operatorname{ext}(Q_1) \cup \operatorname{ext}(Q_2)$. Suppose $d \in \operatorname{ext}(Q)$ and $d = u + v$ where $u \in Q_1$ and $v \in Q_2$. Since $\operatorname{span}(Q_1) \cap \operatorname{span}(Q_2) = \{0\}$, either u or v is 0 and so either $d \in Q_1$ or $d \in Q_2$. Assuming $d \in Q_1$, we easily verify that $d \in \operatorname{ext}(Q_1)$.

Now, we show $\operatorname{ext}(Q_1) \cup \operatorname{ext}(Q_2) \subseteq \operatorname{ext}(Q)$. Without loss of generality, let d be nonzero in $\operatorname{ext}(Q_1)$. If there exist u, $v \in Q$ such that $d = u + v$, then we have three possible cases:

Case 1. Suppose u and v are both in Q_1. Since d is an extreme vector of Q_1, u and v are multiple of d. Thus, $d \in \operatorname{ext}(Q)$.

Case 2. Suppose u and v are both in Q_2. As $d = u + v$ are both in Q_1 and Q_2, $\operatorname{span}(Q_1) \cup \operatorname{span}(Q_2) \neq \{0\}$ leading a contradiction.

Case 3. Suppose, without loss of generality, that $u \in Q_1$ and $v \in Q_2$. From $d = u + v$, we get $d - u = v$ are both in Q_1 and Q_2. This contradicts the fact that $\text{span}(Q_1) \cap \text{span}(Q_2) = \{0\}$.

Considering all possible cases, we conclude that $d \in \text{ext}(Q)$.

Finally, it is clear that $\text{ext}(Q_1) \cap \text{ext}(Q_2) = \emptyset$ as $\text{span}(Q_1) \cap \text{span}(Q_2) = \{0\}$. Hence, $\text{ext}(Q_1)$ and $\text{ext}(Q_2)$ must be disjoint. □

Remark. Note that this lemma can be generalized to the sum of finite number of subcones, that is, if $Q = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_r$ for some r, then

$$\text{ext}(Q) = \bigcup_{i=1}^{r} \text{ext}(Q_i),$$

where all sets $\text{ext}(Q_i)$ are mutually disjoint.

Theorem 5.3.2 Let Q be a permutation invariant proper polyhedral cone in $\mathcal{R}^n$. Then $\beta(Q)$ divides n.

Proof. In the case that Q is irreducible, $\beta(Q) = 1$ because Q is a polyhedral cone, see [16]. Suppose Q is reducible so that

$$Q = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_r.$$

Here, each Q_i is irreducible, $Q_i \neq \{0\}$, and $\text{span}(M) \cap \text{span}(N) = \{0\}$, where M is the direct sum of some Q_i's and N is the direct sum of the rest. Since Q is polyhedral, from the above remark, each Q_i has a finite number of extreme vectors. So, each Q_i is polyhedral as well as irreducible implying that $\beta(Q_i) = 1$ for all $i = 1, 2, \ldots, r$. Since $\beta(Q)$ is the sum of the Lyapunov ranks of its subcones and each subcone has Lyapunov rank 1, we get $\beta(Q) = r$. Note that $1 < r \leq n$.

Now, let $\text{ext}(Q_1) = \{v_1, v_2, \ldots, v_m\}$ and take $\sigma \in \Sigma_n$. As σ is an automorphism of $\mathcal{R}^n$, $\sigma(Q_1)$ is irreducible. Assume that $\sigma(v_1) \in Q_i$ for some i. Then we claim that $\sigma(\text{ext}(Q_1)) \subseteq Q_i$ and hence $\sigma(Q_1) \subseteq Q_i$. To see this, define

$$C_1 = \text{cone}\Big(\sigma\big(\,\text{ext}(Q_1)\big) \cap Q_i\Big) \quad \text{and} \quad C_2 = \text{cone}\left(\sigma\big(\,\text{ext}(Q_1)\big) \cap \Big(\bigcup_{j \neq i} Q_j\Big)\right),$$

where $C_2 \neq \{0\}$ if possible. Note that $C_1 \cap C_2 = \{0\}$ as C_1 is in Q_i while C_2 is in the span of other subcones, and $C_1 \oplus C_2 \subseteq \sigma(Q_1)$. However, any $v \in Q_1$ can be expressed as

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m.$$

Multiplying both sides by σ, we get

$$\sigma(v) = \lambda_1 \sigma(v_1) + \lambda_2 \sigma(v_2) + \cdots + \lambda_m \sigma(v_m) \subseteq C_1 \oplus C_2.$$

This implies $\sigma(Q_1) \subseteq C_1 \oplus C_2$. Thus, we have $\sigma(Q_1) = C_1 \oplus C_2$ implying that $\sigma(Q_1)$ is reducible and so is Q_1. As this is a contradiction, we must have $C_2 = \{0\}$ and hence $\sigma(\text{ext}(Q_1)) \subseteq Q_i$ for some i.

In a similar way, we may assume that $\sigma^{-1}(Q_i) \subseteq Q_j$ for some j. Then we get $Q_1 \subseteq \sigma^{-1}(Q_i) \subseteq Q_j$. As Q_1 and Q_j intersect only at $\{0\}$, we have either $Q_1 = \{0\}$ (which is not the case) or $Q_1 = Q_j$. Thus, we get $\sigma(Q_1) = Q_i$ so Q_1 is isomorphic to Q_i with the isomorphism σ.

Now, a similar statement can be repeated for $Q_2, \ldots, Q_r$. This argument shows that each $\sigma \in \Sigma_n$ maps every Q_m to some Q_l. Grouping (permutation) isomorphic cones together, we may write

$$Q = E_1 \oplus E_2 \oplus \cdots \oplus E_s,$$

where each E_i is a direct sum of such (permutation) isomorphic cones. We now claim that each E_i is permutation invariant. Without loss of generality, consider

$$E_1 = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_{r_1}. \tag{5.4}$$

As $\{\sigma(Q_1), \ldots, \sigma(Q_{r_1})\} \subseteq \{Q_1, \ldots Q_{r_1}\}$ for any $\sigma \in \Sigma_n$, we see that $\sigma(E_1) \subseteq E_1$ for any $\sigma \in \Sigma_n$.

Note that Q is pointed because it is proper. Thus, without loss of generality, we may assume that $\sigma_u > 0$ for all $u \neq 0$ in Q and $\mathbf{1} \in Q$ by Lemma 4.4.2. As $E_1, \ldots, E_s$ are pointed (being contained in Q) and permutation invariant, again by Lemma 4.4.2, we get $\mathbf{1} \in E_i$ for all $i = 1, 2, \ldots, s$. However, since the spans of E_i's have only zero as their common element, we conclude that $s = 1$. Then $Q = E_1 = Q_1 \oplus \cdots \oplus Q_{r_1}$ by (5.4), i.e., $r_1 = r$. Hence,

$$Q = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_r,$$

where all Q_i are irreducible cones which are isomorphic to each other.

Lastly, since all Q_i's are isomorphic to each other, we have $\dim(Q_i) = \dim(Q_j)$ for every $i, j \in \{1, 2, \ldots, r\}$. Since $n = \dim(Q) = \sum_{i=1}^{r} \dim(Q_i) = r \dim(Q_1)$, we conclude that r divides n. □

Corollary 5.3.3 Let Q be a permutation invariant proper polyhedral cone in $\mathcal{R}^n$, where n is a prime number. Then $\beta(Q) = 1$ or n.

Chapter 6

Spectral Functions in Euclidean Jordan Algebras

6.1 Introduction

This chapter deals with spectral and weakly spectral functions on Euclidean Jordan algebra. Let $\mathcal{V}$ be a Euclidean Jordan algebra of rank k. A function $F : \mathcal{V} \to \mathcal{R}$ is spectral if there exists a permutation invariant function $f : \mathcal{R}^n \to \mathcal{R}$ such that $F = f \circ \lambda$. It is weakly spectral if $F(x) = F(\phi(x))$ for all $x \in \mathcal{V}$ and $\phi \in \mathrm{Aut}(\mathcal{V})$.

The spectrality is a generalization of similar a concept that has been extensively studied in the setting of $\mathcal{R}^n$ (where the concepts reduce to permutation invariant functions) and in $\mathcal{S}^n$ ($\mathcal{H}^n$), the space of all $n \times n$ real (respectively, complex) Hermitian matrices. In the case of $\mathcal{S}^n$ ($\mathcal{H}^n$), spectral functions are precisely those that are invariant under linear transformations of the form $X \to UXU^*$, where U is an orthogonal (respectively, unitary) matrix. By realizing that the map $X \to UXU^*$ is an algebra automorphism on $\mathcal{S}^n$ ($\mathcal{H}^n$), we see that spectrality and weak spectrality coincide when $\mathcal{V} = \mathcal{R}^n$, $\mathcal{S}^n$, or $\mathcal{H}^n$. However, this is no longer true when $\mathcal{V}$ is not essentially simple.

There are a few works that deal with spectral functions on general Euclidean Jordan algebras. Baes [1] discusses some properties of f that get transferred to F (such as convexity and differentiability). Sun and Sun [47] deal with the transferability of the semismoothness properties of f to F. Ramirez, Seeger, and Sossa [39] and Sossa [46] deal with a commutation principle and a number of applications.

We show that every convex spectral function on $\mathcal{V}$ is Schur-convex and provide some applications.

We organize this chapter as follows:

- Section 2 focuses on a characterization of spectral functions and a connection between spectral and weakly spectral functions.
- In Section 3, we study the Schur-convexity of spectral functions.

6.2 Spectral functions on Euclidean Jordan algebras

Definition 6.2.1 A function $F : \mathcal{V} \to \mathcal{R}$ is a **spectral function** if there exists a permutation invariant function $f : \mathcal{R}^n \to \mathcal{R}$ such that $F = f \circ \lambda$. F is weakly spectral if it is invariant under algebra automorphisms, i.e., $F(x) = F(\phi(x))$ for all $x \in \mathcal{V}$ and $\phi \in \mathrm{Aut}(\mathcal{V})$.

By way of an example, we observe that $F(x) = \lambda_{max}(x)$ (the maximum of eigenvalues of x) is a spectral function on $\mathcal{V}$ as it corresponds to $f(u) = \max\, u_i$ on $\mathcal{R}^n$.

Theorem 6.2.2 Let E be a nonempty set in V and $F : \mathcal{V} \to \mathcal{R}$ be a function. Then, the following hold:

(1) E is a spectral set in $\mathcal{V}$ if and only if χ_E, the characteristic function of E, is a spectral function.

(2) F is a spectral function if and only if for each $\alpha \in \mathcal{R}$, the level set $F^{-1}(\{\alpha\})$ is a spectral set in $\mathcal{V}$.

Proof.

(1) Suppose E is a spectral set, say $E = \lambda^{-1}(Q)$, where Q is permutation invariant set in $\mathcal{R}^n$. Then $\chi_E = \chi_Q \circ \lambda$. As χ_Q is a permutation invariant, χ_E is a spectral function.

Conversely, if $\chi_E = f \circ \lambda$ is a spectral function, then define Q as the set where f takes the value one. It is obvious that Q is a permutation invariant set. We now we show that $E = \lambda^{-1}(Q)$. Suppose $x \in E$. Then $1 = \chi_E(x) = f(\lambda(x))$, which implies $\lambda(x) \in Q$ by our construction. Thus, $x \in \lambda^{-1}(Q)$. For the reverse implication, suppose $x \in \lambda^{-1}(Q)$. As $\lambda(x) \in Q$, we get $\chi_E(x) = f(\lambda(x)) = 1$, and so $x \in E$.

(2) Suppose F is a spectral function, say $F = f \circ \lambda$, where $f : \mathcal{R}^n \to \mathcal{R}$ is a permutation invariant function. For any $\alpha \in \mathcal{R}^n$, $F^{-1}(\{\alpha\}) = \lambda^{-1}(f^{-1}(\{\alpha\})$. As f is a permutation invariant function, $Q := f^{-1}(\{\alpha\})$ is a permutation invariant set in $\mathcal{R}^n$. Thus, $F^{-1}(\{\alpha\}) = \lambda^{-1}(Q)$ is a spectral set.

For the converse, suppose $F^{-1}(\{\alpha\})$ is a spectral set for all a. Then, for each a, we have $F^{-1}(\{\alpha\}) = \lambda^{-1}(Q_a)$ for some permutation invariant set Q_a. Note that $\bigcup_a Q_a = V$ and $Q_{a'} \cap Q_{a''} = \emptyset$ whenever $a' \neq a''$. Put $f = \sum_a a\chi_a$; the function f is well-defined. Now, take $u \in \mathcal{R}^n$ and suppose $f(u) = a$. This implies $u \in Q_a$.

As Q_a is permutation invariant, $\pi(u) \in Q_a$ and hence $f(\pi(u)) = a$ for any permutation matrix $\pi \in \Sigma_n$ showing f is permutation invariant. Now, we show $F = f \circ \lambda$. Take any $x \in V$ with $F(x) = a$. Then, $x \in F^{-1}(\{\alpha\}) = \lambda^{-1}(Q_a)$. Thus, $\lambda(x) \in Q_a$ or $f(\lambda(x)) = a$. $\square$

Before characterizing spectral functions in $\mathcal{V}$, we first describe a duality relation between permutation invariant functions in $\mathcal{R}^n$ and spectral functions in $\mathcal{V}$.

Definition 6.2.3 Given a function $f : \mathcal{R}^n \to \mathcal{R}$, we define $f^{\diamondsuit} : \mathcal{V} \to \mathcal{R}$ by

$$f^{\diamondsuit}(x) := f(\lambda(x)) \qquad \text{for} \quad x \in \mathcal{V}. \tag{6.1}$$

Fix a Jordan frame $\{\bar{e}_1, \ldots, \bar{e}_n\}$ in $\mathcal{V}$. Then, given a function $F : \mathcal{V} \to \mathcal{R}$, we define $F^{\diamondsuit} : \mathcal{R}^n \to \mathcal{R}$ by

$$F^{\diamondsuit}(u) := F\left(\sum_{i=1}^{n} u_i^{\downarrow} \bar{e}_i\right) \qquad \text{where} \quad u = (u_1, \ldots, u_n)^{\mathsf{T}} \in \mathcal{R}^n. \tag{6.2}$$

For simplicity, we write $f^{\diamondsuit\diamondsuit}$ in place of $(f^{\diamondsuit})^{\diamondsuit}$, etc. In the result below, $F^{\diamondsuit}$ and $(f^{\diamondsuit})^{\diamondsuit}$ are defined as in (6.2) with respect to a (fixed) Jordan frame $\{\bar{e}_1, \ldots, \bar{e}_n\}$.

Proposition 6.2.4 The following statements hold:

(1) For any permutation invariant function $f : \mathcal{R}^n \to \mathcal{R}$, $f^{\diamondsuit}$ is a spectral function and $f^{\diamondsuit\diamondsuit} = f$.

(2) For any function $F : \mathcal{V} \to \mathcal{R}$, $F^{\diamondsuit}$ is permutation invariant.

Proof.

(1) When f is permutation invariant, by definition, $f^\diamondsuit$ is a spectral function. Now, define $(f^\diamondsuit)^\diamondsuit$ as in (6.2). Then, for any $u \in \mathcal{R}^n$, we have

$$f^{\diamondsuit\diamondsuit}(u) = f^\diamondsuit\left(\sum_{i=1}^n u_i^\downarrow \bar{e}_i\right) = f\left(\lambda\left(\sum_{i=1}^n u_i^\downarrow \bar{e}_i\right)\right) = f(u^\downarrow) = f(u).$$

Since u is arbitrary, $f^{\diamondsuit\diamondsuit} = f$.

(2) Take $u \in \mathcal{R}^n$ and $\sigma \in \Sigma_n$. As $u^\downarrow = \sigma(u)^\downarrow$, we get

$$F^\diamondsuit(\sigma(u)) = F\left(\sum_{i=1}^n \sigma(u)_i^\downarrow \bar{e}_i\right) = F\left(\sum_{i=1}^n u_i^\downarrow \bar{e}_i\right) = F^\diamondsuit(u).$$

Hence, $F^\diamondsuit$ is permutation invariant.

$\square$

Example 6.2.5 For the equality $f^{\diamondsuit\diamondsuit} = f$ in Item (i) above, it is essential to have f invariant under permutations. To see this, take $\mathcal{V} = \mathcal{S}^2$ and define $f : \mathcal{R}^2 \to \mathcal{R}$ by $f(u_1, u_2) = u_1$. Then, for any $x \in \mathcal{S}^2$, $f^\diamondsuit(x) = f(\lambda(x)) = \lambda_1(x)$. Hence, for any $u \in \mathcal{R}^2$,

$$f^{\diamondsuit\diamondsuit}(u) = f^\diamondsuit\left(\sum_{i=1}^2 u_i^\downarrow \bar{e}_i\right) = f(u^\downarrow) = \max\{u_1, u_2\}.$$

This proves that $f^{\diamondsuit\diamondsuit} \neq f$.

We now characterize spectral functions.

Theorem 6.2.6 The following are equivalent for a function $F : \mathcal{V} \to \mathcal{R}$:

(a) F is a spectral function.

(b) F is constant on each equivalence class of $\sim$.

(c) $F^{\diamond\diamond} = F$.

Proof. (a) $\Rightarrow$ (b): Suppose F is a spectral function and $x \sim y$. Put $\alpha := F(x)$. Then by Theorem 6.2.2, we see that $F^{-1}(\{\alpha\})$ is a spectral set and $x \in F^{-1}(\{\alpha\})$; hence, by Theorem 3.2.6, $y \in F^{-1}(\{\alpha\})$. This implies $F(y) = \alpha$. It follows that $F(x) = F(y)$ whenever $x \sim y$.

(b) $\Rightarrow$ (c): Let $x \in \mathcal{V}$ with its spectral decomposition $x = \sum_1^n \lambda_i(x) e_i$. Define $y = \sum_1^n \lambda_i(x)\bar{e}_i$. Then, $x \sim y$ and by (c), $F(x) = F(y)$. This implies,

$$F^{\diamond\diamond}(x) = F^{\diamond}(\lambda(x)) = F\left(\sum_{i=1}^{n} \lambda_i(x)\bar{e}_i\right) = F(y) = F(x).$$

Hence, $F^{\diamond\diamond} = F$.

(c) $\Rightarrow$ (a): Let $F^{\diamond\diamond} = F$. Then, $f := F^{\diamond}$ is permutation invariant, by Item (ii) in Proposition 6.2.4. Hence, $F = f^{\diamond}$ is spectral. $\square$

Theorem 6.2.7 Every spectral function is weakly spectral. Converse holds when $\mathcal{V}$ is essentially simple.

Proof. Suppose $F = f \circ \lambda$ for some permutation invariant function f. Note that $\lambda(x) = \lambda(\phi(x))$ for every $x \in \mathcal{V}$ and $\phi \in \mathrm{Aut}(\mathcal{V})$. Thus, $F(\phi(x)) = f(\lambda(\phi(x))) = f(\lambda(x)) = F(x)$. This shows that F is weakly spectral.

For the converse, let F be weakly spectral and $\mathcal{V}$ be essentially simple. If $x \sim y$, then by Proposition 2.1.15, there exists $\phi \in \mathrm{Aut}(\mathcal{V})$ such that $x = \phi(y)$. This implies that $F(x) = F(y)$. We now use Item (c) in Theorem 6.2.6 to see that F is spectral. $\square$

Example 6.2.8 The converse in Theorem 6.2.8 may not hold in general. To see this, take $\mathcal{V} = \mathcal{R} \times \mathcal{S}^2$ and let $F : \mathcal{V} \to \mathcal{R}$ be defined by

$$F\left((a,\, A)\right) = \mathrm{tr}(A), \quad \forall\, a \in \mathcal{R},\, A \in \mathcal{S}^2.$$

Since we know the (explicit) description of automorphisms of $\mathcal{V}$ (via Proposition 2.1.10), we easily see that F is automorphism invariant. However, for x and y of Example 3.2.8, $\lambda(x) = \lambda(y)$ and so $x \sim y$. One can easily check that $F(x) = 0 \neq 1 = F(y)$. As F violates condition (c) in Theorem 6.2.6, F is not a spectral function.

A celebrated result of Davis [10] says that a unitarily invariant function on $\mathcal{H}^n$ is convex if and only if its restriction to diagonal matrices is convex. This result has numerous applications in various fields. A generalization of this result for Euclidean Jordan algebras has already been observed by Baes [1]. In what follows, we consider equivalent formulations of this generalization and give some applications.

Theorem 6.2.9 Let E be a convex spectral set in $\mathcal{V}$ so that $E = \lambda^{-1}(Q)$ with Q convex and permutation invariant in $\mathcal{R}^n$. Let $F : E \to \mathcal{R}$. Consider the following statements:

(a) $F = f \circ \lambda$, where $f : Q \to R$ is convex and permutation invariant.

(b) F is convex, and $x \prec y$ implies $F(x) \leq F(y)$.

(c) F is convex, and $x \sim y$ implies $F(x) = F(y)$.

(d) F is convex and $F \circ \phi = F$ for all $\phi \in \mathrm{Aut}(\mathcal{V})$.

Then $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ and $(c) \Rightarrow (d)$. Moreover, all these statements are equivalent when $\mathcal{V}$ is essentially simple.

Proof. $(a) \Rightarrow (b)$: The convexity of F has already been proved in Theorem 41, [1]. Capturing its essence, we present our proof based on Proposition 2.3.8. For $t \in [0,1]$ and $x, y \in E$, we write $\lambda(tx+(1-t)y) = t\,A\lambda(x)+(1-t)\,B\lambda(y)$, where A and B are doubly stochastic matrices. As these matrices are convex combinations of permutation matrices, by permutation invariance and convexity of f on Q, we see that $f(A\lambda(x)) \leq f(\lambda(x))$ and $f(B\lambda(y)) \leq f(\lambda(y))$. Thus,

$$\begin{aligned} F(tx+(1-t)y) &= f(tA\lambda(x)+(1-t)B\lambda(y)) \\ &\leq tf(A\lambda(x))+(1-t)f(B\lambda(y)) \\ &\leq tf(\lambda(x))+(1-t)f(\lambda(y)) \\ &= tF(x)+(1-t)F(y). \end{aligned}$$

Now for the second part of (b), suppose $x, y \in E$ with $x \prec y$. Writing $u = \lambda(x)$ and $v = \lambda(y)$, we get $u \prec v$ in Q and hence (from Propositions 2.3.2, 2.3.3) $u = \sum_j \alpha_j \sigma_j(v)$ for some $\alpha_j > 0$ with $\sum_j \alpha_j = 1$ and permutation matrices $\sigma_j \in \Sigma_n$. As f is convex and permutation invariant,

$$f(u) = f\left(\sum \alpha_j \sigma_j(v)\right) \leq \sum \alpha_j f(\sigma_j(v)) = \sum \alpha_j f(v) = f(v).$$

Since $F = f \circ \lambda$, we get $F(x) = f(\lambda(x)) = f(u) \leq f(v) = f(\lambda(y)) = F(y)$.

$(b) \Rightarrow (c)$: This is obvious as $x \sim y$ implies $x \prec y$.

$(c) \Rightarrow (a)$: Given F satisfying (c), we define $f : Q \to \mathcal{R}$ as follows. For any $u \in Q$, there exists $x \in E$ such that $\lambda(x) = u^{\downarrow}$. We let $f(u) := F(x)$. This is well defined: if there is a $y \in E$ with $\lambda(y) = u^{\downarrow}$, then $x \sim y$ and so by (c), $F(x) = F(y)$. Now, for any permutation σ, $u^{\downarrow} = \sigma(u)^{\downarrow}$; hence, $f(\sigma(u)) = F(x) = f(u)$, proving the permutation invariance of f. We also see $F(x) = f(u) = f(u^{\downarrow}) = f(\lambda(x))$. Note that these (relations) hold if we start with an $x \in E$ and let $u = \lambda(x)$. Thus we have $F = f \circ \lambda$.

We now show that f is convex. Let u, $v \in Q$ and $t \in [0,1]$. As $w := tu+(1-t)v \in Q$, there exists $z \in E$ such that $\lambda(z) = w^{\downarrow}$. Let $\sum_1^n w_i\, e_i$ be the spectral decomposition of z. Define $x = \sum_1^n u_i\, e_i$ and $y = \sum_1^n v_i\, e_i$. Note that $\lambda(x) = u^{\downarrow} \in Q$ and $\lambda(y) = v^{\downarrow} \in Q$; hence, $x, y \in E$ with $z = tx + (1-t)y$. Then, as f is permutation invariant and F is convex, we get

$$\begin{aligned} f(tu + (1-t)v) &= f(w) = f(w^{\downarrow}) = F(z) \\ &= F(tx + (1-t)y) \\ &\leq tF(x)) + (1-t)F(y) \\ &= tf(u) + (1-t)f(v). \end{aligned}$$

Hence, f is convex.

$(c) \Rightarrow (d)$: Let ϕ be an automorphism. For $x \in \mathcal{V}$, $\lambda(\phi(x)) = \lambda(x)$ and so $\phi(x) \sim x$. Then, by (c), we must have $F(\phi(x)) = F(x)$.

Now, we prove $(d) \Rightarrow (c)$ when $\mathcal{V}$ is essentially simple. Suppose x, $y \in E$ such that $x \sim y$. By Proposition 2.1.15, there exists an automorphism ϕ such that $x = \phi(y)$. Then, from (d), $F(x) = F(\phi(y)) = F(y)$. Thus, condition (c) holds.

This completes the proof. $\square$

Remarks. In the case of a general $\mathcal{V}$, (d) may not imply other statements. This can be seen by taking $\mathcal{V}$ and F as in Example 6.2.8. Clearly, this F is automorphism invariant and convex (indeed, linear). However, as seen in Example 6.2.8, F is not a spectral function.

We now provide two applications of the above theorem.

Example 6.2.10 For $p \in [1, \infty]$, let $F(x) = \|x\|_{sp,\,p} := ||\lambda(x)||_p$, where $||u||_p$ denotes the pth-norm of a vector u in $\mathcal{R}^n$. Then, F is a convex spectral function which is also positive homogeneous. Since $F(x) = 0$ implies $x = 0$, we see that

$$|| \cdot ||_{sp,\,p} \text{ is a norm on } \mathcal{V}.$$

This fact has already been observed in [48], where it is proved via Thompson's triangle inequality, majorization techniques, and case-by-case analysis.

Example 6.2.11 *A Golden-Thompson type inequality.*

The Golden-Thompson inequality ([3], p.261) says that for $A,\, B \in \mathcal{H}^n$,

$$\operatorname{tr}\Big(\exp(A+B)\Big) \leq \operatorname{tr}\Big(\exp(A)\,\exp(B)\Big).$$

It is not known if an analogous result holds in Euclidean Jordan algebras. The Golden-Thompson inequality easily implies the (weaker) inequality

$$\operatorname{tr}\Big(\exp(A+B)\Big) \leq \operatorname{tr}\Big(\exp(A)\Big)\,\operatorname{tr}\Big(\exp(B)\Big).$$

Rivin [40], based on a result of Davis [10], gives a simple proof of this. A modification of this proof, given below, leads to the following generalization:

For any two elements x, y in a Euclidean Jordan algebra $\mathcal{V}$,

$$\operatorname{tr}\Big(\exp(x+y)\Big) \leq \operatorname{tr}\Big(\exp(x)\Big)\ \operatorname{tr}\Big(\exp(y)\Big), \tag{6.3}$$

where $\exp(x)$ is defined by $\exp(x) := \sum_1^n \exp(\lambda_i(x))\, e_i$ when x has the spectral decomposition $x = \sum_1^n \lambda_i(x) e_i$.

We prove (6.3) as follows. For $u \in \mathcal{R}^n$ with components $u_1, \ldots, u_n$, let $f(u) = \ln(\sum_1^n \exp(u_i))$, which is known to be convex [40]. Since f is also permutation invariant, $F := f \circ \lambda$ is convex by the above theorem. It follows that

$$F\left(\frac{x+y}{2}\right) \leq \frac{F(x)+F(y)}{2},$$

for all x, $y \in \mathcal{V}$. As $F(x) = \ln\Big(\operatorname{tr}\Big(\exp(x)\Big)\Big)$, this leads to

$$\left[\operatorname{tr}\left(\exp\left(\frac{x+y}{2}\right)\right)\right]^2 \leq \operatorname{tr}\Big(\exp(x)\Big)\ \operatorname{tr}\Big(\exp(y)\Big).$$

This, with the observation $\operatorname{tr}\Big(\exp(2z)\Big) \leq \Big[\operatorname{tr}\Big(\exp(z)\Big)\Big]^2$ for any $z \in \mathcal{V}$, yields (6.3).

6.3 Schur-convex functions

Definition 6.3.1 A function $F : \mathcal{V} \to \mathcal{R}$ is said to be *Schur-convex* if

$$x \prec y \quad \Longrightarrow \quad F(x) \leq F(y).$$

Theorem 6.2.9 (together with Proposition 2.3.7) immediately yields the following.

Theorem 6.3.2 Every convex spectral function on $\mathcal{V}$ is Schur-convex. In particular, for any doubly stochastic transformation Ψ on $\mathcal{V}$, and for any convex spectral function F on $\mathcal{V}$, we have

$$F(\Psi(x)) \leq F(x) \qquad \text{for all} \quad x \in \mathcal{V}.$$

We illustrate the above result with a number of examples.

Example 6.3.3 For $p \in [1, \infty]$, as in Example 6.2.10, consider $F(x) = \|x\|_{sp,\,p} := ||\lambda(x)||_p$. Then, F is a convex spectral function and hence, for any doubly stochastic transformation Ψ on $\mathcal{V}$,

$$\|\Psi(x)\|_{sp,\,p} \leq \|x\|_{sp,\,p}\,.$$

This extends Proposition 2 in [18], where it is shown that $||\Psi(x)||_{sp,\,2} \leq ||x||_{sp,\,2}$ for all x in a simple Euclidean Jordan algebra.

Example 6.3.4 Consider an idempotent c $(\neq 0, e)$ in $\mathcal{V}$ and the corresponding orthogonal decomposition [11]

$$\mathcal{V} = \mathcal{V}(c,\,1) + \mathcal{V}(c,\,\tfrac{1}{2}) + \mathcal{V}(c,\,0),$$

where $\mathcal{V}(c,\,\gamma) = \{x \in \mathcal{V} : x \circ c = \gamma\, x\}$, $\gamma \in \{0, \frac{1}{2}, 1\}$. For each $x \in \mathcal{V}$, we write

$$x = u + v + w \qquad \text{where} \quad u \in \mathcal{V}(c,\,1),\, v \in \mathcal{V}(c,\,\tfrac{1}{2}),\, w \in \mathcal{V}(c,\,0).$$

For any $a \in \mathcal{V}$, let P_a denote the corresponding quadratic representation defined by $P_a(x) = 2a \circ (a \circ x) - a^2 \circ x$. Then for $\varepsilon = 2c - e$, one verifies that $P_\varepsilon(x) = u - v + w$ and $(\frac{P_\varepsilon + I}{2})(x) = u + w$. As $\varepsilon^2 = e$, P_ε is an automorphism of $\mathcal{V}$ ([11], Prop. II.4.4).

Thus, P_ε and $\frac{P_\varepsilon+I}{2}$ are doubly stochastic on $\mathcal{V}$. Then, for any convex spectral function F on $\mathcal{V}$,

$$F(u-v+w) \leq F(x) \quad \text{and} \quad F(u+w) \leq F(x).$$

We note that the process of going from $x = w+v+w$ to $u+w$ is a 'pinching' process; in the context of block matrices, this 'pinching' is obtained by setting the off-diagonal blocks to zero.

Example 6.3.5 Let A be an $n \times n$ real symmetric positive semidefinite matrix with each diagonal entry one. In $\mathcal{V}$, we fix a Jordan frame $\{e_1, e_2, \ldots, e_n\}$ and consider the corresponding Peirce decomposition of any $x \in \mathcal{V}$ (as in Proposition 2.1.11): $x = \sum_{i \leq j} x_{ij} = \sum_{i=1}^n x_i e_i + \sum_{i<j} x_{ij}$. Then the transformation

$$\Psi(x) := A \bullet x = \sum_{i \leq j} a_{ij} x_{ij}$$

is doubly stochastic, see [18]. It follows that when F is a convex spectral function on $\mathcal{V}$,

$$F(A \bullet x) \leq F(x).$$

By taking A to be the identity matrix, this inequality reduces to

$$F\left(\sum_{i=1}^n x_i e_i\right) \leq F(x).$$

Example 6.3.6 Let $g : \mathcal{R} \to \mathcal{R}$ be any function. Then the corresponding *Löewner mapping* $L_g : \mathcal{V} \to \mathcal{V}$ [47] is defined as follows: For any $x \in \mathcal{V}$ with spectral decomposition $x = \sum \lambda_i(x) e_i$,

$$L_g(x) := \sum_{i=1}^{n} g(\lambda_i(x))\, e_i.$$

Clearly, $F(x) := \mathrm{tr}(L_g(x)) = \sum g(\lambda_i(x)) = f \circ \lambda$, where $f(u_1, \ldots, u_n) = \sum g(u_i)$, is a spectral function. Thus, when g is convex, F is a convex spectral function. Hence, the function $x \mapsto \mathrm{tr}(L_g(x))$ is Schur-convex for any convex function $g : \mathcal{R} \to \mathcal{R}$.

The second part of Theorem 6.3.2 says that a doubly stochastic transformation decreases the value of a convex spectral function. Below, we present a converse to this statement:

Theorem 6.3.7 The following statements hold:

(1) If $x, y \in \mathcal{V}$ with $F(x) \leq F(y)$ for all convex spectral functions F on $\mathcal{V}$, then $x \prec y$.

(2) If $x, y \in \mathcal{V}$ with $F(x) = F(y)$ for all convex spectral functions F on $\mathcal{V}$, then $x \sim y$.

(3) If Ψ is a linear transformation on $\mathcal{V}$ such that $F(\Psi(x)) \leq F(x)$ for all $x \in \mathcal{V}$ and for all convex spectral functions F, then Ψ is a doubly stochastic transformation on $\mathcal{V}$.

This result is based on two lemmas. The first lemma (noted on page 159 in [36]) is a consequence of a result of Hardy, Littlewood, and Pólya.

Lemma 6.3.8 Suppose $u, v \in \mathcal{R}^n$ with the property that $f(u) \leq f(v)$ for all convex permutation invariant functions $f : \mathcal{R}^n \to \mathcal{R}$. Then, $u \prec v$.

Our second lemma is a generalization of a a well-known result: An $n \times n$ matrix A is doubly stochastic if and only if $Ax \prec x$ for all $x \in \mathcal{R}^n$, see Theorem A.4 in [36].

Lemma 6.3.9 A linear transformation $\Psi : \mathcal{V} \to \mathcal{V}$ is doubly stochastic if and only if $\Psi(x) \prec x$ for all $x \in \mathcal{V}$.

Proof. First, suppose Ψ is doubly stochastic. Then by Proposition 2.3.7, we have $\Psi(x) \prec x$. To prove the converse, suppose $\Psi(x) \prec x$ for all $x \in \mathcal{V}$. First, let $x \in \mathcal{V}_+$. Then all the eigenvalues of x are nonnegative. Now, $\Psi(x) \prec x$ implies, from the inequalities (2.2) in Section 2.3.2, $\lambda_n(\Psi(x)) \geq \lambda_n(x) \geq 0$. As $\lambda_n(\Psi(x))$ is the smallest of the eigenvalues of $\Psi(x)$, we see that all eigenvalues of $\Psi(x)$ are nonnegative; hence $\Psi(x) \in \mathcal{V}_+$. Thus, $\Psi(\mathcal{V}_+) \subseteq \mathcal{V}_+$.

Next, we show that $\Psi(e) = e$. As $\Psi(e) \prec e$ and every eigenvalue of e is one, from the inequalities (2.2) again, the smallest and largest eigenvalues of $\Psi(e)$ coincide with 1. Thus, all eigenvalues of $\Psi(e)$ are equal to one. By the spectral decomposition of e, we see that $\Psi(e) = e$.

We now show Ψ is trace-preserving: For any Jordan frame $\{e_1, \ldots, e_n\}$, we have $\Psi(e_i) \prec e_i$, so $\mathrm{tr}(\Psi(e_i)) = \mathrm{tr}(e_i) = 1$. Thus for $x \in \mathcal{V}$ with (spectral decomposition) $x = \sum_i x_i e_i$,

$$\mathrm{tr}(\Psi(x)) = \mathrm{tr}\left(\Psi\left(\sum_{i=1}^{n} x_i e_i\right)\right) = \sum_{i=1}^{n} x_i \,\mathrm{tr}(\Psi(e_i)) = \sum_{i=1}^{n} x_i = \mathrm{tr}(x).$$

Thus, Ψ is doubly stochastic on $\mathcal{V}$. $\square$

We now come to the proof the theorem.

Proof of the Theorem.

(1) Suppose $x, y \in \mathcal{V}$ with $F(x) \leq F(y)$ for all convex spectral functions F on $\mathcal{V}$. Then, for any f that is convex and permutation invariant on $\mathcal{R}^n$, we let $F = f \circ \lambda$ and get $f(\lambda(x)) \leq f(\lambda(y))$. Now from Lemma 6.3.8, $\lambda(x) \prec \lambda(y)$. This means that $x \prec y$.

(2) From the proof of Item (i), $\lambda(x) \prec \lambda(y)$ and $\lambda(y) \prec \lambda(x)$. Since $\lambda(x)$ and $\lambda(y)$ have decreasing order, defining (majorization) inequalities give $\lambda(x) = \lambda(y)$. Thus, $x \sim y$.

(3) Now suppose that Ψ is linear and $F(\Psi(x)) \leq F(x)$ for all $x \in \mathcal{V}$ and all convex spectral functions F on $\mathcal{V}$. By (i), $\Psi(x) \prec x$ for all x. By Lemma 6.3.9, Ψ is doubly stochastic. $\square$

It has been observed before that automorphisms preserve eigenvalues. The following result shows that in the setting of essentially simple algebras, automorphisms are the only linear transformations that preserve eigenvalues.

Corollary 6.3.10 Suppose $\mathcal{V}$ is essentially simple and $\phi : \mathcal{V} \to \mathcal{V}$ is linear. If $\phi(x) \sim x$ for all $x \in \mathcal{V}$, then $\phi \in \mathrm{Aut}(\mathcal{V})$.

Proof. Suppose that $\phi(x) \sim x$ for all $x \in \mathcal{V}$. Then, ϕ is invertible: If $\phi(x) = 0$ for some x, then $0 \sim x$ and so $x = 0$. Now, take any convex spectral function F on $\mathcal{V}$. As $\phi(x) \sim x$, we have $F(x) = F(\phi(x))$. From Theorem 6.3.7, we see that ϕ and ϕ^{-1} are doubly stochastic on $\mathcal{V}$. Recall, by definition, these are positive transformations, that is, $\phi(\mathcal{V}_+) \subseteq \mathcal{V}_+$ and $\phi^{-1}(\mathcal{V}_+) \subseteq \mathcal{V}_+$. Hence, $\phi(\mathcal{V}_+) = \mathcal{V}_+$, that is, ϕ belongs to $\mathrm{Aut}(\mathcal{V}_+)$. Since $\mathcal{V}$ is essentially simple, from Theorem 8 in [18] we have

$$\mathrm{Aut}(\mathcal{V}_+) \cap \mathrm{DS}(\mathcal{V}) = \mathrm{Aut}(\mathcal{V}),$$

where $\mathrm{DS}(\mathcal{V})$ denotes the set of all doubly stochastic transformations on $\mathcal{V}$. This proves that $\phi \in \mathrm{Aut}(\mathcal{V})$. $\square$

Bibliography

[1] M. Baes, *Convexity and differentiability of spectral functions on Euclidean Jordan algebra*, Linear Algebra and its Applications, 422 (2007) 664-700.

[2] A. Berman and N. Shaked-Monderer, *Completely positive matrices*, New Jersey, World Scienfic, 2003.

[3] R. Bhatia, *Matrix Analysis*, Springer Graduate Texts in Mathematics, Springer-Verlag, New York, 1997.

[4] S.C. Billups and K.G. Murty, *Complementarity Problems*, Journal of Computational and Applied Mathematics, 124 (2000) 303-318.

[5] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, Cambridge, 2004.

[6] V. Chandrasekaran, P.A. Parrilo, A.S. Willsky, *Convex graph invariants*, SIAM Review, 54 (2012) 513-541.

[7] C.K. Li, P.P. Mehta, *Permutation invariant norms*, Linear Algebra and its Applications, 219 (1995) 93-110.

[8] A. Daniilidis, A. Lewis, J. Malick, and H. Sendov, *Prox-regularity of spectral functions and spectral sets*, Journal of Convex Analysis, 15 (2008) 547-560.

[9] A. Daniilidis, D. Drusvyatskiy, and A.S. Lewis, *Orthogonal invariance and identifiability*, SIAM Journal of Matrix Analysis and Applications, 35 (2014) 580-598.

[10] C. Davis, *All convex invariant functions of hermitian matrices*, Archiv der Mathematik, 8 (1957) 276-278.

[11] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford, 1994.

[12] M.S. Gowda, R. Sznajder, and J. Tao, *Some P-properties for linear transformations on Euclidean Jordan algebras*, Linear Algebra and its Applications, 393 (2004) 203-232.

[13] J. Tao and M.S. Gowda, *Some P-properties for nonlinear transformations on Euclidean Jordan algebras*, Mathematics of Operations Research, 30 (2005) 985-1004

[14] M.S. Gowda and R. Sznajder, *Some Global Uniqueness and Solvability Results for Linear Complementarity Problems over Symmetric Cones*, SIAM Journal on Optimization, 18 (2007) 461-481.

[15] M.S. Gowda and J. Tao, *The Cauchy interlacing theorem in simple Euclidean Jordan algebras and some consequences*, Linear and Multilinear Algebra, 59 (2011) 65-86.

[16] M.S. Gowda and J. Tao, *On the bilinearity rank of a proper cone and Lyapunov-like transformations*, Mathematical Programming Series A, 147 (2014) 155-170.

[17] M.S. Gowda and D. Trott, *On the Irreducibility, Lyapunov rank, and Automorphisms of Special Bishop-Phelps Cones*, Journal of Mathematical Analysis and Applications, 419 (2014) 172-184.

[18] M.S. Gowda, *Positive and doubly stochastic maps, and majorization in Euclidean Jordan algebras*, To appear in Linear Algebra and its Applications, DOI 10.1016/j.laa.2016.02.024.

[19] M.S. Gowda and J. Jeong, *Commutation principles in Euclidean Jordan algebras and normal decomposition systems*, SIAM Journal of Optimization, to appear.

[20] R. Henrion and A. Seeger, *Inradius and circumradius of various convex cones arising in applications*, Set Valued Analysis, 18 (2010) 483-511.

[21] F. Hiai, *Similarity preserving linear maps on matrices*, Linear Algebra and its Applications, 97 (1987) 127-139.

[22] A. Iusem and A. Seeger, *Angular analysis of two classes of non-polyhedral convex cones: the point of view of optimization theory*, Computational and Applied Mathematics, 26 (2007) 191-214.

[23] J. Jeong and M.S. Gowda, *Spectral sets and functions on Euclidean Jordan algebras*, Linear Algebra and its Applications, 518 (2017) 31-56.

[24] J. Jeong and M.S. Gowda, *Spectral cons on Euclidean Jordan algebras*, Linear Algebra and its Applications, 509 (2016) 286-305.

[25] M. Koecher, *The Minnesota Notes on Jordan Algebras and Their Applications*, Lecture Notes in Mathematics 1710, Springer, Berlin 1999.

[26] J. Lahtonen, *Subspace generated by permutations of a vector in a vector space*, http://math.stackexchange.com/questions/75427.

[27] A.S. Lewis, *Group invariance and convex matrix analysis*, SIAM Journal on Matrix Analysis and Applications, 17 (1996) 927-949.

[28] A. S. Lewis, *Convex analysis on the Hermitian matrices*, SIAM Journal on Optimization, 6 (1996) 164-177.

[29] A. S. Lewis, *Derivatives of Spectral Functions*, Mathematics of Operations Research, 6 (1996) 164-177.

[30] A. S. Lewis and H. S. Sendov, *Twice differentiable spectral functions*, SIAM Journal on Matrix Analysis and Applications, 23 (2001) 368-386.

[31] R. Loewy, *Positive operators on the n-dimensional ice cream cone*, Journal of Mathematical Analysis and Applications, 49 (1975) 375-392.

[32] C. K. Li and N. K. Tsing, *Linear preserver problems: A brief introduction and some special techniques*, Linear Algebra and its Applications, 162 (1992) 217-235.

[33] C. K. Li and S. Pierce, *Linear preserver problems*, The American Mathematical Monthly, 108 (2001) 591-605.

[34] C. K. Li and P. Mehta, *Permutation invariant norms*, Linear Algebra and its Applications, 219 (1995) 93-110.

[35] Y. Lim, J. Kim, and L. Faybusovich, *Simultaneous diagonalization on simple Euclidean Jordan algebras and its applications*. Forum Mathematicum, 15 (2003) 639-644.

[36] A. W. Marshall, I. Olkin and B.C. Arnold, *Inequalities: Theory of Majorization and its Applications*, Springer, 2010.

[37] M. Moldovan, *A Gersgorin type theorem, spectral inequalities, and simultaneous stability in Euclidean Jordan algebras*, PhD Thesis, University of Maryland, Baltimore County, 2009.

[38] M. Orlitzky and M.S. Gowda, *An Improved Bound for the Lyapunov Rank of a Proper Cone*, Research Rdport, Dept. of Math & Stat, UMBC, 2014

[39] H. Ramírez, A. Seeger, and D. Sossa, *Commutation principle for variational Problems on Euclidean Jordan algebras*, SIAM Journal on Optimization, 23 (2013) 687-694.

[40] I. Rivin, *Golden-Thompson from Davis*, Arxiv: http://arxiv.org/abs/1010.2193, 2010.

[41] R. T. Rockafellar, *Convex analysis*, Princeton University Press, 1970.

[42] G. Rudolf, N. Noyan, D. Papp, and F. Alizadeh, *Bilinear optimality constraints for the cone of positive polynomials*, Mathematical Programming, Series B, 129 (2011) 5-31.

[43] A. Seeger, *Convex analysis of spectrally defined matrix functions*, SIAM Journal on Optimization, 7 (1997) 679-696.

[44] A. Seeger, *Orbital equations for extremal problems under various invariance assumptions*, Set-Valued and Variational Analysis, 21 (2013) 503-516.

[45] A. Seeger, *Extremal problems involving isotropic sets and functions on spaces of rectangular matrices*, TOP, 22 (2014) 1017-1027.

[46] D. Sossa, *Euclidean Jordan algebras and variational problems under conic constraints*, PhD Thesis, University of Chile, 2014.

[47] D. Sun and J. Sun, *Löwner's operator and spectral functions in Euclidean Jordan algebras*, Mathematics of Operations Research, 33 (2008) 421-445.

[48] J. Tao, L. Kong, Z. Luo, and N. Xiu, *Some majorization inequalities in Euclidean Jordan algebras*, Linear Algebra and its Applications, 161 (2014) 92-122.

[49] G. A. Watson, *Characterization of the subdifferential of some matrix norms*, Linear Algebra and its Applications, 170 (1992) 33-45.