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CONTINUOUS DATA ASSIMILATION FOR THE THREE-DIMENSIONAL NAVIER–STOKES EQUATIONS*

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Abstract. In this paper, we identify conditions, *based solely on the observed data*, for the global well-posedness, regularity, and the asymptotic tracking property of solutions of the Newtonian relaxation (nudging) algorithm for data assimilation for the three-dimensional incompressible Navier–Stokes equations (3D NSE). A rigorous analysis of this algorithm for dissipative partial differential equations was first provided by Azouani, Olson, and Titi [*J. Nonlinear Sci.*, 24 (2014), pp. 277–304] in the context of the two-dimensional Navier–Stokes equations. In that analysis, as also in each of the subsequent ones of other dissipative systems including the 3D Boussinesq system with a large Prandtl number, the primitive equations of the ocean and atmosphere, and several α-models of turbulence, a crucial role is played by the known uniform $\mathbb{H}^1$-norm bound of the absorbing ball (i.e., an eventual $\mathbb{H}^1$-norm bound on a solution of each of these systems). However, in the 3D case, even for a globally regular solution, *no such (eventual) uniform* $\mathbb{H}^1$*-norm bound is known*. The starting point of our analysis is a Leray–Hopf weak solution, satisfying a certain condition *based on observations*, which subsequent work has shown to imply eventual regularity (and regularity in case the solution is on the weak attractor). To the best of our knowledge, this is the first such rigorous analysis of the Azouani–Olson–Titi data assimilation algorithm for the 3D NSE for which an a priori eventual uniform $\mathbb{H}^1$-norm bound is unknown, even if the solution is regular.

Key words. three-dimensional Navier–Stokes equations, continuous data assimilation, determining modes, determining volume elements and nodes, signal synchronization

AMS subject classifications. Primary, 35Q30, 93C20; Secondary, 35Q35, 76B75

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1. Introduction.

For a given dynamical system, which is believed to accurately describe some aspect(s) of an underlying physical reality, the problem of forecasting is often hindered by inadequate knowledge of the initial state and/or model parameters describing the system. However, in many cases, this is compensated by the fact that one has access to data from (possibly noisy) measurements of the system, collected on a *much coarser spatial grid than the desired resolution of the forecast.* The objective of data assimilation and signal synchronization is to use this coarse scale observational measurements to fine tune our knowledge of the state and/or model to improve the accuracy of the forecasts [24, 45].

Due to its ubiquity in scientific applications, data assimilation has been the subject of a very large body of work. Classically, these techniques are based on linear quadratic estimation, also known as the Kalman filter. The Kalman filter has the drawback of assuming that the underlying system and any corresponding observation models are linear. It also assumes that measurement noise is Gaussian distributed. This has been mitigated by practitioners via modifications, such as the ensemble Kalman filter, the extended Kalman filter, and the unscented Kalman filter, and consequently, there has been a recent surge of interest in developing a rigorous mathematical framework for these approaches; see, for instance, [5, 39, 45, 46, 48, 57] and

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the references therein. These works provide a Bayesian and variational framework for the problem, with emphasis on analyzing variational and Kalman filter based methods. It should be noted, however, that the problems of stability, accuracy, and *catastrophic filter divergence*, particularly for *infinite dimensional chaotic dynamical systems governed by PDEs,* continue to pose serious challenges to rigorous analysis and are far from being resolved [15, 16, 39, 65, 66].

An alternative approach to data assimilation has recently been rigorously analyzed in the context of dissipative PDEs in [7, 8]. Motivated by earlier work, mainly in the context of finite dimensional dynamical systems governed by ordinary differential equations and early work in meteorology [4, 6, 14, 42, 54, 64, 62, 68], this algorithm employs a feedback control paradigm via a *Newtonian relaxation scheme* (*nudging*). This was in turn predicated on the notion of finite determining functionals (modes, nodes, volume elements) for dissipative systems, the rigorous existence of which was first established in [35, 36, 37, 43]. Due to the fact that it was first studied rigorously in fluid dynamics in [7, 8], we henceforth refer to the system associated to the nudging algorithm as the *Azouani–Olson–Titi (AOT) system.* The AOT system was later generalized to include various other models and convergence in stronger norms (e.g., the analytic Gevrey class) [2, 13, 28, 30, 31, 32, 33, 51, 55], as well as to more general situations such as discrete in time and error-contaminated measurements and to statistical solutions [10, 12, 34]. This method has been shown to perform remarkably well in numerical simulations [3, 29, 38, 40, 41, 47] and has recently been successfully implemented for the first time for efficient dynamical downscaling of a global atmospheric reanalysis [25]. Recent applications include its implementation in *reduced order modeling (ROM)* of turbulent flows to mitigate inaccuracies in ROM [67], and in inferring flow parameters and turbulence configurations [26, 18].

We will now give a schematic description of the AOT system. Assuming that the observations are generated from a continuous dynamical system given by

$$\frac{d}{dt}u = F(u), u(0) = u_0,$$

the AOT algorithm entails solving an associated system

$$\frac{d}{dt}w = F(w) - \mu(I_h w - I_h u), w(0) = w_0 \text{ (arbitrary)}, \tag{1.1}$$

where I_h is a finite rank linear operator acting on the phase space, called an *interpolant operator*, constructed *solely from observations* on u (e.g., low (Fourier) modes of u or values of u measured in a coarse spatial grid). Here h refers to the size of the spatial grid or, in case of the *modal interpolant*, the reciprocal of h stands for the number of observed modes. Moreover, $\mu > 0$ is the *relaxation/nudging parameter. An appropriate choice of μ is crucial* in establishing that the AOT system (1.1) is well-posed and that its solution *tracks* the solution of the original system asymptotically, i.e., $\|w - u\| \longrightarrow 0$ as $t \to \infty$ in a suitable norm.

Here, we consider the well-posedness, stability, and convergence/asymptotic tracking property of solutions of the AOT system for the three-dimensional incompressible Navier–Stokes equations (3D NSE). Although numerical simulations demonstrating the efficacy of the AOT algorithm for the 3D NSE have recently been demonstrated in [27], to the best of our knowledge, this is the first such rigorous analytical result for the 3D NSE. Our starting point in the analysis is observations on *finitely many* modes or volume elements (more generally, type 1 observations [7]) of a given solution u of the 3D NSE *with an arbitrarily large Grashoff number* and with either space periodic or Dirichlet boundary conditions. We consider a Leray–Hopf weak solution, satisfying a certain condition *based on observations* (more precisely (1.6) in case of *modal* observations or (3.8) for *volume element* observation), which subsequent work has shown to imply eventually regularity (and regularity in case the solution is on the weak attractor). More importantly from our perspective, unlike in data assimilation for the 2D NSE or the α-models of turbulence in the 3D case [1, 2, 7, 28, 33] *we do not know the bound*

$$M = \sup_{[T_0,\infty)} \|u\|_{\mathbb{H}^1} \quad (T_0 \geq 0), \tag{1.2}$$

where T_0 is the time beyond which the solution is regular. Note that global regularity simply means that $\sup_{[0,T]} \|u\|_{\mathbb{H}^1} < \infty$ for all $T < \infty$, and unlike in the 2D case, *it is quite possible for a globally regular solution* to satisfy $\limsup_{t\to\infty} \|u\|_{\mathbb{H}^1} = \infty$ (i.e., $M = \infty$, where M is as in (1.2)). Furthermore, in all the cases mentioned before where a rigorous analysis of the AOT system is available, including the 3D cases such as the Navier–Stokes-α models [1, 2, 33] and the Boussinesq system with a large Prandtl number [28], it is known that an absorbing ball exists with an explicit estimate (in terms of the physical parameters) of a uniform bound of its $\mathbb{H}^1$-norm. This in turn means that not only M as in (1.2) is finite for a sufficiently large T_0, an estimate of it in terms of the physical parameters of the system is available. This fact is critical in setting the value of the parameter μ in (1.1) that guarantees well-posedness, stability, and the asymptotic tracking property of solutions of the AOT system. Moreover, these bounds are used in providing an upper bound on the spatial resolution h of the observations (or lower bound on the number of observed low modes) necessary for data assimilation. Such a bound M is unknown for the 3D NSE, even for a globally regular solution.

In this work, we circumvent this difficulty by *identifying a condition on the observed data* (e.g., (1.6) for modal observations or (3.23) for a more general type 1 observation operator) which leads us to appropriately set the value of μ in (1.1) such that the AOT system is well-posed and its solution asymptotically tracks the solution u exponentially. We note further that unlike in the 2D case where one uses the 2D embedding inequalities, the global well-posedness of the AOT system does not follow from the usual Sobolev embedding inequalities in three dimensions; instead it crucially depends on the choice of μ, which in turn depends on condition (1.6) or (3.23). Moreover, our condition does not depend on any a priori knowledge of the regularity, or the bound M in (1.2), which may not even be finite. We further emphasize *that our result applies quite generally to an arbitrarily large Grashoff number.*

1.1. Connection to regularity. At the time of submission of this manuscript, it was unknown whether the conditions (1.6) or (3.23) imply regularity, although as noted in the discussion below, an (eventual) uniform bound as in (1.2) implies our condition. In a follow-up work [9] of the first author (together with another collaborator), we have established that (1.6) and (3.23) are in fact *new, observable regularity criteria* on the weak global attractor of the 3D NSE. This result, which is obtained a posteriori, and without any a priori assumption of regularity on the Leray–Hopf weak solution u, essentially uses the techniques developed in this paper. Additionally, for general Leray–Hopf weak solutions not necessarily on the weak attractor, it is now known that our conditions also imply *eventual regularity.* More precisely, if either of (1.6) or (3.23) holds with $T = \infty$, then there exists a time T_0 such that the solution of the 3D NSE u is regular on $[T_0, \infty)$, i.e., $\sup_{[T_0,T_1]} \|u\|_{\mathbb{H}^1} < \infty$ for all fixed T_1 satisfying $T_0 < T_1 < \infty$. This result was obtained by Cheskidov and Titi [21] and involves a contradiction argument that crucially makes use of the asymptotic tracking property (e.g., Theorems 3.3 and 3.6) obtained here.

Examples of nonstationary 3D flows with an arbitrarily large Grashoff number which are either regular or eventually regular include Navier–Stokes equations in a thin 3D domain [52, 53, 56], large Prandtl number flows in case of the Boussinesq system [69, 70], as well as 2D flows which can be considered as "embedded" in 3D flows.

1.2. Overview. We will give a brief description of our results in the case of the *modal interpolant (observable)*, even though our results are applicable to the more general case of a *type* 1 *interpolant operator* (see section 2 or [8, 7] for a definition). Let A denote the Stokes operator (see section 2 or [23, 63]) with either the space periodic or the homogeneous Dirichlet boundary condition. It is well-known that A is a positive self-adjoint operator with a compact inverse (on appropriate functional spaces as described in section 2) with eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$, with $\lambda_K \to \infty$. Let the modal interpolant P_K be orthogonal projection on the (finite dimensional) space spanned by eigenvectors corresponding to eigenvalues $\lambda_1, \ldots, \lambda_K$. Let the Navier–Stokes equations be given by (2.1) (or in its functional form (2.2)) and let G denote the (nondimensional) Grashoff number

$$G = \frac{\|f\|_{L^2}}{\nu^2 \lambda_1^{3/4}},$$

where ν is the kinematic viscosity and f is the body force. Let u be a Leray–Hopf weak solution of the 3D NSE (see Definition 2.1), satisfying the energy bound $\|u(t)\|_{L^2} \lesssim G\nu\lambda_1^{-1/4}$ for $t \geq t_*$ [23]. Henceforth, $a \lesssim b$ means $a \leq Cb$ and $a \gtrsim b$ means $a \geq Cb$, for some nondimensional constant C which may depend on the Sobolev constants (i.e., the domain), but not on any other physical parameters. Moreover, the notation $a \sim b$ means that both $a \lesssim b$ and $b \lesssim a$ hold.

Following Theorem 3.2 (see (3.7)), define

$$M_{K,u}^2 := 8\left(\frac{\|f\|_{L^2}^2}{\nu^2\lambda_1} + \sup_{t\in[t_0,T)} \|P_K(u)\|_{\mathbb{H}^1}^2\right), \text{ where } t_0 < T \leq \infty.$$

We first proceed to show that $M_{K,u}$ as defined above is finite. To that end, using the fact that [23, 63]

$$\|P_K u\|_{\mathbb{H}^1} \sim \|A^{1/2} P_K u\|_{L^2} \leq \lambda_K^{1/2} \|u\|_{L^2},$$

we have

$$\sup_{t\in[t_0,\infty)} \|P_K u\|_{\mathbb{H}^1} \lesssim \lambda_K^{1/2} \sup_{t\in[t_0,\infty)} \|u\|_{L^2} \lesssim \lambda_K^{1/2}\lambda_1^{-1/4} G\nu,$$

where in the last inequality, we used the uniform bound on the L^2-norm of a Leray–Hopf weak solution u noted before. We thus conclude that in the definition above, $M_{K,u} < \infty$.

With the above notation, our sufficient condition for the AOT algorithm (1.1) to be well-posed, admitting a "regular" solution w, and possessing the (exponential) tracking property (i.e., $\|w - u\|_{L^2} \lesssim e^{-c\mu t}$) on the time interval $[t_0, T)$ with $t_0 < T \leq \infty$, is given by (see (3.8))

$$\nu \max\left\{\frac{2cM_K^4}{\nu^4}, \lambda_1\right\} \leq \frac{\nu\lambda_K}{4}. \tag{1.3}$$

This is equivalent to

$$\frac{2\sqrt{2c}M_{K,u}^2}{\nu^2} \leq \lambda_K^{1/2} \text{ and } 4\lambda_1 \leq \lambda_K.$$

In other words, denoting $c_1 = 16\sqrt{2c}$ and using the definition of $M_{K,u}$ above, we need K satisfying

$$c_1 \left(\frac{\|f\|_{L^2}^2}{\nu^4 \lambda_1} + \frac{\sup_{t\in[t_0,T)} \|P_K(u)\|_{\mathbb{H}^1}^2}{\nu^2} \right) \leq \lambda_K^{1/2} \,\text{and}\, 4\lambda_1 \leq \lambda_K. \tag{1.4}$$

Let K_0 be the smallest natural number such that we have

$$\frac{2c_1\|f\|_{L^2}^2}{\nu^4\lambda_1} \leq \lambda_K^{1/2} \text{ and } 4\lambda_1 \leq \lambda_K \ \forall\ K \geq K_0. \tag{1.5}$$

Since λ_K is strictly increasing in K and $\lambda_K \to \infty$ as $K \to \infty$, the existence of such a K_0 (which in fact is unique since λ_K is strictly monotone) is guaranteed. Note that no smallness condition on f is necessary for this to hold. However, a larger value of $\|f\|_{L^2}$ will necessitate a larger value of K_0. Now, once K_0 is thus defined, provided there exists $K \geq K_0$ which satisfies

$$\frac{2c_1 \sup_{t\in[t_0,T)} \|P_K(u)\|_{\mathbb{H}^1}^2}{\nu^2} \leq \lambda_K^{1/2},$$

then λ_K satisfies (1.4) and, consequently, (1.3). Thus, with K_0 as in (1.5), one may reformulate (1.3) as

$$\exists\ K \in \mathbb{N}, K \geq K_0 \text{ such that } \frac{\lambda_K^{-1/4}}{\nu} \sup_{[t_0,T)} \|P_K u\|_{\mathbb{H}^1} \leq \frac{c}{2}. \tag{1.6}$$

A similar condition can be formulated for a more general type 1 interpolant (e.g., volume element) as well (see Theorems 3.5 and 3.6).

Remark 1.1. We now note the following concerning condition (1.6), and more generally concerning Theorems 3.2, 3.3, 3.5, and 3.6.

1. The size of $\|f\|_{L^2}$ (i.e., the Grashoff number G) occurring in the definition of $M_{K,u}$ above *can be arbitrarily large*. A larger Grashoff number will in turn necessitate a larger value of K_0 in (1.5). We note that in the $2D$ *case*, where regularity of u is known, with $\|u\|_{\mathbb{H}^1} \lesssim \nu\lambda_1^{1/2}G$ in the space periodic case, (1.6) holds for $K \sim K_0$. This is indeed the case due to

$$\|P_K u\|_{\mathbb{H}^1} \sim \|A^{1/2}P_K u\|_{L^2} \leq \|A^{1/2}u\|_{L^2} \sim \|u\|_{\mathbb{H}^1}. \tag{1.7}$$

   Therefore, in the 2D space periodic case, (1.5) and (1.6) simplify to a single condition, namely $\lambda_K \geq C_{\nu,\lambda_1}G^4$. A similar bound for the number of modal observations necessary for the AOT algorithm to converge *in the* $2D$ *case* was obtained in their seminal work by Azouani, Olson, and Titi [7] (though their bound is much sharper for the 2D space periodic case, namely, $\lambda_K \geq C(\nu,\lambda_1)G^{1/2}$, due to the fact that they used 2D Sobolev embedding inequalities in their derivation). Thus, (1.3) may be viewed as a generalization of the 2D result in [7] to the 3D setting.

2. The condition (1.6) *depends only on the observed part of the data* $P_K u$ (i.e., the low modes) and does not involve the high modes. This ensures that we can construct the data assimilation system (1.1) based only on the observed data, provided (1.6) holds, which can in turn be verified from the observations. Moreover, the value of the tuning parameter μ, crucial in setting up (1.1), can be determined from the observed data as prescribed in (3.8).
3. Denote

$$K_{inf} = \min\{K : K \text{ satisfies } (1.3)\,(\text{or } (1.6))\}. \tag{1.8}$$

Suppose that u is regular on the interval $[t_0, T)$ where $t_0 < T \le \infty$ and satisfies

$$\sup_{[t_0,T)} \|u\|_{\mathbb{H}^1} < \infty. \tag{1.9}$$

Due to (1.7) and the fact that $\lambda_K \to \infty$ as $K \to \infty$, clearly, in this case, we have $K_{inf} < \infty$. However, K_{inf} (equivalently $\lambda_{K_{inf}}$) as defined in (1.8) may be much smaller than the one determined by the upper bound provided by (1.9).
4. At the time of the submission of the manuscript, the converse direction of part 3 above was unknown. Subsequently, using the techniques developed here, we have shown in [9] that (1.6) implies regularity on the weak attractor of the 3D NSE. More precisely, if $\{u(t) : t \in \mathbb{R}\}$ is a Leray–Hopf weak solution on the weak global attractor of the 3D NSE satisfying $\frac{\lambda_K^{-1/4}}{\nu} \sup_{(-\infty,T]} \|P_K u\|_{\mathbb{H}^1} \le c$, then u is regular on $(-\infty, T]$. Additionally, it is now also known that condition (1.6) with $T = \infty$ implies eventual regularity, i.e., if (1.6) holds with $T = \infty$, then there exists T_0 such that u is regular on $[T_0, \infty)$, i.e., for all T_1 satisfying $T_0 < T_1 < \infty$, $\sup_{[T_0,T_1]} \|u\|_{\mathbb{H}^1} < \infty$. This result, which uses a contradiction argument and the asymptotic tracking property established here in Theorems 3.3 and 3.6, was obtained by Cheskidov and Titi [21].
5. Assume that u satisfies (1.6) with $T = \infty$. Then, due to the eventual regularity result due to Cheskidov and Titi [21], we know that u is regular on some interval $[T_0, \infty)$ and $\sup_{[T_0,\infty)} \|u\|_{\mathbb{H}^1} < \infty$. An interesting question that arises here is whether it is possible for $\sup_{[T_0,\infty)} \|u(t)\|_{\mathbb{H}^1} = \infty$ for a solution that is regular on $[T_0, \infty)$. Although we do not know the answer to this question, we note that if such a regular solution u exists, then there exists another solution $\tilde{u}$ of the 3D NSE with initial data $\tilde{u}_0 \in \mathbb{H}^1$ and a finite time $T_0 < \infty$ such that $\tilde{u}$ is regular on $[t_0, T_0)$ but not at T_0 (i.e., $\limsup_{t \nearrow T_0} \|u(t)\|_{\mathbb{H}^1} = \infty$) [23]. In other words, if solutions of the 3D NSE are globally regular *for all initial data* in $\mathbb{H}^1$, then in fact (1.9) holds.
6. It is shown in Theorem 3.4 that the wave number K_{inf} identified in (1.8) is a determining wave number for the 3D NSE. A different notion of a time-varying determining wave number $\Lambda(t)$ for the 3D, space periodic NSE is given in [19, 20]. One crucial difference between them is that K_{inf} can be determined by observing finitely many modes while determining the time dependent wave number in [19, 20] requires the knowledge of all Fourier modes of u; a detailed discussion and comparison is provided in Remark 3.4. However, $\Lambda(t)$ is known to be a determining wave number for any Leray–Hopf weak solution. On the other hand, the pointwise analogue of K_{inf} is not known to be a determining

wave number for a possibly non-regular Leray–Hopf solutions (if they exist) at this point.

The organization of the paper is as follows. In section 2, we establish the requisite notation and state preliminary results and facts. In section 3, we state and prove our main results, while in section 4, we provide an adaptive version of our algorithm which might be useful for computational purposes when the flow is turbulent in certain intervals of time.

2. Notation and preliminaries. The 3D NSE on a domain Ω with time independent forcing (assumed for simplicity) is given by

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f,\ \nabla\cdot u = 0\ for\ t\in(0,\infty), \tag{2.1}$$

with the initial condition $u(0) = u_0$. Here u denotes the velocity of the fluid, p denotes the pressure, ν is the kinematic viscosity, and f is the body force. Concerning the boundary conditions, we assume either that $\Omega\subset\mathbb{R}^3$ is bounded with boundary $\partial\Omega$ of class C^2 and $u|_{\partial\Omega} = 0$ or that $\Omega = [0, L]^3$ and u is space periodic with space period L in all variables with space average zero, i.e., $\int_\Omega u = 0$. For simplicity, we also take the body force f to be time independent with $f \in (L^2(\Omega))^3$.

We briefly introduce the functional framework for (2.1); for a more detailed discussion see [23, 63]. For $\alpha \geq 0$, we will denote by $H^\alpha(\Omega)$ the usual Sobolev space of order α. In the case of the *periodic boundary conditions* we consider $\mathcal{V}$ as the set of all L-periodic trigonometric polynomials from $\mathbb{R}^3$ to $\mathbb{R}^3$ that are divergence free and have zero average. In the case of the *no-slip Dirichlet boundary conditions* we consider $\mathcal{V}$ as the set of all C^∞ vector fields from Ω to $\mathbb{R}^3$ that are divergence free and compactly supported. Then H is the closure of $\mathcal{V}$ with respect to the norm in $(L^2(\Omega))^3$ and V is the closure of $\mathcal{V}$ with respect to the norm in $(\mathbb{H}^1(\Omega))^3$. The inner products in H and V are given by

$$(u, v) = \int_\Omega u(x)\cdot v(x)dx\ \forall\ u, v \in H$$

and

$$((u, v)) = \int_\Omega \sum_{i=1}^{3} \frac{\partial u}{\partial x_i}\cdot\frac{\partial v}{\partial x_i}dx\ \forall u, v \in V,$$

and the corresponding norms are given by $|u| = (u, u)^{1/2}$ and $\|u\| = ((u, u))^{1/2}$, respectively. We also denote by $\mathbb{P}_\sigma$ the Leray–Hopf orthogonal projection operator from $L^2(\Omega)$ to H. The Stokes operator is given by

$$Av = \mathbb{P}_\sigma(-\Delta)v, v \in H^2(\Omega)\cap V.$$

We recall that A is a positive self-adjoint operator with a compact inverse and $D(A) = \{u \in V : Au \in H\}$. Moreover, there exists a complete orthonormal set of eigenfunctions $\phi_i \in H$ such that $A\phi_i = \lambda_i\phi_i$ where $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ are the eigenvalues of A repeated according to multiplicity. In case $u \in D(A^\alpha), \alpha \geq 0$, then $u \in H^{2\alpha}(\Omega)$ and $\|u\|_{H^{2\alpha}} \sim |A^\alpha u|$. With the notation above, by applying the projection $\mathbb{P}_\sigma$ to (2.1), we may express the NSE in the functional form as

$$\frac{du}{dt} + \nu Au + B(u, u) = f,\ \nabla\cdot u = 0\ for\ t \in (0,\infty), u(0) = u_0, \tag{2.2}$$

where $B(u, u) = \mathbb{P}_\sigma(u\cdot\nabla)u$ an for simplicity and where, by an abuse of notation, we denote $\mathbb{P}_\sigma f$ by f.

We denote by H_N the space spanned by the first N eigenvectors of A and the orthogonal projection from H onto H_N is denoted by P_N. We also recall the Poincaré inequality

$$\lambda_1^{1/2}|v| \le \|v\|, v \in V. \tag{2.3}$$

Let V' denote the dual space of V. The bilinear continuous operator B from $V \times V$ to V' is defined by

$$\langle B(u,v), w\rangle = \sum_{i,j} \int_\Omega u_i(\partial_i v_j) w_j, u, v, w \in V,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between V and V'. The bilinear term B satisfies the orthogonality property

$$\langle B(u,w), w\rangle = 0 \ \forall u, w \in V.$$

Moreover, the bilinearity of B implies

$$B(u,u) - B(w,w) = B(u-w,u) - B(w,w-u) = B(\widetilde{w},u) + B(w,\widetilde{w}). \tag{2.4}$$

We recall some well-known bounds on the bilinear term in the 3D case which appear in [58, 63].

Proposition 2.1. *If $u \in V$, $v \in D(A)$ and $w \in H$, then $B(u,v) \in H$ and*

$$|(B(u,v),w)| \le c\|u\|_{L^6}\|\nabla v\|_{L^3}\|w\|_{L^2} \le c\|u\|\|v\|^{1/2}|Av|^{1/2}|w|. \tag{2.5}$$

Moreover, if $u, v, w \in V$, then $B(u,v) \in H$ and

$$|(B(u,v),w)| \le c\|u\|_{L^4}\|\nabla v\|_{L^2}\|w\|_{L^4} \le c|u|^{1/4}\|u\|^{3/4}\|v\||w|^{1/4}\|w\|^{3/4}. \tag{2.6}$$

Definition 2.1. *u is said to be a* weak solution *of* (2.2) *if for all $T>0$, u belongs to $L^\infty([0,T];H) \cap L^2([0,T];V) \cap C([0,T];V') \cap C([0,T];H_w)$, $\frac{du}{dt} \in L^1((0,T);V')$ and satisfies a.e. t*

$$\frac{d}{dt}(u,v) + \nu((u,v)) + (B(u,u),v) = (f,v) \ \forall v \in V \ \text{and} \ u(0) = u_0.$$

In the above definition, H_w denotes the Hilbert space H equipped with its weak topology.

A weak solution u is said to be a Leray–Hopf weak solution if it additionally satisfies for a.e. $t_0 \in [0,\infty)$, including at $t_0 = 0$, the energy inequality

$$\frac{1}{2}|u(t)|^2 + \nu\int_{t_0}^t \|u(s)\|^2\,ds \le \frac{1}{2}|u(t_0)|^2 + \int_{t_0}^t (f,u(s))\,ds, \quad t \ge t_0, \tag{2.7}$$

as well as the global (in time) energy bound

$$|u(t)|^2 \le e^{-\nu\lambda_1 t}|u_0|^2 + \frac{|f|^2}{\nu^2\lambda_1^2}\left(1 - e^{-\nu\lambda_1 t}\right), t \ge 0. \tag{2.8}$$

A weak solution is said to be a strong solution *if it also belongs to $L^\infty((0,T);V) \cap L^2((0,T);D(A))$.*

Note that the equality $u(0) = u_0$ makes sense as $u \in C([0,T]; V')$. Moreover, it follows immediately from (2.8) that for any Leray–Hopf weak solution, there exists a time $t_* = t_*(u_0)$ such that

$$|u(t)|^2 \leq 2G^2\nu^2\lambda_1^{-1/2} \ \forall\, t \geq t_*, \text{ where the Grashoff number } G := \frac{|f|}{\nu^2\lambda_1^{3/4}}. \tag{2.9}$$

The Galerkin approximation corresponding to (2.2) is given by the solution u_N of the following Galerkin system:

$$\begin{aligned} \frac{du_N}{dt} + \nu A u_N + P_N B(u_N, u_N) &= P_N f, \\ \nabla \cdot u_N &= 0, \\ u_N(0) &= P_N u(0). \end{aligned} \tag{2.10}$$

The following theorem due to Leray [23, 58, 63] gives us the existence of weak solutions to (2.2) in three dimensions.

THEOREM 2.1. *Let $T > 0$ and $f \in L^2_{loc}([0,T]; V^*)$. Then if $u_0 \in H$, there is a weak solution of* (2.2) *such that*

$$u \in L^\infty([0,T]; H) \cap L^2([0,T]; V),$$

and the equation holds as an equality in $L^{4/3}([0,T]; V')$. Moreover, there exists a subsequence $\{u_{N_k}\}$ which converges to a weak solution u, in the topology of $L^2([0,T]; H) \cap C([0,T]; V') \cap C([0,T]; H_w)$, as well as weakly in $L^2([0,T]; V)$.

DEFINITION 2.2. *Following* [49]*, we will say that u is a* restricted Leray–Hopf weak solution *if it is obtained as a subsequential limit of a Galerkin system where the convergence is as given in Theorem* 2.1. *We will denote by $\mathfrak{W}_{u_0}$ the set of all restricted weak solutions of* (2.2) *with initial data $u_0 \in H$ and we denote $\mathfrak{W} = \bigcup_{u_0 \in H} \mathfrak{W}_{u_0}$.*

It can be shown [23] that any $u \in \mathfrak{W}$ is in fact a Leray–Hopf weak solution, thus justifying the terminology in Definition 2.2.

Remark 2.1. The existence of weak solutions via the Galerkin construction shows that the class $\mathfrak{W}_{u_0}$, which is contained in the set of all Leray–Hopf weak solutions, is nonempty. It is presently unknown whether the class of restricted weak solutions (and therefore Leray–Hopf weak solutions) are unique, i.e., whether or not the cardinality of $\mathfrak{W}_{u_0}$ is one. However, recent results in [17] (see also [50] for the hyperdissipative NSE) show that weak solutions are not unique, although the problem of uniqueness for Leray–Hopf weak solutions (i.e., weak solutions which additionally satisfy the energy inequality (2.7)) remain open as of this writing. On the other hand, the well-known weak-strong uniqueness result of Sather and Serrin [61] says that if a strong solution of (2.2) exists on $[0,T]$, then all Leray–Hopf weak solutions coincide with the strong solution.

2.1. Interpolant operators.

DEFINITION 2.3. *A* finite rank, bounded linear operator $I_h : H \to L^2(\Omega)$ *is said to be a type* 1 *interpolant observable if there exists a dimensionless constant $c > 0$ such that*

$$|I_h v| \leq c|v| \ \forall\, v \in H \ \textit{and} \ |I_h v - v| \leq ch\|v\| \ \forall\, v \in V. \tag{2.11}$$

The orthogonal projection operator P_K, also known as the *modal interpolant*, provides such an example. Indeed, it is easy to check that it satisfies (2.11):

$$|P_K v| \le |v| \ \forall\ v \in H \text{ and } |P_K v - v| \le \frac{1}{\lambda_K^{1/2}}\|v\| \ \forall\ v \in V. \tag{2.12}$$

Thus (2.11) is satisfied with $h = \frac{1}{\sqrt{\lambda_K}}$.

Another physically relevant example of a type 1 interpolant, which is important from the point of view of applications, is the *volume element interpolant* [7, 43]. For the case of the periodic boundary conditions, the volume element interpolant is given by

$$I_h(\phi)(x) = \sum_{j=1}^{N} \bar{\phi}_j \left(\chi_{Q_j}(x) - \frac{h^3}{L^3}\right), \quad \textit{where} \quad \bar{\phi}_j = \frac{1}{h^3}\int_{Q_j} \phi(x)dx, \tag{2.13}$$

where the periodic domain with side length L has been divided into equal cubes Q_j of side length h and χ_E denotes the characteristic function of a Borel set E. As shown in [37, 43, 44], the volume element interpolant satisfies (2.11). One can analogously define the volume element interpolant in case of the homogeneous Dirichlet boundary condition, with appropriate modifications.

3. Well-posedness and the tracking property of the data assimilation system. For the remainder of the paper, *we will assume that* $u \in \mathfrak{W}$ *is a restricted global weak solution* of (2.2) corresponding to initial data $u_0 \in V$, i.e., u can be approximated by a sequence of solutions $\{u_N\}$ of the Galerkin system in the following way: $u_N \to u$ weakly in $L^2(0,T;V)$, strongly in $L^2(0,T;H)$, and in $C(0,T;V')$ (equipped with the sup-norm on $[0,T]$). We begin by describing the AOT data assimilation system that we consider here. The observations are given by

$$\text{Observation } \mathcal{O} = \{I_h u(t)\}_{t\ge 0} \tag{3.1}$$

where I_h is a type 1 interpolant (e.g., either a modal or volume interpolant). Since I_h is of finite rank and $u \in C([0,T];V')$, the mapping $t \to I_h u(t)$ from $[0,\infty)$ to $L^2(\Omega)$ is continuous.

Our data assimilation algorithm is given by the solution w of the equation

$$\begin{aligned} \frac{dw}{dt} + \nu A w + B(w,w) &= f - \mu \mathbb{P}_\sigma I_h(w-u), \\ \nabla \cdot w &= 0, \\ w(0) &= 0, \end{aligned} \tag{3.2}$$

where, as in (2.2), f is time independent and $f \in (L^2(\Omega))^3$. Furthermore, since $u(\cdot)$ is weakly continuous in t, i.e., $u \in C([0,\infty);H_w)$, it follows that $I_h u(\cdot)$ is well-defined and continuous in t with values in $L^2(\Omega)$. We made the choice $w(0)=0$ for specificity. However, the AOT data assimilation system can be initialized by any initial value. Recall that for notational simplicity, we have assumed that f is time independent. The Galerkin approximation of w is given by the solution of the equation

$$\begin{aligned} \frac{dw_N}{dt} + \nu A w_N + P_N B(w_N,w_N) &= P_N f - \mu P_N I_h(w_N-u), \\ \nabla \cdot w_N &= 0, \\ w_N(0) = P_N w(0) &= 0. \end{aligned} \tag{3.3}$$

3.1. Global existence of a weak solution. We will now show existence of a global (in time) weak solution of (3.2), where the definition of a weak solution is similar to Definition 2.1. As in the case of the 3D NSE, we proceed by establishing a priori bounds on the Galerkin system (3.3). Henceforth, by translating time if necessary, we will assume that the weak solution $u \in \mathfrak{W}$ satisfies (2.9) for all $t \geq 0$.

THEOREM 3.1. *Let $u \in \mathfrak{W}$ satisfy* (2.9) *for all $t \geq 0$ and let I_h be any type* 1 *interpolant satisfying* (2.11). *Then, provided*

$$\nu\lambda_1 \leq \mu \leq \frac{\nu}{4ch^2} \quad (c \text{ as in } (2.11)), \tag{3.4}$$

there is a weak solution w of (3.2) *such that for any $T > 0$,*

$$w \in L^\infty(0,T;H) \cap L^2(0,T;V) \text{ with } |w(t)|^2 \lesssim G^2\nu^2\lambda_1 \ \forall t \geq 0,$$

and the equation holds as an equality in $L^{4/3}(0,T;V')$. Moreover, there exists a subsequence $\{w_{N_k}\}$ which converges to a weak solution w weakly in $L^2([0,T];V)$, strongly in $L^2([0,T];H)$, and in $C([0,T];V')$.

Proof. We will start by establishing a priori estimates on the Galerkin system. Taking inner product of (3.3) with w_N, we readily obtain (after some elementary algebra)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&|w_N|^2 + \nu\|w_N\|^2 \\ &= (f,w_N) - \mu|w_N|^2 + \mu(I_h w_N - w_N, w_N) + \mu(I_h u, w_N) \\ &\leq \frac{|f|^2}{\nu\lambda_1} + \frac{\nu\lambda_1}{4}|w_N|^2 + \frac{\mu}{4}|w_N|^2 + \mu c h^2\|w_N\|^2 \\ &\quad + \mu|I_h u|^2 + \frac{\mu}{4}|w_N|^2 - \mu|w_N|^2,\end{aligned} \tag{3.5}$$

where to obtain (3.5), we used Cauchy–Schwarz and Young inequalities, in conjunction with the second inequality in (2.11). Using the inequalities (2.3), (2.9) and the first inequality in (2.11), we readily obtain

$$\frac{d}{dt}|w_N|^2 + \mu|w_N|^2 + \nu\|w_N\|^2 \leq \frac{|f|^2}{\nu\lambda_1} + c\mu G^2\nu^2\lambda_1. \tag{3.6}$$

Dropping the last term from the left and applying Gronwall inequality together with (3.4) and recalling $w_N(0) = 0$, we get

$$|w_N|^2 \leq \frac{|f|^2}{\mu\nu\lambda_1} + cG^2\nu^2\lambda_1 \lesssim G^2\nu^2\lambda_1.$$

Integrating both sides of (3.6) and inserting the above bound, we immediately obtain

$$\nu\int_0^T \|w_N\|^2 \lesssim (1+\mu T)\, G^2\nu^2\lambda_1.$$

The remainder of the proof is similar to the proof of existence of weak solutions of the 3D NSE [23, 63]. ∎

3.2. Global existence of a strong solution and tracking property. Thus far, no assumption on the solution u was necessary to establish existence of a weak solution of (3.2). In order to ensure global existence of a (hence the) regular solution of (3.2) and to establish the tracking property (i.e., to show that it tracks u asymptotically), we need to impose conditions on the *observed data* coming from the solution u. For clarity of exposition, we will consider the case of modal interpolant first before proceeding to a more general type 1 interpolant.

3.2.1. Modal interpolant case (i.e., $I_h = P_K$). Before we proceed, we first note that $\sup_{t\geq 0} \|P_K u\| < \infty$. Indeed,

$$\|P_K u\| \leq \lambda_K^{1/2} |P_K u| \lesssim \lambda_K^{1/2} G^2 \nu^2 \lambda_1,$$

where the last inequality follows from (2.9). However, as we will see below (see Remark 3.3 and (3.17)), this bound is insufficient to guarantee that w tracks u. We require a more stringent bound as given in (3.8) or, equivalently, as in (3.17).

THEOREM 3.2. *Suppose $I_h = P_K$ is a modal interpolant which satisfies* (2.12). *Let $0 < T \leq \infty$ and denote*

$$M_K^2 = M_{K,u}^2 := 8\left(\frac{|f|^2}{\nu^2\lambda_1} + \sup_{t\in[0,T)} \|P_K(u)\|^2\right). \tag{3.7}$$

Assume that

$$\nu \max\left\{\frac{2cM_K^4}{\nu^4}, \lambda_1\right\} \leq \mu \leq \frac{\nu\lambda_K}{4}. \tag{3.8}$$

Then any weak solution of (3.2) *constructed in Theorem* 3.1 *as a subsequential limit of w_N satisfying* (3.3) *is regular on* $[0, T]$, *i.e., it satisfies*

$$\|w(t)\| \leq M_K, t \in [0, T]. \tag{3.9}$$

Additionally, any two strong solutions w_1, w_2 of (3.2) *on the interval* $[0, T]$ *(i.e., solutions satisfying* $\sup_{t\in[0,T]} \|w_i(t)\|_V = \sup_{t\in[0,T]} \|w_i(t)\| < \infty, i = 1, 2$*) coincide.*

Proof. As is customary, we will obtain a priori estimates on the Galerkin system and then pass to the limit. We begin by rearranging (3.3) with the assumption $N \geq K$ to first obtain

$$\frac{dw_N}{dt} + \nu A w_N + P_N B(w_N, w_N) = f - \mu P_N(P_K(w_N) - w_N) + \mu P_K(u) - \mu w_N.$$

Now taking the inner product with Aw_N yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w_N\|^2 + \nu|Aw_N|^2 + \mu\|w_N\|^2 \\
&= (f, Aw_N) - (B(w_N, w_N), Aw_N) \\
&\quad - \mu(P_K(w_N) - w_N, Aw_N) + \mu(P_K(u), Aw_N).
\end{aligned} \tag{3.10}$$

Each term on the right-hand side is estimated below. First, using Cauchy–Schwarz and Young's inequalities, we have

$$|(f, Aw_N)| \leq \frac{1}{\nu}|f|^2 + \frac{\nu}{4}|Aw_N|^2.$$

Next, using (2.5) and Young's inequality, we obtain

$$|(B(w_N,w_N),Aw_N)| \le c\|w_N\|^{3/2}|Aw_N|^{3/2} \le \frac{c}{\nu^3}\|w_N\|^6 + \frac{\nu}{4}|Aw_N|^2.$$

Observe now that from (2.12) and Young's inequality, we obtain

$$\begin{aligned}\mu|(P_K(w_N)-w_N,Aw_N)| &\le \mu|P_K(w_N)-w_N||Aw_N| \le \mu\lambda_K^{-1/2}\|w_N\||Aw_N|\\ &\le \frac{\mu^2}{\nu\lambda_K}\|w_N\|^2 + \frac{\nu}{4}|Aw_N|^2 \le \frac{\mu}{4}\|w_N\|^2 + \frac{\nu}{4}|Aw_N|^2.\end{aligned}$$

Moreover,

$$\begin{aligned}&\mu|(P_K(u),Aw_N)| = \mu|(A^{1/2}P_K(u),A^{1/2}w_N)|\\ &\le \mu\|P_K(u)\|\|w_N\| \le \mu\|P_K(u)\|^2 + \frac{\mu}{4}\|w_N\|^2.\end{aligned}$$

Inserting these estimates into (3.10), we obtain

$$\frac{d}{dt}\|w_N\|^2 + \left(\mu - \frac{c}{\nu^3}\|w_N\|^4\right)\|w_N\|^2 \le \frac{2}{\nu}|f|^2 + 2\mu\|P_K(u)\|^2. \tag{3.11}$$

Let $[0,T_1]$ be the maximal interval on which $\|w_N(t)\| \le M_K$ holds for $t \in [0,T_1]$ where M_K as in (3.7). Note that $T_1 > 0$ exists because we have $w_N(0) = 0$. Assume that $T_1 < T$. Then by continuity, we must have $\|w_N(T_1)\| = M_K$. Using the lower bound for μ in (3.8), for all $t \in [0,T_1]$, we obtain

$$\frac{d}{dt}\|w_N\|^2 + \frac{\mu}{2}\|w_N\|^2 \le \frac{2}{\nu}|f|^2 + 2\mu \sup_{s\in[0,T]}\|P_K(u(s))\|^2.$$

Since $w_N(0) = 0$, by the Gronwall inequality we immediately obtain

$$\|w_N\|^2 \le \frac{4}{\nu^2\lambda_1}|f|^2 + 4\sup_{s\in[0,T)}\|P_K(u(s))\|^2 \le \frac{1}{2}M_K^2 \ \forall t \in [0,T_1].$$

This contradicts $\|w_N(T_1)\| = M_K$. Therefore we conclude $T_1 \ge T$, and consequently, $\|w_N(t)\| \le M_K$ for all $t \in [0,T]$. Passing to the limit as $N \to \infty$, we obtain the desired conclusion for w, i.e., w satisfies the bound in (3.9).

We will now prove uniqueness of strong solutions. Observe that $\|B(w,w)\|_{V'} \lesssim \|w\|_{L^4}^2 \le |w|^{1/2}\|w\|^{3/2}$. It is now easy to see from (3.2) that if w is a strong solution on $[0,T]$, then $\frac{d}{dt}w \in L^2([0,T];V')$ and consequently, $|w(t)|^2$ is differentiable a.e. on $[0,T]$. Let $\widetilde{w} = w_2 - w_1$. Then $\widetilde{w}$ satisfies

$$\frac{d}{dt}\widetilde{w} + \nu A\widetilde{w} + B(w_2,\widetilde{w}) + B(\widetilde{w},w_1) = -\mu I_h\widetilde{w}. \tag{3.12}$$

Let

$$M = \sup_{[0,T]}\|w_1\| < \infty. \tag{3.13}$$

By taking inner product with $\widetilde{w}$ with (3.12), using the properties of the type 1 interpolant in (2.11) and the estimate of the nonlinear term in (2.6), we readily obtain

$$\begin{aligned}&\frac{d}{dt}|\widetilde{w}|^2 + \nu\|\widetilde{w}\|^2\\ &\le c|\widetilde{w}|^{1/2}\|\widetilde{w}\|^{3/2}\|w_1\| - \mu|\widetilde{w}|^2 + \mu ch^2\|\widetilde{w}\|^2 + \frac{\mu}{2}|\widetilde{w}|^2\\ &\le \frac{cM}{\nu^3}|\widetilde{w}|^2 + \frac{\nu}{2}\|\widetilde{w}\|^2 + \frac{\mu}{2}\|\widetilde{w}\|^2 - \frac{\mu}{2}|\widetilde{w}|^2,\end{aligned}$$

where to obtain the inequality in the line above, we used Young's inequality together with (3.13) and the condition on μ in (3.4). Since $\widetilde{w}(0) = 0$, using the Gronwall inequality, we readily conclude that $\widetilde{w} = 0$ on $[0, T]$, i.e., w_1 and w_2 coincide on $[0, T]$. ∎

Remark 3.1. Observe that in the proof of the uniqueness presented above, the bound in V of only one of the two solutions appears explicitly. Indeed, one can prove a weak-strong uniqueness as in the case of the 3D NSE, a result due to Sather and Serrin [61]. More precisely, if there exists a strong solution w of (3.2) on $[0, T]$, then it coincides with any other Leray–Hopf weak solution of (3.2). The proof is similar to that of the Sather–Serrin result [61].

We will now prove the tracking property of the solution given in Theorem 3.2.

THEOREM 3.3. *Assume that the hypotheses of Theorem* 3.2 *hold. Let* $\widetilde{w} = w - u$. *Then* $|\widetilde{w}(t)|^2 \le e^{-\frac{\mu}{2}t}|\widetilde{w}(0)|^2$ *for all* $t \in [0, T]$. *In particular, if in the statement of Theorem* 3.2*,* $T = \infty$*, then*

$$\lim_{t\to\infty} |\widetilde{w}(t)|^2 = 0.$$

Proof. Since $u \in \mathfrak{B}$, there exists a Galerkin sequence $\{u_{N_k}\}_{k=1}^{\infty}$ converging to u in the sense of Theorem 2.1 (in particular, in the topology of $L^2([0, T]; H) \cap C([0, T]; V'))$. Note that by Theorem 3.2 and Remark 3.1, w is *the unique, strong solution* of (3.2). Therefore, the entire Galerkin system $\{w_N\}$ converges to w. (To see this, note that any subsequence of $\{w_N\}$ has a further subsequence which converges (in the metric space $L^2([0, T]; H) \cap C([0, T]; V')$) to a strong solution of (3.2), which by uniqueness (see Theorem 3.2) must be w.) Therefore, by relabeling the subsequence N_k, we can assume that $\{u_N\}$ converges to the weak solution u while $\{w_N\}$ converges to the *unique strong solution* w. We will now fix this sequence and assume $N \ge K$, where recall that P_K is the modal interpolant.

Note that $\widetilde{w}_N = w_N - u_N$ satisfies

$$\frac{d\widetilde{w}_N}{dt} + \nu A\widetilde{w}_N + P_N B(w_N, w_N) - P_N B(u_N, u_N) = -\mu P_K(\widetilde{w}_N) + \mu P_K(u - u_N),$$

which can be rearranged to

$$\begin{aligned}\frac{d\widetilde{w}_N}{dt} + \nu A\widetilde{w}_N + P_N B(\widetilde{w}_N, w_N) + P_N B(u_N, \widetilde{w}_N) = -\mu Q_K(\widetilde{w}_N) - \mu\widetilde{w}_N \\ + \mu P_K(u - u_N).\end{aligned}$$

We take the inner product with $\widetilde{w}_N$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\widetilde{w}_N|^2 + \nu\|\widetilde{w}_N\|^2 + \mu|\widetilde{w}_N|^2 = -(B(\widetilde{w}_N, w_N), \widetilde{w}_N) - \mu(Q_K(\widetilde{w}_N), \widetilde{w}_N) \\ + \mu(P_K(u - u_N), \widetilde{w}_N)\end{aligned} \tag{3.14}$$

and estimate each term on the right-hand side as follows:

$$\begin{aligned}|(B(\widetilde{w}_N, w_N), \widetilde{w}_N)| &\le c|\widetilde{w}_N|^{1/2}\|w_N\|\|\widetilde{w}_N\|^{3/2} \le \frac{c}{\nu^3}\|w_N\|^4|\widetilde{w}_N|^2 + \frac{\nu}{2}\|\widetilde{w}_N\|^2, \\ \mu|(Q_K(\widetilde{w}_N), \widetilde{w}_N)| &\le \mu|Q_K(\widetilde{w}_N)||\widetilde{w}_N| \le \frac{\mu}{\lambda_K^{1/2}}\|\widetilde{w}_N\||\widetilde{w}_N| \\ &\le \frac{\mu}{\lambda_K}\|\widetilde{w}_N\|^2 + \frac{\mu}{4}|\widetilde{w}_N|^2 \le \frac{\nu}{2}\|\widetilde{w}_N\|^2 + \frac{\mu}{4}|\widetilde{w}_N|^2, \\ \mu|(P_K(u - u_N), \widetilde{w}_N)| &\le \mu|P_K(u - u_N)||\widetilde{w}_N| \le \mu|P_K(u - u_N)|^2 + \frac{\mu}{4}|\widetilde{w}_N|^2.\end{aligned}$$

Inserting the estimates into (3.14),

$$\frac{d}{dt}|\widetilde{w}_N|^2 + \left(\mu - \frac{c}{\nu^3}\|w_N\|^4\right)|\widetilde{w}_N|^2 \le \mu|P_K(u-u_N)|^2.$$

Since μ satisfies (3.8), we get

$$\frac{d}{dt}|\widetilde{w}_N|^2 + \frac{\mu}{2}|\widetilde{w}_N|^2 \le \mu|P_K(u-u_N)|^2.$$

Applying Gronwall, for all $t \in [0,T]$, we get

$$|\widetilde{w}_N(t)|^2 \le e^{-(\mu/2)t}|\widetilde{w}(0)|^2 + 2\sup_{t\in[0,T]}|P_K(u-u_N)|^2. \tag{3.15}$$

Recall that $u_N \to u$ in $C(0,T;V')$ for all $T<\infty$. Therefore, denoting by ϕ_i the ith eigenvector of the Stokes operator A (which implies that $\phi_i \in V$), we have for each i, $\limsup_{N\to\infty}\sup_{[0,T]}|(u-u_N,\phi_i)| = 0$. Therefore,

$$\limsup_{N\to\infty}\sup_{t\in[0,T]}|P_K(u-u_N)|^2 \le \limsup_{N\to\infty}\sup_{t\in[0,T]}\sum_{i=1}^{K}|(u-u_N,\phi_i)|^2 = 0. \tag{3.16}$$

Since $\widetilde{w}_N$ converges to $\widetilde{w}$ in $L^2([0,T];H)$, there exists a subsequence $\widetilde{w}_{N_k}$ which converges a.e. $t\in[0,T]$. Now taking the limit and using (3.15) and (3.16), it follows that a.e. t, we have

$$|\widetilde{w}(t)|^2 \le e^{-(\mu/2)t}|\widetilde{w}(0)|^2.$$

This proves the result. ∎

Remark 3.2. The fact that u is a restricted Leray–Hopf weak solution, i.e., u is obtained as a suitable limit of a Galerkin system (or some other appropriate limiting procedure), is used only in the proof of Theorem 3.3. This technicality is due to the fact that it is unknown whether for an arbitrary Leray–Hopf weak solution, the quantity $|u(t)|^2$ (and hence $|\widetilde{w}(t)|^2 = |(w-u)|^2$ is differentiable, although $|w(t)|^2$ is due to its being a strong solution. However, subsequently in [9], we have been able to extend this result to the case of an arbitrary Leray–Hopf weak solution (i.e., in Theorem 3.3, the assumption that u is a *restricted* Leray–Hopf weak solution can be relaxed to u being an arbitrary Leray–Hopf weak solution). Since w is a strong solution, the proof is an appropriate (but nontrivial) modification of the Sather–Serrin weak-strong uniqueness theorem for the Navier–Stokes equations.

Remark 3.3. Note that the conclusions of Theorems 3.2 and 3.3 rest on the choice of μ satisfying (3.8). This is possible if there exists K such that

$$C \ge \lambda_K^{-1}\max\left\{\frac{M_K^4}{\nu^4}, \lambda_1\right\}, \tag{3.17}$$

where M_K is as defined in (3.7) and for suitable $C>0$. We emphasize that *this condition is expressed purely in terms of the observed data which in this case are the low modes of the solution* u and does not involve information on the unknown high modes.

Suppose now that the solution u is regular on $[0,T]$. For initial data $u_0\in V$, it is well-known (see, for instance, [60]) that this happens if for some $\frac{1}{2}<\theta\le 1$ (here we take $\theta \le 1$ as $u_0 \in V = D(A^{1/2})$)

$$\sup_{t\in[0,T]}|A^{\theta/2}u| = M_\theta < \infty.$$

In this case,

$$\|P_K u\| = |P_K A^{1/2} u| \le \lambda_K^{(1-\theta)/2} |A^{\theta/2} u| \le \lambda_K^{(1-\theta)/2} M_\theta.$$

Therefore,

$$M_K^2 \lesssim \left(\frac{|f|^2}{\nu^2 \lambda_1} + \lambda_K^{(1-\theta)} M_\theta^2 \right).$$

Since $\theta > \frac{1}{2}$, a choice of λ_K satisfying (3.17) is indeed possible if K is chosen large enough. In the borderline case $\theta = \frac{1}{2}$, by proceeding in an analogous manner, we get that (3.17) can be satisfied if $M_\theta = \sup_{t\in[0,T]} |A^{1/4}u|$ is small. It should be noted that it is well-known that in case f is small and $|A^{1/4}u_0|$ is small, then the solution u is globally regular and additionally $|A^{1/4}u|$ remains small for all times. Thus we conclude that (3.17) holds for sufficiently large K for $T < \infty$ when the solution is regular on $[0,T]$. We show below that this condition when $T = \infty$ also implies that K satisfying (3.17) is *asymptotically determining.*

THEOREM 3.4. *Let u_1, u_2 be two restricted Leray–Hopf weak solutions with $M_{K,u_i}, i = 1,2$ defined as in* (3.7)*. Assume moreover that on* $[0,\infty)$*, we have*

$$\lambda_K \gtrsim \max\left\{ \frac{M_{K,u_1}^4}{\nu^4}, \frac{M_{K,u_2}^4}{\nu^4}, \lambda_1 \right\}, \tag{3.18}$$

where $M_{K,u_i}, i = 1,2,$ as defined in (3.7) *with $T = \infty$. If*

$$\lim_{t\to\infty} |P_K(u_1 - u_2)| = 0, \tag{3.19}$$

then $\lim_{t\to\infty} |u_1(t) - u_2(t)| = 0$.

Proof. Let w_1 and w_2 be two strong solutions of the data assimilation equation (3.2) corresponding to u_1 and u_2 for μ satisfying (3.8) for both u_1 and u_2. Denote $\widetilde{w}_i = w_i - u_i, i = 1,2$. Then by Theorem 3.3, $\lim_{t\to\infty} |\widetilde{w}_i| = 0$. Let $\widetilde{w} = w_1 - w_2$. Proceeding exactly as in the proof of Theorem 3.3, and noting that $\|w_i\| \le M_K$ and that μ satisfies (3.8), we conclude

$$\frac{d}{dt}|\widetilde{w}|^2 + \frac{\mu}{2}|\widetilde{w}|^2 \le \mu |P_K(u_2 - u_1)|^2,$$

which yields, upon integrating between s to T, that

$$|\widetilde{w}(T)|^2 \lesssim e^{-\frac{\mu}{2}(T-s)} |\widetilde{w}(s)|^2 + \sup_{t\in[s,T]} |P_K(u_2 - u_1)|.$$

Letting $T \to \infty$ and using (3.19), we conclude that $\lim_{t\to\infty} |\widetilde{w}| = 0$. Thus, $\lim_{t\to\infty} |u_1 - u_2| = 0$. ∎

Remark 3.4. Due to Theorem 3.4 and the discussion in the introduction, K_{inf} as given in (1.8) defines a notion of a determining wave number in the 3D case. This definition applies both to the space periodic as well as the homogeneous Dirichlet boundary conditions. A different notion of a time-varying determining wave number for two Leray–Hopf weak solutions, based on the Littlewood–Paley decomposition and applicable to the space periodic boundary condition on the domain $\Omega = [0,L]^3$, is provided in [19, 20]. It is given by

$$\begin{aligned}\Lambda(t) = \min\{\lambda_q : (L\lambda_{p-q})^{\delta-1/2} \|u_p\|_{L^\infty} < c_0\nu \ \forall\ p > q \text{ and} \\ \lambda_q^{-2} \|\nabla u_{\le q}\|_{L^\infty} < c_0\nu\},\end{aligned} \tag{3.20}$$

where recall that $\lambda_q = 2^q/L$, $u_q = \Delta_q u$, where Δ_q is the Littlewood–Paley projection, while $u_{\le q} = \sum_{-1\le j\le q} \Delta_q u$ (see, e.g., [22] for a description of the Littlewood–Paley decomposition). In case u is regular, the set over which the infimum is being taken in (1.8) is nonempty. It should be noted that Λ as defined in (3.20) is in $L^1[T_1,T_2])$ for any Leray–Hopf weak solution and its time average is bounded by the Kolmogorov's dissipation wave number. Additionally, Λ is bounded on $[T_1,T_2]$ if and only if u is regular on that interval.

However, in order to verify whether a λ_K satisfies (1.6), we only need all lower modes of u (i.e., $P_K u$). On the other hand, in order to verify the first condition in (3.20), namely

$$(L\lambda_{p-q})^{\delta-1/2}\|u_p\|_{L^\infty} < c_0\nu \ \forall\ p > q,$$

one needs the knowledge of *all the higher modes* of u. This is an important point from the data assimilation perspective as the choice of μ in the AOT algorithm (3.2) depends on this wave number and should be verifiable from observed data (i.e., low modes only). We cannot define the AOT algorithm (3.2) based on (3.20) (at least as expounded in [20]), because in order to define Λ in (3.20), we need to observe all modes. A similar comment holds for the definition of the determining wave number presented in [19] which is based on the L^r-norm instead of the L^∞-norm as discussed here in (3.20).

We also note that our approach results in a suitable definition of *determining volume elements* in the 3D case (Theorem 3.7), no analogue of which exists in [19, 20].

3.2.2. General type 1 interpolant. In this section, we assume that u and w satisfy the space periodic boundary condition. Thus, the Stokes operator $A = (-\Delta)$ on $V = \mathbb{H}^1(\Omega)$. Moreover, we also assume that in addition to (2.11), I_h satisfies the condition $Ran(I_h) \subset V = \mathbb{H}^1(\Omega)$ and

$$\|I_h v\| \le C\|v\| \ \forall v \in V. \tag{3.21}$$

This is clearly satisfied for the modal interpolant P_K. In case of the volume interpolant, one may apply a suitable mollification procedure to obtain a modified volume interpolant, $\tilde{I}_h$, that satisfies both (2.11) and (3.21). In fact, this is achieved by replacing the term $(\chi_{Q_j}(x) - \frac{h^3}{L^3})$ by $\big(\psi_j(x) - \int_\Omega \psi_j(y)dy\big)$, where $\psi_j = \rho_\epsilon * \chi_{Q_j}$ for

$$\rho(\xi) = \begin{cases} K_0 \exp\left(\frac{1}{1-|\xi|^2}\right) & for\ |\xi| < 1, \\ 0 & for\ |\xi| \ge 1, \end{cases}$$

and

$$(K_0)^{-1} = \int_{|\xi|<1} \exp\left(\frac{1}{1-|\xi|^2}\right) d\xi.$$

The mollification parameter ϵ is chosen to be a fraction of h. $\tilde{I}_h$ is a C^∞ function and it can be show that it satisfies (3.21). For more details see the appendix of [7], which proves the corresponding result for the type 2 case.

THEOREM 3.5. *Assume that u as in* (2.2) *and w as in* (3.2) *satisfy the space periodic boundary condition. Suppose I_h is a general type* 1 *interpolant and satisfies* (2.11). *Let $0 < T \le \infty$ and denote*

$$M_h^2 = M_{h,u}^2 := 8\left(\frac{1}{\nu^2\lambda_1}|f|^2 + \sup_{t\in[0,T)} \|I_h(u)\|^2\right). \tag{3.22}$$

Assume that

$$\max\left\{\frac{2cM_h^4}{\nu^3}, \nu\lambda_1\right\} \le \mu \le \frac{\nu}{4ch^2}. \tag{3.23}$$

Then any weak solution of (3.2) *constructed in Theorem* 3.1 *as a subsequential limit of* w_N *satisfying* (3.3) *is regular on* $[0,T]$, *i.e., it satisfies*

$$\|w(t)\| \le M_h, t \in [0,T]. \tag{3.24}$$

Proof. We proceed as in Theorem 3.2 by rearranging (3.3)

$$\frac{dw_N}{dt} + \nu A w_N + P_N B(w_N, w_N) = f - \mu P_N(I_h(w_N) - w_N) + \mu P_N I_h(u) - \mu w_N$$

and taking the inner product with Aw_N

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w_N\|^2 + \nu|Aw_N|^2 + \mu\|w_N\|^2 \\ &\quad = (f, Aw_N) - (B(w_N,w_N), Aw_N) \\ &\qquad -\mu(I_h(w_N) - w_N, Aw_N) + \mu(I_h(u), Aw_N). \end{aligned} \tag{3.25}$$

Each term on the right-hand side is estimated below as in Theorem 3.2. First, we have by Cauchy–Schwarz and Young's inqualities,

$$|(f, Aw_N)| \le |f||Aw| \le \frac{1}{\nu}|f|^2 + \frac{\nu}{4}|Aw_N|^2.$$

Next, the nonlinear term is estimated using (2.5) and Young's inequalities as

$$|(B(w_N,w_N), Aw_N)| \le c\|w_N\|^{3/2}|Aw_N|^{3/2} \le \frac{c}{\nu^3}\|w_N\|^6 + \frac{\nu}{4}|Aw_N|^2.$$

Next, using (2.11),

$$\begin{aligned} \mu|I_h(w_N) - w_N, Aw_N)| &\le \mu|I_h(w_N) - w_N||Aw_N| \\ &\le \mu ch\|w_N\||Aw_N| \le \frac{(\mu ch)^2}{\nu}\|w_N\|^2 + \frac{\nu}{4}|Aw_N|^2 \\ &\le \frac{\mu}{4}\|w_N\|^2 + \frac{\nu}{4}|Aw_N|^2, \end{aligned}$$

where to obtain the last inequality, we used (3.23). Observe now that since $A = (-\Delta)$ in the space periodic case, we can integrate by parts to obtain

$$\mu|(I_h(u), Aw_N)| \le \mu\|I_h(u)\|\|w_N\| \le \mu\|I_h(u)\|^2 + \frac{\mu}{4}\|w_N\|^2.$$

Inserting the above estimates into (3.25) we obtain

$$\frac{d}{dt}\|w_N\|^2 + \left(\mu - \frac{c}{\nu^3}\|w_N\|^4\right)\|w_N\|^2 \le \frac{2}{\nu}|f|^2 + 2\mu\|I_h(u)\|^2.$$

Let $[0,T_1]$ be the maximal interval on which $\|w_N(t)\| \le M_h$ holds for $t \in [0,T_1]$ where M_h as in (3.22). Note that T_1 exists because we have $w_N(0) = 0$. Assume that $T_1 < T$. Using the lower bound for μ in (3.23) and the Gronwall inequality we obtain

$$\|w_N\|^2 \le \frac{4}{\nu^2\lambda_1}|f|^2 + 4\sup_{s\in[0,T]}\|I_h(u(s))\|^2 = \frac{1}{2}M_h^2 \ \forall\ t \in [0,T_1].$$

Arguing as in Theorem 3.2 by contradiction, we obtain the desired conclusion for w, i.e., $\|w(t)\| \le M_h$ for all $t \in [0,T]$. ∎

We now can deduce the following result regarding the tracking property of w.

THEOREM 3.6. *Assume that the hypotheses of Theorem* 3.5 *hold. Let* $\widetilde{w} = w - u$. *Then* $|\widetilde{w}(t)|^2 \le e^{\frac{-\mu}{2}t}|\widetilde{w}(0)|^2$ *for all* $t \in [0,T]$. *In particular, if in the statement of Theorem* 3.5 *we have* $T = \infty$, *then*

$$\lim_{t\to\infty} |\widetilde{w}(t)|^2 = 0.$$

Proof. We proceed as in the proof of Theorem 3.3 by considering Galerkin sequences $\{u_N\}$ and $\{w_N\}$. Note that the difference $\widetilde{w}_N$ satisfies

$$\frac{d\widetilde{w}_N}{dt} + vA\widetilde{w}_N + P_N B(w_N, w_N) - P_N B(u_N, u_N) = -\mu P_N I_h(\widetilde{w}_N) + \mu P_N I_h(u - u_N),$$

which can be rearranged to

$$\begin{aligned}\frac{d\widetilde{w}_N}{dt} + vA\widetilde{w}_N + P_N B(\widetilde{w}_N, w_N) + P_N B(u_N, \widetilde{w}_N) = -\mu(I_h(\widetilde{w}_N) - \widetilde{w}_N) - \mu\widetilde{w}_N \\ + \mu P_N I_h(u - u_N).\end{aligned}$$

We take the inner product with $\widetilde{w}_N$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\widetilde{w}_N|^2 + \nu\|\widetilde{w}_N\|^2 + \mu|\widetilde{w}_N|^2 &= -(B(\widetilde{w}_N, w_N), \widetilde{w}_N) - \mu(I_h(\widetilde{w}_N) - \widetilde{w}_N, \widetilde{w}_N) \\ &\quad + \mu(I_h(u - u_N), \widetilde{w}_N)\end{aligned} \tag{3.26}$$

and estimate each term on the right-hand side as follows:

$$\begin{aligned}|(B(\widetilde{w}_N, w_N), \widetilde{w}_N)| &\le c|\widetilde{w}_N|^{1/2}\|w_N\|\|\widetilde{w}_N\|^{3/2} \le \frac{c}{\nu^3}\|w_N\|^4|\widetilde{w}_N|^2 + \frac{\nu}{2}\|\widetilde{w}_N\|^2 \\ \mu|(I_h(\widetilde{w}_N) - \widetilde{w}_N, \widetilde{w}_N)| &\le \mu|I_h(\widetilde{w}_N) - \widetilde{w}_N||\widetilde{w}_N| \le \mu ch\|\widetilde{w}_N\||\widetilde{w}_N| \\ &\le \mu c^2h^2\|\widetilde{w}_N\|^2 + \frac{\mu}{4}|\widetilde{w}_N|^2 \le \frac{\nu}{2}\|\widetilde{w}_N\|^2 + \frac{\mu}{4}|\widetilde{w}_N|^2 \\ \mu|(I_h(u - u_N), \widetilde{w}_N)| &\le \mu|I_h(u - u_N)||\widetilde{w}_N| \le \mu|I_h(u - u_N)|^2 + \frac{\mu}{4}|\widetilde{w}_N|^2.\end{aligned}$$

Inserting the estimates into (3.26),

$$\frac{d}{dt}|\widetilde{w}_N|^2 + \left(\mu - \frac{c}{\nu^3}\|w_N\|^4\right)|\widetilde{w}_N|^2 \le \mu|I_h(u - u_N)|^2$$

and since μ satisfies (3.23) we get

$$\frac{d}{dt}|\widetilde{w}_N|^2 + \frac{\mu}{2}|\widetilde{w}_N|^2 \le \mu|I_h(u - u_N)|^2.$$

Applying Gronwall, for all $t \in [0,T]$, we get

$$|\widetilde{w}_N(t)|^2 \le e^{-(\mu/2)t}|\widetilde{w}(0)|^2 + 2\sup_{t\in[0,t]}|I_h(u - u_N)|^2.$$

Recall $u_N \to u$ in $C(0,T;V')$ and the range of I_h is a finite dimensional vector space with a basis $\{\psi\}$. Therefore,

$$\liminf_{N\to\infty} |I_h(u - u_N)| \le \liminf_{N\to\infty} \sum_{i=1}^{K'} |(u - u_N, \psi_i)| = 0.$$

Since $\widetilde{w}_N$ converges to $\widetilde{w}$ weakly,

$$|\widetilde{w}(t)|^2 \leq e^{-(\mu/2)t}|\widetilde{w}(0)|^2$$

which proves the result. ∎

So far, we have not used the bound on the interpolant assumed in (3.21) in the proof of the above results. Its role is clarified in the remark below. More precisely, we show using (3.21) that in case the solution u from which the observations are obtained is regular, then a parameter μ satisfying the condition (3.23) can be chosen, thus ensuring that the corresponding data assimilation solution given by (3.2) tracks u.

Remark 3.5. A choice of μ satisfying (3.23) exists provided the condition

$$\max\left\{\frac{2cM_h^4}{\nu^4}, \lambda_1\right\} \lesssim \frac{1}{h^2} \tag{3.27}$$

holds. Due to (3.21), this is clearly satisfied for sufficiently small h if u is regular and $\sup_{t\in[t_0,\infty)} \|u\| < \infty$. Thus global regularity and uniform boundedness in V of u guarantee the existence of a globally regular solution for the AOT algorithm (3.2) and the unique solution w tracks u.

We now show that Theorem 3.6 implies the existence of *asymptotically determining volume elements*, similar to the modal case.

THEOREM 3.7. *Let u_1, u_2 be two restricted Leray-Hopf weak solutions with $M_{h,u_i}, i = 1, 2$, defined as in* (3.22). *Assume moreover that on* $[0,\infty)$, *we have*

$$h^{-2} \gtrsim \max\left\{\frac{M_{h,u_1}^4}{\nu^4}, \frac{M_{h,u_2}^4}{\nu^4}, \lambda_1\right\}, \tag{3.28}$$

where $M_{h,u_i}, i = 1, 2$, as defined in (3.22) *with $T = \infty$. If*

$$\lim_{t\to\infty} |I_h(u_1 - u_2)| = 0, \tag{3.29}$$

then $\lim_{t\to\infty} |u_1(t) - u_2(t)| = 0$.

Proof. Let w_1 and w_2 be two strong solutions of the data assimilation equation (3.2) corresponding to u_1 and u_2 for μ satisfying (3.23) for both u_1 and u_2. Denote $\widetilde{w}_i = w_i - u_i, i = 1, 2$. Then by Theorem 3.6, $\lim_{t\to\infty} |\widetilde{w}_i| = 0$. Let $\widetilde{w} = w_1 - w_2$. Proceeding exactly as in the proof of Theorem 3.6, and noting that $\|w_i\| \leq M_h$ and that μ satisfies (3.23), we conclude

$$\frac{d}{dt}|\widetilde{w}|^2 + \frac{\mu}{2}|\widetilde{w}|^2 \leq \mu|I_h(u_2 - u_1)|^2,$$

which yields, upon integrating between s to T, that

$$|\widetilde{w}(T)|^2 \lesssim e^{-\frac{\mu}{2}(T-s)}|\widetilde{w}(s)|^2 + \sup_{t\in[s,T]} |I_h(u_2 - u_1)|.$$ ∎

Letting $T \to \infty$ and using (3.29), we conclude that $\lim_{t\to\infty} |\widetilde{w}| = 0$. Thus, $\lim_{t\to\infty} |u_1 - u_2| = 0$.

4. Adaptive algorithm. Since (3.2) becomes stiff for larger values of μ, in this section we define an adaptive algorithm so that the value of the nudging parameter can be adjusted to a higher value only in the time intervals where the flow is turbulent. This data assimilation algorithm is iteratively defined by

$$\begin{aligned} \frac{dw}{dt} + \nu Aw + B(w,w) = f - \mu_{k+1} I_h(w-u),\ t \in (T_k, T_{k+1}], \\ \nabla\cdot w = 0, \\ w(T_k) = \lim_{t\to T_k} w(t),\ w(T_0) = 0 \end{aligned} \tag{4.1}$$

for $k \in \{0,1,\dots,j\}$.

Although the following theorems can be proven for a general type 1 interpolant I_h, for simplicity of exposition, we take $I_h = P_K$.

THEOREM 4.1. *Suppose $I_h = P_K$ is a modal interpolant (and consequently satisfies* (2.12)*). Denote for $k \in \{0,1,\dots,j\}$*

$$M_{k+1}^2 = 2\times\max\left\{\|w(T_k)\|^2, \left(\frac{4|f|^2}{\nu^2\lambda_1} + 2\widetilde{M}_{k+1}^2\right)\right\}, \tag{4.2}$$

where

$$\widetilde{M}_{k+1} = \sup_{t\in[T_k,T_{k+1}]} \|P_K u(t)\|. \tag{4.3}$$

Suppose a choice of $\{\mu_k\}_{k=1}^{j+1}$ exists satisfying

$$\max\left\{\frac{2c}{\nu^3}M_k^4, \nu\lambda_1\right\} \le \mu_k \le \frac{\nu\lambda_K}{8},\ k\in\{1,\dots,j+1\}. \tag{4.4}$$

Then the solution of (4.1) *satisfies*

$$\|w(t)\|^2 \le M_{k+1}^2 \quad \forall t\in[T_k,T_{k+1}],\ k\in\{0,1,\dots,j\}, \tag{4.5}$$

and

$$\|w(t)\|^2 \le \frac{8|f|^2}{\nu^2\lambda_1} + 4\times\sup_{t\in[T_0,T_{j+1}]}\|P_K u(t)\|^2 \le \frac{\nu^2\lambda_K^{1/2}}{4c} \quad \forall t\in[T_0,T_{j+1}]. \tag{4.6}$$

Proof. We restrict time to be in an arbitrary interval $[T_k, T_{k+1}]$ and take the inner product of (4.1) with Aw

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|^2 + \nu|Aw|^2 + \mu_{k+1}\|w\|^2 = (B(w,w),Aw) + (f,Aw) \\ + \mu_{k+1}(Q_K w, Aw) + \mu_{k+1}(P_K u, Aw). \end{aligned} \tag{4.7}$$

Each term on the right-hand side is estimated as before:

$$\begin{aligned} &|(B(w,w),Aw)| \le c\|w\|^{3/2}|Aw|^{3/2} \le \frac{\nu}{8}|Aw|^2 + \frac{c}{\nu^3}\|w\|^6, \\ &|(f,Aw)| \le \frac{1}{\nu}|f|^2 + \frac{\nu}{4}|Aw|^2, \\ &\mu_{k+1}|(Q_K w, Aw)| = \mu_{k+1}\|Q_K w\|^2 \le \frac{\mu_{k+1}}{\lambda_K}|Aw|^2 \le \frac{\nu}{8}|Aw|^2, \\ &\mu_{k+1}|(P_K u, Aw)| \le \frac{\mu_{k+1}}{2}\|P_K u\|^2 + \frac{\mu_{k+1}}{2}\|w\|^2. \end{aligned}$$

Inserting the estimates into (4.7),

$$\frac{d}{dt}\|w\|^2 + \left(\mu_{k+1} - \frac{c}{\nu^3}\|w\|^4\right)\|w\|^2 \leq \frac{2}{\nu}|f|^2 + \mu_{k+1}\|P_K u\|^2.$$

Assume $[T_k, T^*],\ T^* < T_{k+1}$, is the largest interval starting from T_k such that $\|w\| \leq M_{k+1}$ holds. Note by construction we have $\|w(T_k)\| \leq M_{k+1}$ so T^* exists. Then using the lower bound for μ_{k+1} in (4.4) and defining $\tau = t - T_k$ we obtain by the Gronwall inequality for all $t \in [T_k, T^*]$,

$$\|w(t)\|^2 \leq e^{-(\mu_{k+1}/2)\tau}\|w(T_k)\|^2 + \left(1 - e^{-(\mu_{k+1}/2)\tau}\right)\left(\frac{4|f|^2}{\nu^2\lambda_1} + 2\widetilde{M}_{k+1}^2\right) \leq \frac{1}{2}M_{k+1}^2,$$

a contradiction, thus $\|w(t)\| \leq M_{k+1}$ for $t \in [T_k, T_{k+1}],\ k \in \{0, 1, \ldots, j\}$. The bound in (4.6) is obtained by finding an upper bound of the collection $\{M_k\}$ and using (4.4) as follows. Recall $M_1^2 = \frac{8|f|^2}{\nu^2\lambda_1} + 4\widetilde{M}_1^2$.

$$\begin{aligned}
M_{j+1}^2 &\leq \max\left\{M_j^2, \left(\frac{8|f|^2}{\nu^2\lambda_1} + 4\widetilde{M}_{j+1}^2\right)\right\} \\
&\leq \max\left\{M_{j-1}^2, \left(\frac{8|f|^2}{\nu^2\lambda_1} + 4\widetilde{M}_j^2\right), \left(\frac{8|f|^2}{\nu^2\lambda_1} + 4\widetilde{M}_{j+1}^2\right)\right\} \\
&\cdots \\
&\leq \frac{8|f|^2}{\nu^2\lambda_1} + 4 \times \max\{\widetilde{M}_1^2, \ldots, \widetilde{M}_{j+1}^2\} = \frac{8|f|^2}{\nu^2\lambda_1} + 4 \times \sup_{t\in[T_0, T_{j+1}]}\|P_K u(t)\|^2.
\end{aligned}$$

We finish by noting that (4.4) implies $\frac{8|f|^2}{\nu^2\lambda_1} + 4 \times \max\{\widetilde{M}_1^2, \ldots, \widetilde{M}_{j+1}^2\} \leq \frac{\nu^2\lambda_K^{1/2}}{4c}$. ∎

Remark 4.1. In order for this iterative construction to work up to time interval $[T_j, T_{j+1}]$, i.e., for a choice of $\{\mu_k\}$ satisfying (4.4) to exist, we need to satisfy $M_k^4\lambda_K^{-1} \leq \frac{\nu^4}{16c},\ k \in \{1, \ldots, j+1\}$. Using the upper bound on $\{M_k\}$, such a choice of $\{\mu_k\}_{k=1}^{j+1}$ is possible if we assume

$$\sup_{t\in[T_0,T_{j+1}]}\left(\frac{128|f|^4}{\nu^4\lambda_1^2} + 32\|P_K u(t)\|^4\right)\lambda_K^{-1} \leq \frac{\nu^4}{16c}. \tag{4.8}$$

The theorem below establishes the tracking property of the solution w.

THEOREM 4.2. *Suppose $I_h = P_K$ is a modal interpolant and satisfies* (2.12). *Assume the same conditions from Theorem* 4.1. *Then the solutions of* (2.2) *and* (4.1) *satisfy*

$$|\widetilde{w}(t)|^2 \leq e^{-(\mu_{k+1}/2)t}|\widetilde{w}(T_k)|^2\ \forall t \in [T_k, T_{k+1}],\ \forall k \in \{0, 1, \ldots, j\},$$

where $\widetilde{w} = w - u$.

Proof. Assume $N \geq K$, $t \in [T_k, T_{k+1}]$, and $\widetilde{w}_N$ satisfies

$$\begin{aligned}
&\frac{d\widetilde{w}_N}{dt} + vA\widetilde{w}_N + P_N B(w_N, w_N) - P_N B(u_N, u_N) \\
&\qquad = -\mu_{k+1}P_K(\widetilde{w}_N) + \mu_{k+1}P_K(u - u_N),
\end{aligned}$$

which can be rearranged to

$$\frac{d\widetilde{w}_N}{dt} + vA\widetilde{w}_N + P_N B(\widetilde{w}_N, w_N) + P_N B(u_N, \widetilde{w}_N)$$
$$= -\mu_{k+1} Q_K(\widetilde{w}_N) - \mu_{k+1}\widetilde{w}_N + \mu_{k+1} P_K(u - u_N).$$

We take the inner product with $\widetilde{w}_N$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\widetilde{w}_N|^2 + \nu\|\widetilde{w}_N\|^2 + \mu_{k+1}|\widetilde{w}_N|^2\\ &\qquad = -(B(\widetilde{w}_N, w_N), \widetilde{w}_N) - \mu_{k+1}(Q_K(\widetilde{w}_N), \widetilde{w}_N)\\ &\qquad + \mu_{k+1}(P_K(u - u_N), \widetilde{w}_N).\end{aligned} \tag{4.9}$$

We now estimate each term on the right-hand side as follows:

$$\begin{aligned}&|(B(\widetilde{w}_N, w_N), \widetilde{w}_N)| \le c|\widetilde{w}_N|^{1/2}\|w_N\|\|\widetilde{w}_N\|^{3/2}\\ &\qquad \le \frac{c}{\nu^3}\|w_N\|^4|\widetilde{w}_N|^2 + \frac{\nu}{2}\|\widetilde{w}_N\|^2,\\ &\mu_{k+1}|(Q_K(\widetilde{w}_N), \widetilde{w}_N)| \le \mu_{k+1}|Q_K(\widetilde{w}_N)||\widetilde{w}_N|\\ &\qquad \le \frac{\mu_{k+1}}{\lambda_K^{1/2}}\|\widetilde{w}_N\||\widetilde{w}_N| \le \frac{\mu_{k+1}}{\lambda_K}\|\widetilde{w}_N\|^2 + \frac{\mu_{k+1}}{4}|\widetilde{w}_N|^2\\ &\qquad \le \frac{\nu}{2}\|\widetilde{w}_N\|^2 + \frac{\mu_{k+1}}{4}|\widetilde{w}_N|^2,\\ &\mu_{k+1}|(P_K(u - u_N), \widetilde{w}_N)| \le \mu_{j+1}|P_K(u - u_N)||\widetilde{w}_N|\\ &\qquad \le \mu_{k+1}|P_K(u - u_N)|^2 + \frac{\mu_{k+1}}{4}|\widetilde{w}_N|^2.\end{aligned}$$

Inserting the estimates into (4.9),

$$\frac{d}{dt}|\widetilde{w}_N|^2 + (\mu_{k+1} - \frac{c}{\nu^3}\|w_N\|^4)|\widetilde{w}_N|^2 \le \mu_{k+1}|P_K(u - u_N)|^2,$$

and since μ satifies (4.4),

$$\frac{d}{dt}|\widetilde{w}_N|^2 + \frac{\mu_{k+1}}{2}|\widetilde{w}_N|^2 \le \mu_{k+1}|P_K(u - u_N)|^2.$$

By the Gronwall inequality,

$$|\widetilde{w}_N(t)|^2 \le e^{-(\mu_{k+1}/2)t}|\widetilde{w}(T_k)|^2 + 2\sup_{t\in[T_k,T_{k+1}]}|P_K(u - u_N)|^2.$$

Recall $u_N \to u$ in $C(0, T; V')$ and ϕ_i is the ith eigenvector associated with A. Therefore,

$$\liminf_{N\to\infty} |P_K(u - u_N)| \le \liminf_{N\to\infty}\sum_{i=1}^{K}|(u - u_N, \phi_i)| = 0.$$

Since $\widetilde{w}_N$ converges to $\widetilde{w}$ weakly,

$$|\widetilde{w}(t)|^2 \le e^{-(\mu_{k+1}/2)t}|\widetilde{w}(T_k)|^2 \ \forall t \in [T_k, T_{k+1}],$$

which proves the result. ∎

Remark 4.2. In certain applications, particularly for noisy observations, replacing the scalar damping operator μ_{k+1} (i.e., $\mu_{k+1}I$) in (4.1) with a positive semidefinite operator Σ_{k+1} may be beneficial. This is precisely the case in 3D Var or variants of the ensemble Kalman filter; see [15, 16, 46]. For instance, in the context of the 2D Navier–Stokes equation, a convergence analysis for 3D Var is presented in [15] for modal observations where the damping operator is $\mu P_N A^{2\alpha} = \mu P_N A^{2\alpha} P_N$ for adequate α and where A is the Stokes operator. One can suitably modify our techniques to address this situation for the 3D case as well [11].

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