

# On the connectedness of spectral sets and irreducibility of spectral cones in Euclidean Jordan algebras

M. Seetharama Gowda

Department of Mathematics and Statistics  
University of Maryland, Baltimore County  
Baltimore, Maryland 21250, USA  
gowda@umbc.edu

and

Juyoung Jeong

Department of Mathematics and Statistics  
University of Maryland, Baltimore County  
Baltimore, Maryland 21250, USA  
juyoung1@umbc.edu

May 7, 2018

## Abstract

Let  $\mathcal{V}$  be a Euclidean Jordan algebra of rank  $n$ . A set  $E$  in  $\mathcal{V}$  is said to be a *spectral set* if there exists a permutation invariant set  $Q$  in  $\mathcal{R}^n$  such that  $E = \lambda^{-1}(Q)$ , where  $\lambda : \mathcal{V} \rightarrow \mathcal{R}^n$  is the *eigenvalue map* that takes  $x \in \mathcal{V}$  to  $\lambda(x)$  (the vector of eigenvalues of  $x$  written in the decreasing order). If the above  $Q$  is also a convex cone, we say that  $E$  is a *spectral cone*. This paper deals with connectedness and arcwise connectedness properties of spectral sets. By relying on the result that in a simple Euclidean Jordan algebra, every eigenvalue orbit  $[x] := \{y : \lambda(y) = \lambda(x)\}$  is arcwise connected, we show that if a permutation invariant set  $Q$  is connected (arcwise connected), then  $\lambda^{-1}(Q)$  is connected (respectively, arcwise connected). A related result is that in a simple Euclidean Jordan algebra, every pointed spectral cone is irreducible.

**Key Words:** Euclidean Jordan algebra, spectral set, connectedness, irreducible cone

**AMS Subject Classification:** 54D05, 17C20, 17C30, 22E99, 15B48.

# 1 Introduction

Let  $\mathcal{V}$  be a Euclidean Jordan algebra of rank  $n$  [8]. The *eigenvalue map*  $\lambda : \mathcal{V} \rightarrow \mathcal{R}^n$  takes  $x \in \mathcal{V}$  to  $\lambda(x)$ , the vector of eigenvalues of  $x$  written in the decreasing order. A set  $E$  in  $\mathcal{V}$  is said to be a *spectral set* if there exists a permutation invariant set  $Q$  in  $\mathcal{R}^n$  such that  $E = \lambda^{-1}(Q)$ . If the above  $Q$  is also a convex cone, we say that  $E$  is a *spectral cone*. A function  $F : \mathcal{V} \rightarrow \mathcal{R}$  is said to be a *spectral function* if it is of the form  $F = f \circ \lambda$ , where  $f : \mathcal{R}^n \rightarrow \mathcal{R}$  is a permutation invariant function. The above concepts are generalizations of similar concepts that have been studied in the settings of Euclidean  $n$ -space  $\mathcal{R}^n$  and  $\mathcal{S}^n(\mathcal{H}^n)$ , the space of all  $n \times n$  real (respectively, complex) Hermitian matrices, see for example, [2], [3], [4], [5], [12], [13], [17], [18], [19], [22], and the references therein. In the case of  $\mathcal{R}^n$ , spectral sets/cones/functions are related to permutation invariance, and in  $\mathcal{S}^n(\mathcal{H}^n)$  they are precisely those that are invariant under linear transformations of the form  $X \rightarrow UXU^*$ , where  $U \in \mathcal{R}^{n \times n}$  is an orthogonal (respectively, unitary) matrix. In the general setting of Euclidean Jordan algebras, they have been studied in several works, see [1], [9], [14], [15], [16], [20], [23], and [24].

Focusing on topological/convexity/linearity properties of spectral sets/cones, in two recent papers Jeong and Gowda [15], [16] show that the multivalued map  $\lambda^{-1}$  from  $\mathcal{R}^n$  to  $\mathcal{V}$  behaves like a linear isomorphism on certain types of permutation invariant sets. Specifically, the following results are shown (where part of the first result is due to Baes [1]):

**Proposition 1.1** *Let  $Q$ ,  $Q_1$ , and  $Q_2$  be permutation invariant sets in  $\mathcal{R}^n$  with  $Q_1$  and  $Q_2$  convex. Let  $\alpha \in \mathcal{R}$ . Then*

- (a)  $\lambda^{-1}(Q)$  is open/closed/compact/convex/cone in  $\mathcal{V}$  if and only if  $Q$  is so in  $\mathcal{R}^n$ . Moreover,  $\lambda^{-1}(Q^\#) = [\lambda^{-1}(Q)]^\#$ , where  $\#$  denotes any operation of taking closure, interior, boundary, or convex/conic hull.
- (b)  $\lambda^{-1}(Q_1 + Q_2) = \lambda^{-1}(Q_1) + \lambda^{-1}(Q_2)$  and  $\lambda^{-1}(\alpha Q_1) = \alpha \lambda^{-1}(Q_1)$ .
- (c) When  $Q$  is a convex cone,  $\lambda^{-1}(Q)$  is pointed if and only if  $Q$  is pointed.

Motivated by these results, we ask if connectedness and arcwise (=pathwise) connected properties of permutation invariant  $Q$  carry over to  $\lambda^{-1}(Q)$ . In this paper, we answer these affirmatively by relying on the result that in a simple Euclidean Jordan algebra, for any element  $x$ , the eigenvalue orbit  $[x] := \{y : \lambda(y) = \lambda(x)\}$  is arcwise connected. This result will also be used to show that in any simple Euclidean Jordan algebra, every pointed spectral cone is irreducible.

## 2 Preliminaries

We let  $\mathcal{R}^n$  denote the Euclidean  $n$ -space (where the vectors are regarded as column vectors or row vectors depending on the context). In  $\mathcal{R}^n$ , we denote the standard coordinate vectors by  $c_1, c_2, \dots, c_n$ , where  $c_i$  is the vector with one in the  $i$ th slot and zeros elsewhere. We use the notation  $\Sigma_n$  to denote the set of all  $n \times n$  permutation matrices. For any set  $S$  in  $\mathcal{R}^n$ , we let  $\Sigma_n(S) := \{\sigma(s) : \sigma \in \Sigma_n, s \in S\}$ . For any vector  $q \in \mathcal{R}^n$  and a set  $Q \subseteq \mathcal{R}^n$ ,  $q^\downarrow$  denotes the decreasing rearrangement of  $q$  (i.e., the entries of  $q^\downarrow$  satisfy  $q_1^\downarrow \geq q_2^\downarrow \geq \dots \geq q_n^\downarrow$ ) and  $Q^\downarrow := \{q^\downarrow : q \in Q\}$ . A non-empty set  $Q$  in  $\mathcal{R}^n$  is said to be *permutation invariant* if  $\sigma(Q) = Q$  for all  $\sigma \in \Sigma_n$ .

We assume that the reader is familiar with standard topological notions/results dealing with (arcwise) connectedness, components, etc. Recall that a set  $S$  (say, in a topological space) is *connected* if it is not the union of two nonempty separated sets (where separation takes the form  $\overline{A} \cap B = \emptyset = A \cap \overline{B}$ , with the ‘overline’ indicating the closure) [21]. The set  $S$  is *arcwise* (= *pathwise*) *connected* if any two points of  $S$  can be joined by a continuous arc (= continuous image of an interval in  $\mathcal{R}$ ) that lies inside  $S$ . Connected (arcwise connected) components of a set  $S$  are maximal connected (respectively, arcwise connected) sets in  $S$ .

For basic things related to Euclidean Jordan algebras, we refer to [8] or [16] for a summary. *Throughout this paper*,  $\mathcal{V}$  denotes a Euclidean Jordan algebra of rank  $n$ . Recall that a Euclidean Jordan algebra  $\mathcal{V}$  is *simple* if it is not a direct product of nonzero Euclidean Jordan algebras (or equivalently, if it does not contain any non-trivial ideal). It is known (see [8], Prop. III.4.4) that any nonzero Euclidean Jordan algebra is, in a unique way, a direct product/sum of simple Euclidean Jordan algebras. Moreover, there are (in the isomorphic sense) only five simple algebras, two of which are:  $\mathcal{S}^n$ , the algebra of  $n \times n$  real symmetric matrices, and  $\mathcal{H}^n$ , the algebra of  $n \times n$  complex Hermitian matrices. The other three are:  $n \times n$  quaternion Hermitian matrices,  $3 \times 3$  octonion Hermitian matrices, and the Jordan spin algebra. In  $\mathcal{V}$ , each element  $x$  has a spectral decomposition:  $x = q_1 e_1 + q_2 e_2 + \dots + q_n e_n$ , where  $\{e_1, e_2, \dots, e_n\}$  is a Jordan frame in  $\mathcal{V}$  and the real numbers  $q_1, q_2, \dots, q_n$  are (called) the eigenvalues of  $x$ . Then  $\lambda(x) = (\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x))$  denotes the vector of eigenvalues of  $x$  written in the decreasing order. We note that

$$\lambda : \mathcal{V} \rightarrow \mathcal{R}^n \text{ is continuous ([1], Corollary 24) and } \lambda(\mathcal{V}) = (\mathcal{R}^n)^\downarrow.$$

For any  $x \in \mathcal{V}$ , we let

$$[x] := \{y \in \mathcal{V} : \lambda(y) = \lambda(x)\}$$

denote the ‘eigenvalue orbit’ of  $x$ . (The notation  $[x]_{\mathcal{V}}$  will be used when more than one algebra is involved.) Suppose  $\mathcal{V}$  is a Cartesian product (or a direct sum)  $\mathcal{V} = \mathcal{V}^{(1)} \times \mathcal{V}^{(2)} \times \dots \times \mathcal{V}^{(N)}$ , where each  $\mathcal{V}^{(i)}$  is simple. It is easy to see that in  $\mathcal{V}$ , an element  $c$  is a primitive idempotent if and only if it is of the form  $(0, 0, \dots, 0, c_i, 0, \dots, 0)$ , where  $c_i$  is a primitive idempotent in  $\mathcal{V}^{(i)}$  for some  $i$ . Applying this to elements of a Jordan frame in  $\mathcal{V}$ , we see that the eigenvalues of any

$x = (x^{(1)}, x^{(2)}, \dots, x^{(N)}) \in \mathcal{V}$  comprise of the eigenvalues of  $x^{(i)}$  in  $\mathcal{V}^{(i)}$ ,  $i = 1, 2, \dots, N$ . For such an  $x$ , we define the ‘restricted eigenvalue orbit’ by

$$[x]_r := \left\{ y = (y^{(1)}, y^{(2)}, \dots, y^{(N)}) : y^{(i)} \in [x^{(i)}]_{\mathcal{V}^{(i)}} \text{ for all } i \right\}.$$

We note that  $[x]_r \subseteq [x]$  with equality when  $\mathcal{V}$  is simple. To see an example, let  $\mathcal{V} = \mathcal{R}^n = \mathcal{R} \times \mathcal{R} \times \dots \times \mathcal{R}$  ( $n > 1$ ). Recalling that  $\{c_1, c_2, \dots, c_n\}$  denotes the set of standard coordinate vectors in  $\mathcal{R}^n$ , we see that  $[c_1] = \{c_1, c_2, \dots, c_n\}$ , while  $[c_1]_r = \{c_1\}$ .

Given a Jordan frame  $\mathcal{E} = \{e_1, e_2, \dots, e_n\}$  in  $\mathcal{V}$ , we assume that its listing/enumeration is fixed and define, for any  $q = (q_1, q_2, \dots, q_n) \in \mathcal{R}^n$ ,

$$q * \mathcal{E} := q_1 e_1 + q_2 e_2 + \dots + q_n e_n.$$

We note that

$$\lambda(q * \mathcal{E}) = q^\downarrow$$

and when  $\mathcal{E}$  is fixed,  $\Theta : q \mapsto q * \mathcal{E}$  is a continuous map from  $\mathcal{R}^n$  to  $\mathcal{V}$ .

Recall that a nonempty set in  $\mathcal{V}$  is a *spectral set* if it is of the form  $\lambda^{-1}(Q)$  for some permutation invariant set  $Q$  in  $\mathcal{R}^n$ . We will freely use the results in the following (easily verifiable) proposition.

**Proposition 2.1** *Let  $P$  and  $Q$  be nonempty subsets of  $\mathcal{R}^n$  with  $Q$  permutation invariant. Then,*

- $Q = \Sigma_n(Q^\downarrow)$ .
- $x \in \lambda^{-1}(Q)$  if and only if  $x = q * \mathcal{E}$  for some Jordan frame  $\mathcal{E}$  and  $q \in Q$ .
- $\lambda(\lambda^{-1}(P)) = P \cap P^\downarrow$  and  $\lambda^{-1}(P) = \lambda^{-1}(P \cap P^\downarrow)$ . In particular, for any  $\Omega \subseteq (\mathcal{R}^n)^\downarrow$ ,  $\lambda(\lambda^{-1}(\Omega)) = \Omega$  and  $\lambda^{-1}(Q) = \lambda^{-1}(Q^\downarrow)$ .
- Spectral sets can be generated by taking  $\lambda$ -inverse images of subsets of  $(\mathcal{R}^n)^\downarrow$ . In fact, any set of the form  $\lambda^{-1}(P)$  is a spectral set.
- The correspondence  $\Omega \mapsto \lambda^{-1}(\Omega)$  is one-to-one and onto between nonempty subsets of  $(\mathcal{R}^n)^\downarrow$  and spectral sets in  $\mathcal{V}$ .
- For a Jordan frame  $\mathcal{E} = \{e_1, e_2, \dots, e_n\}$ , the set  $\Theta(Q) = \{q * \mathcal{E} : q \in Q\}$  is independent of the listing of elements in  $\mathcal{E}$ .
- $\lambda^{-1}(Q)$  is the union of sets of the form  $\Theta(Q)$  as  $\mathcal{E}$  varies over all Jordan frames.

Recall that an *automorphism*  $\phi$  of  $\mathcal{V}$  is an invertible linear transformation on  $\mathcal{V}$  that satisfies the condition  $\phi(x \circ y) = \phi(x) \circ \phi(y)$  for all  $x, y \in \mathcal{V}$ . The set of all such transformations is denoted by  $\text{Aut}(\mathcal{V})$  and we let  $G$  denote the connected component of the identity transformation (henceforth called the *component of identity*) in  $\text{Aut}(\mathcal{V})$ . For example,  $\text{Aut}(\mathcal{S}^n)$  consists of transformations of the form  $\phi(X) := UXU^T$  ( $X \in \mathcal{S}^n$ ) where  $U \in \mathcal{R}^{n \times n}$  is an orthogonal matrix. When  $U$  has deter-

minant one, such a  $\phi$  belongs to the corresponding  $G$ . Also,  $\text{Aut}(\mathcal{H}^n)$  consists of transformations of the form  $\phi(X) := UXU^*$  ( $X \in \mathcal{H}^n$ ), where  $U$  is a unitary matrix. In this case,  $G = \text{Aut}(\mathcal{H}^n)$  ([10], page 15).

We need the following result connecting eigenvalue orbits and spectral sets.

**Proposition 2.2** (i) *The following inclusions hold:*

$$\{\phi(x) : \phi \in G\} \subseteq \{\phi(x) : \phi \in \text{Aut}(\mathcal{V})\} \subseteq [x] \quad (x \in \mathcal{V}).$$

*These become equalities when  $\mathcal{V}$  is simple.*

(ii) *A set  $E$  in  $\mathcal{V}$  is a spectral set if and only if for all  $x \in E$ ,  $[x] \subseteq E$ .*

(iii) *If  $E$  is a spectral set in  $\mathcal{V}$ , then for all  $\phi \in \text{Aut}(\mathcal{V})$ ,  $\phi(E) = E$ . Converse holds when  $\mathcal{V}$  is simple or  $\mathcal{R}^n$ .*

**Proof.** (i) As automorphisms preserve Jordan frames and eigenvalues, the stated inclusions follow. Now suppose  $\mathcal{V}$  is simple and let  $y \in [x]$  so that  $\lambda(x) = \lambda(y)$ . We write the spectral decompositions  $x = \sum_1^n \lambda_i(x)e_i$  and  $y = \sum_1^n \lambda_i(y)f_i$ , where  $\{e_1, e_2, \dots, e_n\}$  and  $\{f_1, f_2, \dots, f_n\}$  are Jordan frames in  $\mathcal{V}$ . Since  $\mathcal{V}$  is simple, by Corollary IV.2.7 in [8], there is an automorphism in  $G$  that takes one Jordan frame to the other. Hence, we may write  $y = \phi(x)$  for some  $\phi \in G$ . Thus, by (i),  $[x] = \{\phi(x) : \phi \in G\}$ .

(ii) This is given in Theorem 1, [16].

(iii) This is given in Theorem 2, [16]. □

In view of Item (iii) in the above theorem, we see that in  $\mathcal{S}^n$ , a set  $E$  is spectral if and only if

$$X \in E \Rightarrow UXU^T \in E \text{ for all orthogonal matrices } U \in \mathcal{R}^{n \times n}.$$

### 3 Main connectedness results

To motivate our discussion and results, we start with two examples.

**Example (1)** Let  $n > 1$  and consider the non-simple algebra  $\mathcal{V} = \mathcal{R}^n$ . Then,  $\text{Aut}(\mathcal{R}^n) = \Sigma_n$  and  $G$  consists of just the identity matrix. For any  $q \in \mathcal{R}^n$ ,  $\lambda(q) = q^\downarrow$ . Also, for any permutation invariant set  $Q$  in  $\mathcal{R}^n$ ,  $\lambda^{-1}(Q) = Q$ . Thus, in this setting, if a permutation invariant set is connected (arcwise connected), then its  $\lambda$ -inverse image is connected (respectively, arcwise connected). Can we improve this by assuming only the connectedness (arcwise connectedness) of  $Q^\downarrow$ ? To answer this, let  $Q := \{c_1, c_2, \dots, c_n\}$  denote the set of standard coordinate vectors in  $\mathcal{R}^n$ . Then  $Q$  is permutation invariant and  $Q^\downarrow = \{c_1\}$ . We see that while  $Q^\downarrow$  is connected (arcwise connected),  $\lambda^{-1}(Q)$  is not connected.

**Example (2)** Let  $n > 1$  and consider the simple algebra  $\mathcal{V} = \mathcal{S}^n$ . As noted previously,  $\text{Aut}(\mathcal{S}^n)$  consists of transformations of the form  $\phi(X) := UXU^T$  ( $X \in \mathcal{S}^n$ ) where  $U \in \mathcal{R}^{n \times n}$  is an orthogonal matrix. Now consider the set  $Q := \{c_1, c_2, \dots, c_n\}$  of Example 1. As noted earlier,  $Q$  is a permutation invariant set with  $Q^\perp$  connected. With  $D$  denoting the diagonal matrix with  $c_1$  on its diagonal, we have

$$\lambda^{-1}(Q) = \lambda^{-1}(Q^\perp) = \left\{ UDU^T : U \in \mathcal{R}^{n \times n} \text{ is orthogonal} \right\} = \left\{ uu^T : u \in \mathcal{R}^n, \|u\| = 1 \right\}.$$

(The vector  $u$  that appears above is the first column of  $U$ .) As  $n > 1$ , the unit sphere in  $\mathcal{R}^n$  is arcwise connected. Hence,  $\lambda^{-1}(Q)$  is arcwise connected. Thus, in contrast to Example 1, a weaker hypothesis involving connectedness (arcwise connectedness) of  $Q^\perp$  suffices to get the connectedness (arcwise connectedness) of  $\lambda^{-1}(Q)$ . We show below that in the setting of a simple algebra, such a statement holds for any permutation invariant set.

The following is a basic (possibly known) result about the connectedness of eigenvalue orbits. Recall that  $G$  is the connected component of identity in  $\text{Aut}(\mathcal{V})$ .

**Theorem 3.1** *The following statements hold:*

- (a)  $G$  is arcwise connected.
- (b) When  $\mathcal{V}$  is simple, for any  $x \in \mathcal{V}$ ,  $[x]$  is arcwise connected.
- (c) For any  $x \in \mathcal{V}$ ,  $[x]_r$  is arcwise connected.

**Proof.** (a) As  $\text{Aut}(\mathcal{V})$  is a (matrix) Lie group (see [8], page 36), its connected component of identity, namely  $G$ , is arcwise connected, see [6], Theorem 1.9.1.

(b) Now suppose  $\mathcal{V}$  is simple and consider any  $x \in \mathcal{V}$ . Then, by Item (i) in the previous proposition,  $[x] = \{\phi(x) : \phi \in G\}$ . As  $G$  is arcwise connected and the map  $\phi \mapsto \phi(x)$  is continuous,  $[x]$  is also arcwise connected.

(c) If  $\mathcal{V}$  is simple,  $[x]_r = [x]$  for any  $x$ . Then, the result follows from (b). Suppose  $\mathcal{V}$  is non-simple, and let  $\mathcal{V}$  be a product of simple algebras:  $\mathcal{V} = \mathcal{V}^{(1)} \times \mathcal{V}^{(2)} \times \dots \times \mathcal{V}^{(N)}$ , where each  $\mathcal{V}^{(i)}$  is simple. Let  $x = (x^{(1)}, x^{(2)}, \dots, x^{(N)}) \in \mathcal{V}$ . Then, by (b), each  $[x^{(i)}]_{\mathcal{V}^{(i)}}$  is arcwise connected. Hence, the set

$$[x]_r = [x^{(1)}]_{\mathcal{V}^{(1)}} \times [x^{(2)}]_{\mathcal{V}^{(2)}} \times \dots \times [x^{(N)}]_{\mathcal{V}^{(N)}},$$

being a product of arcwise connected sets, is arcwise connected.  $\square$

To illustrate Items (b) and (c) in the above result, we let  $x$  be a primitive idempotent in a simple algebra  $\mathcal{V}$ . Then,  $[x]$ , which is the set of all primitive idempotents, is arcwise connected. (This result is known, see [8], page 71.) This may fail if  $\mathcal{V}$  is not simple: let  $\mathcal{V} = \mathcal{R}^n$  ( $n > 1$ ) and  $x = c_1$ .

Then,  $[x] = \{c_1, c_2, \dots, c_n\}$  is not connected. However,  $[x]_r = \{c_1\}$  (a singleton set) is arcwise connected.

We now state a necessary condition for a  $\lambda$ -inverse image to be connected (arcwise connected).

**Proposition 3.2** *Let  $P \subseteq \mathcal{R}^n$ . If  $\lambda^{-1}(P)$  is connected (arcwise connected) in  $\mathcal{V}$ , then  $P \cap P^\downarrow$  is connected (arcwise connected) in  $\mathcal{R}^n$ . In particular, if  $Q$  is permutation invariant and  $\lambda^{-1}(Q)$  is connected (arcwise connected) in  $\mathcal{V}$ , then  $Q^\downarrow$  is connected (arcwise connected) in  $\mathcal{R}^n$ .*

**Proof.** The first statement follows from the continuity of the eigenvalue map  $\lambda$  and the equality  $P \cap P^\downarrow = \lambda(\lambda^{-1}(P))$ . The second one follows from the equality  $Q \cap Q^\downarrow = Q^\downarrow$ .  $\square$

The next two results deal with sufficient conditions.

**Theorem 3.3** *Suppose  $Q \subseteq \mathcal{R}^n$  is permutation invariant and one of the following conditions holds.*

- (a)  $\mathcal{V}$  is simple and  $Q^\downarrow$  is connected (arcwise connected).
- (b)  $Q$  is connected (arcwise connected).

*Then  $\lambda^{-1}(Q)$  is connected (respectively, arcwise connected) in  $\mathcal{V}$ .*

**Proof.** Suppose condition (a) holds so that  $\mathcal{V}$  is simple and  $Q^\downarrow$  is connected (arcwise connected). We fix a Jordan frame  $\mathcal{E} = \{e_1, e_2, \dots, e_n\}$  and consider the continuous map  $\Theta : \mathcal{R}^n \rightarrow V$  defined by  $\Theta(q) := q * \mathcal{E}$ , see Section 2. Then, the image  $\Theta(Q^\downarrow)$ , which is a subset of  $\lambda^{-1}(Q^\downarrow)$ , is connected (respectively, arcwise connected). Now let  $x \in \lambda^{-1}(Q^\downarrow)$  be arbitrary. Then,  $\lambda(x) \in Q^\downarrow$  and there is a Jordan frame  $\{f_1, f_2, \dots, f_n\}$  such that  $x = \lambda_1(x)f_1 + \lambda_2(x)f_2 + \dots + \lambda_n(x)f_n$ . As  $\mathcal{V}$  is simple, from Theorem 3.1,  $[x]$  is arcwise connected. Also,

$$\lambda_1(x)e_1 + \lambda_2(x)e_2 + \dots + \lambda_n(x)e_n \in \Theta(Q^\downarrow) \cap [x].$$

So the sets  $\Theta(Q^\downarrow)$  and  $[x]$  are connected (respectively, arcwise connected) and their intersection is nonempty. It follows that their union is also connected (respectively, arcwise connected). As  $x$  is arbitrary in  $\lambda^{-1}(Q^\downarrow)$ , the connected component (respectively, arcwise connected component) of  $\lambda^{-1}(Q^\downarrow)$  that contains  $\Theta(Q^\downarrow)$  must be  $\lambda^{-1}(Q^\downarrow)$  itself. This proves that  $\lambda^{-1}(Q^\downarrow)$  is connected (respectively, arcwise connected) under the condition (a). The stated assertion follows since  $\lambda^{-1}(Q) = \lambda^{-1}(Q^\downarrow)$ .

Now suppose condition (b) holds. If  $\mathcal{V}$  is simple, we can use the previous argument as  $Q^\downarrow$  (which is the image of  $Q$  under the continuous map  $q \mapsto q^\downarrow$ ) is connected (arcwise connected). So, assume that  $\mathcal{V}$  is non-simple and write it as an orthogonal direct sum  $\mathcal{V} = \mathcal{V}^{(1)} \oplus \mathcal{V}^{(2)} \oplus \dots \oplus \mathcal{V}^{(N)}$ , where each  $\mathcal{V}^{(i)}$  is simple. In  $\mathcal{V}$ , we fix a Jordan frame  $\mathcal{E} := \{e_1, e_2, \dots, e_n\}$  and consider the continuous map  $\Theta : \mathcal{R}^n \rightarrow \mathcal{V}$  defined by  $\Theta(q) = q * \mathcal{E}$ . As  $Q$  is connected (arcwise connected) and  $\Theta$

is continuous,  $\Theta(Q)$  is connected (arcwise connected). Also, as  $Q$  is permutation invariant, any rearrangement/listing of elements of  $\mathcal{E}$  will not alter the set  $\Theta(Q)$ . Hence, we may assume without loss of generality that  $\mathcal{E} = \bigcup_1^N \mathcal{E}^{(i)}$ , where each  $\mathcal{E}^{(i)}$  is a Jordan frame in  $V^{(i)}$ . Writing  $q \in \mathcal{R}^n$  in the block form  $q = (q^{(1)}, q^{(2)}, \dots, q^{(N)})$ , we see that  $q * \mathcal{E} = q^{(1)} * \mathcal{E}^{(1)} + q^{(2)} * \mathcal{E}^{(2)} + \dots + q^{(N)} * \mathcal{E}^{(N)}$ . Now, let  $x \in \lambda^{-1}(Q)$ . Then, there exist a Jordan frame  $\mathcal{F}$  in  $\mathcal{V}$  and a  $p \in Q$  such that  $x = p * \mathcal{F}$ . Similar to the above, we may write  $x = p * \mathcal{F} = p^{(1)} * \mathcal{F}^{(1)} + p^{(2)} * \mathcal{F}^{(2)} + \dots + p^{(N)} * \mathcal{F}^{(N)}$ , where  $\mathcal{F}^{(i)}$  is a Jordan frame in  $V^{(i)}$ . Let  $y := p * \mathcal{E} = p^{(1)} * \mathcal{E}^{(1)} + p^{(2)} * \mathcal{E}^{(2)} + \dots + p^{(N)} * \mathcal{E}^{(N)}$ . As  $\lambda(p^{(i)} * \mathcal{E}^{(i)}) = \lambda(p^{(i)} * \mathcal{F}^{(i)})$  for all  $i$ , it follows that  $y \in [x]_r$ . Hence,  $y \in \Theta(Q) \cap [x]_r$ . As  $\Theta(Q)$  is connected (arcwise connected) and  $[x]_r$  is arcwise connected (by Theorem 3.1), it follows that  $\Theta(Q) \cup [x]_r$  is connected (respectively, arcwise connected). Since  $x$  is arbitrary in  $\lambda^{-1}(Q)$ , the connected (arcwise connected) component of  $\lambda^{-1}(Q)$  must be  $\lambda^{-1}(Q)$ . This proves that  $\lambda^{-1}(Q)$  is connected (respectively, arcwise connected).  $\square$

**Corollary 3.4** *When  $\mathcal{V}$  is simple, the following statements hold:*

- (i) *If  $\Omega$  is any subset of  $(\mathcal{R}^n)^\downarrow$  that is connected (arcwise connected), then  $\lambda^{-1}(\Omega)$  is connected (respectively, arcwise connected).*
- (ii) *If  $P$  is any subset of  $\mathcal{R}^n$  such that  $P \cap P^\downarrow$  is connected (arcwise connected), then  $\lambda^{-1}(P)$  is connected (respectively, arcwise connected).*

**Proof.** (i) Let  $\Omega$  be connected (arcwise connected) subset of  $(\mathcal{R}^n)^\downarrow$ . Then  $Q := \Sigma_n(\Omega)$  is permutation invariant and  $Q^\downarrow = \Omega$ . Hence condition (a) in the above theorem applies. Since  $\lambda^{-1}(\Omega) = \lambda^{-1}(Q)$ , we see that  $\lambda^{-1}(\Omega)$  is connected (arcwise connected).

(ii) Suppose  $P$  is any subset of  $\mathcal{R}^n$  such that  $P \cap P^\downarrow$  is connected (arcwise connected). As  $P \cap P^\downarrow$  is a subset of  $(\mathcal{R}^n)^\downarrow$  and  $\lambda^{-1}(P) = \lambda^{-1}(P \cap P^\downarrow)$ , the stated result follows from (i) applied to  $\Omega := P \cap P^\downarrow$ .  $\square$

**Remarks (1)** Suppose  $\mathcal{V}$  is simple. In view of Proposition 3.2, for any set  $P$  in  $\mathcal{R}^n$ , connectedness (arcwise connectedness) of  $P \cap P^\downarrow$  is necessary and sufficient for that of  $\lambda^{-1}(P)$ . Thus, *a spectral set  $E$  in a simple algebra  $\mathcal{V}$  is connected (arcwise connected) if and only if  $\lambda(E)$  is connected (arcwise connected) in  $\mathcal{R}^n$ .*

**(2)** The conclusions in the above corollary may fail if  $\mathcal{V}$  is not simple. For example in  $\mathcal{R}^3$ ,  $c_1 = (1, 0, 0)$  is on the boundary of  $(\mathcal{R}^3)^\downarrow$ . Letting  $\mathcal{V} = \mathcal{R}^3$  and  $\Omega = \{c_1\}$ , we see that  $\lambda^{-1}(\Omega) = \{c_1, c_2, c_3\}$  (set of standard coordinate vectors in  $\mathcal{R}^3$ ) is not connected. In the same setting, it is easy to construct an arcwise connected set  $P$  such that  $P \cap P^\downarrow = \{c_1\}$ . (For example,  $P$  could be a circle through  $c_1$  such that  $P \setminus \{c_1\}$  is outside of  $(\mathcal{R}^3)^\downarrow$ .) For such a  $P$ ,  $\lambda^{-1}(P) = \lambda^{-1}(P \cap P^\downarrow) = \{c_1, c_2, c_3\}$  is not connected.



(3) Motivated by the above two results, one may ask if  $\lambda$ -inverse image of a simply connected permutation invariant set  $Q$  in  $\mathcal{R}^n$  is simply connected in  $\mathcal{V}$ . While the answer to this is not clear, we note that in  $\mathcal{S}^2$ , the set of all primitive idempotents is given by ([8], page 71)

$$\left\{ \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}, 0 \leq \theta \leq \pi \right\}.$$

This is homeomorphic to a circle, hence not simply connected. However, it is of the form  $\lambda^{-1}(Q^\downarrow)$ , where  $Q$  is the set of standard coordinate vectors in  $\mathcal{R}^2$ . While this  $Q$  is not simply connected,  $Q^\downarrow$ , consisting of only one element, is simply connected. So the counterpart of (a) in Theorem 3.3 may not hold for simple connectedness.

We end this section by mentioning a classical result of Ky Fan. Suppose  $Q$  is a permutation invariant set that satisfies one of the conditions of Theorem 3.3. Assume further that  $Q$  is compact. Then,  $\lambda^{-1}(Q)$  is connected and (by Proposition 1.1) compact in  $\mathcal{V}$ . Hence, for any  $c \in \mathcal{V}$ , the image of  $\lambda^{-1}(Q)$  under the continuous function  $x \mapsto \langle c, x \rangle$  is connected and compact in  $\mathcal{R}$ . So, there exists real numbers  $\delta$  and  $\Delta$  such that

$$\left\{ \langle c, x \rangle : x \in \lambda^{-1}(Q) \right\} = [\delta, \Delta]. \quad (1)$$

(With some additional work, it is possible to describe the forms of  $\delta$  and  $\Delta$ .) To see a special case, consider two matrices  $C$  and  $A$  in  $\mathcal{H}^n$  and let  $Q = \Sigma_n(\{\lambda(A)\})$ . Clearly,  $Q$  is permutation invariant and compact (actually, finite). Since  $\mathcal{H}^n$  is simple and  $Q^\downarrow$  (a singleton set) is connected, by Theorem 3.3 (or by Theorem 3.1),  $\lambda^{-1}(Q) = [A] = \{UAU^* : U \in \mathcal{C}^{n \times n} \text{ is unitary}\}$  is connected in  $\mathcal{H}^n$ . As  $\langle X, Y \rangle = \text{tr}(XY)$  in  $\mathcal{H}^n$ , (1) reads:

$$\left\{ \text{tr}(CUAU^*) : U \in \mathcal{C}^{n \times n} \text{ is unitary} \right\} = [\delta, \Delta].$$

With  $\Delta := \langle \lambda(C), \lambda(A) \rangle$  and  $\delta := \langle \lambda^\uparrow(C), \lambda(A) \rangle$ , where  $\lambda^\uparrow(C)$  denotes the increasing rearrangement of  $\lambda(C)$ , this statement is due to Fan [7] (see also [25], Corollary 1.6).

## 4 Components of a spectral set

We now describe connected (arcwise connected) components of a spectral set in a simple algebra.

**Theorem 4.1** *Let  $\mathcal{V}$  be simple and  $E := \lambda^{-1}(Q)$  be a spectral set in  $\mathcal{V}$ , where  $Q$  is permutation invariant in  $\mathcal{R}^n$ . Then, every connected (arcwise connected) component of  $E$  is a spectral set. Moreover,  $C \rightarrow \lambda(C)$  is a one-to-one correspondence between connected (arcwise connected) components of  $E$  and those of  $Q^\downarrow$ .*

**Proof.** We consider the case of connected components. The case of arcwise connected components

is similar. Let  $C$  be a connected component of  $E$ . As  $\mathcal{V}$  is simple, for any  $x \in C$ ,  $[x]$  is (arcwise) connected (by Theorem 3.1) and  $[x] \cap C$  contains  $x$ . Hence,  $[x] \subseteq C$ . By Proposition 2.2(ii),  $C$  is a spectral set; we may now write  $C$  as  $C = \lambda^{-1}(P)$ , where  $P$  is a permutation invariant set in  $\mathcal{R}^n$ . As  $C$  is connected, its image  $\lambda(C) = P^\downarrow$  is connected. We claim that this set is a connected component of  $Q^\downarrow$ . To simplify the notation, let  $\Omega := P^\downarrow$ . Then,

$$C \subseteq E \Rightarrow P^\downarrow = \lambda(C) \subseteq \lambda(E) = Q^\downarrow \Rightarrow \Omega \subseteq Q^\downarrow.$$

So,  $\Omega$  is a connected subset of  $Q^\downarrow$ . Let  $\Omega^*$  be the connected component of  $Q^\downarrow$  that contains  $\Omega$ , so

$$\Omega \subseteq \Omega^* \subseteq Q^\downarrow.$$

Then,

$$C = \lambda^{-1}(\Omega) \subseteq \lambda^{-1}(\Omega^*) \subseteq \lambda^{-1}(Q^\downarrow) = \lambda^{-1}(Q) = E.$$

Since  $\mathcal{V}$  is simple, by Item (i) in Corollary 3.4,  $\lambda^{-1}(\Omega^*)$  is connected. By our assumption that  $C$  is a connected component of  $E$  we get  $C = \lambda^{-1}(\Omega^*)$ , leading to  $\Omega = \lambda(C) = \lambda(\lambda^{-1}(\Omega^*)) = \Omega^*$ , the last equality is due to Proposition 2.1. Hence,  $\Omega$  is a connected component of  $Q^\downarrow$ .

Now we show that every connected component of  $Q^\downarrow$  arises this way. Let  $\Omega$  be a connected component of  $Q^\downarrow$ . By Item (i) in Corollary 3.4,  $C := \lambda^{-1}(\Omega)$  is connected in  $E$ . Suppose  $C^*$  is the connected component of  $E$  that contains  $C$ . We show that  $C = C^*$ . Now,

$$C \subseteq C^* \subseteq E \Rightarrow \lambda(C) \subseteq \lambda(C^*) \subseteq \lambda(E).$$

We have  $\lambda(C) = \lambda(\lambda^{-1}(\Omega)) = \Omega$ , where the second equality is due to Proposition 2.1. It follows that  $\Omega \subseteq \lambda(C^*) \subseteq Q^\downarrow$ . As  $\Omega$  is a connected component and  $\lambda(C^*)$  is connected, we must have  $\Omega = \lambda(C^*)$ , that is,  $\Omega = \lambda(C) = \lambda(C^*)$ . So,  $C = \lambda^{-1}(\Omega) = \lambda^{-1}(\lambda(C^*)) \supseteq C^*$ . Since  $C \subseteq C^*$ , we conclude that  $C = C^*$  and that  $C$  is a connected component of  $E$ . Finally, the concluding statement in the theorem about one-to-one and onto correspondence is easily verified.  $\square$

**Remarks (4)** The conclusions in the above result may fail if  $\mathcal{V}$  is not simple, see Example 1.

## 5 Irreducibility of spectral cones

We now provide another application of Theorem 3.1. Recall that a (nonempty) set  $K$  is a convex cone if it is convex and  $tx \in K$  for all  $x \in K$  and  $t \geq 0$  in  $\mathcal{R}$ . If, in addition,  $K \cap -K = \{0\}$ , then  $K$  is said to be a pointed convex cone. We say that a convex cone  $K$  is *reducible* if it can be written as a sum  $K = K_1 + K_2$  where  $K_1$  and  $K_2$  are nonzero convex cones with  $\text{span}(K_1) \cap \text{span}(K_2) = \{0\}$ . A convex cone that is not reducible is said to be *irreducible*. (These concepts hold in  $\mathcal{R}^n$  as well.) We now address the irreducibility of spectral cones. In  $\mathcal{R}^n$ , spectral cones (which are just permutation invariant convex cones) can be reducible or irreducible. In fact,  $\mathcal{R}_+^n$  ( $n > 1$ ) is reducible, while

$\mathcal{R}_+ \mathbf{1}$  is irreducible, where  $\mathbf{1}$  represents the vector of ones in  $\mathcal{R}^n$ . In a simple Euclidean Jordan algebra, the corresponding symmetric cone (=the cone of squares which is a closed convex self-dual homogeneous cone) is irreducible (see [8], Prop. III.4.5), even though it is of the form  $\lambda^{-1}(\mathcal{R}_+^n)$  with  $\mathcal{R}_+^n$  reducible for  $n > 1$ . Below, we show that any pointed spectral cone in a simple Euclidean Jordan algebra is irreducible. First, we recall a well-known result ([11], slightly modified to suit our setting): *If  $K$  is a nonzero pointed reducible convex cone in  $\mathcal{V}$ , then there exists a unique set of nonzero irreducible convex cones  $K_i$ ,  $i = 1, 2, \dots, r$ , such that*

$$K = K_1 + K_2 + \dots + K_r$$

*with  $\text{span}(M) \cap \text{span}(N) = \{0\}$ , whenever  $M$  denotes the sum of some  $K_i$ s and  $N$  denotes the sum of the rest of the  $K_i$ s. Moreover, the above representation is unique up to permutation of indices.* In this setting, we say that  $K$  is a *direct sum* of  $K_i$ s.

**Theorem 5.1** *Suppose  $\mathcal{V}$  is simple. Then, every pointed spectral cone in  $\mathcal{V}$  is irreducible.*

**Proof.** If possible, suppose  $K$  is a nonzero pointed spectral cone in  $\mathcal{V}$  which is a direct sum of irreducible convex cones:

$$K = K_1 + K_2 + \dots + K_r$$

with  $r > 1$ . We claim that each  $K_i$  is a pointed spectral cone. The pointedness of  $K_i$  is clear, as  $K$  is pointed. We show that  $K_1$  is a spectral cone. Since  $\mathcal{V}$  is simple, because of Items (i) and (ii) in Proposition 2.2, it is enough to show that for all  $x \in K_1$ ,  $[x] = \{\phi(x) : \phi \in G\} \subseteq K_1$ . Let  $\phi \in G$ . Since  $K$  is a spectral cone,  $\phi(K) = K$  (by Item (iii) in Proposition 2.2). This implies that  $K = \phi(K_1) + \phi(K_2) + \dots + \phi(K_r)$  is another direct sum representation in terms of irreducible convex cones. By the uniqueness of factors in the decomposition,  $\phi(K_1)$  must be equal to some  $K_j$ . So,

$$\phi(K_1) \subseteq \bigcup_{j=1}^r K_j, \quad \forall \phi \in G.$$

Now, let  $0 \neq x \in K_1$ . Then, all the elements in  $[x]$  are nonzero and

$$[x] = \{\phi(x) : \phi \in G\} \subseteq A \cup B,$$

where  $A := K_1 \setminus \{0\}$  and  $B := \bigcup_{j=2}^r K_j \setminus \{0\}$ . Since  $K_1 \cap \overline{B} \subseteq \text{span}(K_1) \cap \text{span}(B) = \{0\}$ , we see that  $A \cap \overline{B} = \emptyset$ . (Here, ‘overline’ denotes the closure.) Similarly,  $B \cap \overline{A} = \emptyset$ . So the sets  $A$  and  $B$  are separated [21]. Now, by Theorem 3.3 (or by Theorem 3.1),  $[x]$  is connected. As  $[x] \subseteq A \cup B$  and  $0 \neq x \in A$ , we must have  $[x] \subseteq A$ . We conclude that  $[x] \subseteq K_1$ . This inclusion also holds for  $x = 0$  so

$$[x] \subseteq K_1, \quad \forall x \in K_1.$$

Thus,  $K_1$  is a spectral cone by Item (ii) in Proposition 2.2. A similar argument works for all other cones  $K_i$ . Now, all the spectral cones  $K, K_1, K_2, \dots, K_r$  are pointed spectral cones. Hence, by Theorem 7.3 in [15],  $e$  (the unit element of  $\mathcal{V}$ ) or  $-e$  belongs to all of them. Since  $K$  is pointed, either  $e$  or  $-e$ , but not both, can belong to all. Moreover, since the cones  $K_i$  have zero as the only common element, we conclude that  $r = 1$ . This violates our assumption that  $r > 1$ . Hence,  $K$  is irreducible.  $\square$

The above result gives an alternate proof of the fact that in any simple algebra, the symmetric cone is irreducible. Also, every pointed convex cone  $K$  in  $\mathcal{S}^n$  satisfying the condition

$$X \in K \Rightarrow UXU^T \in K \text{ for all orthogonal matrices } U \in \mathcal{R}^{n \times n}$$

is irreducible. We provide two more examples.

**Example (3)** Let  $n \geq 3$  and  $m \in \{1, 2, \dots, n-1\}$ . For each  $q \in \mathcal{R}^n$ , let  $s_m(q)$  be the sum of the smallest  $m$  entries of  $q$ . Then, the ‘rearrangement cone’

$$Q_m^n = \{q \in \mathcal{R}^n : s_m(q) \geq 0\},$$

is a permutation invariant pointed closed convex cone in  $\mathcal{R}^n$  [14]. Hence, by Item (c) in Proposition 1.1,

$$\lambda^{-1}(Q_m^n) = \{x \in \mathcal{V} : \lambda_n(x) + \lambda_{n-1}(x) + \dots + \lambda_{n-m+1}(x) \geq 0\}$$

is a pointed convex cone. (Note that when  $m = 1$ , this cone is the symmetric cone of  $\mathcal{V}$ .) By the above theorem, this cone, in a simple algebra, is irreducible.

**Example (4)** Let  $n \geq 3$  and consider the following permutation invariant cone (see Example 2 in [15])

$$Q := \left\{ q \in \mathcal{R}^n : \text{tr}(q) \geq \sqrt{\frac{n}{2}} \|q\|_2 \right\}$$

where  $\text{tr}(q)$  denotes the sum of all entries of  $q$  and  $\|q\|_2$  denotes the 2-norm of  $q$ . This cone is a proper cone (that is, it is a closed convex pointed cone with nonempty interior). By the above theorem, when  $\mathcal{V}$  is a simple algebra of rank  $n$ , the proper spectral cone

$$K := \lambda^{-1}(Q) = \left\{ x \in \mathcal{V} : \text{tr}(x) \geq \sqrt{\frac{n}{2}} \|x\|_2 \right\}$$

is irreducible, where  $\text{tr}(x)$  denotes the sum of all eigenvalues of  $x$  and  $\|x\|_2 := \|\lambda(x)\|_2$ . Using the linearity of the trace and the strict convexity of  $\|\cdot\|_2$ , it is easy to show that every boundary vector in  $K$  is an extreme vector. As  $n$  (the rank of  $\mathcal{V}$ ) is at least 3, such a property is false for the symmetric cone of  $\mathcal{V}$  (take a Jordan frame  $\{e_1, e_2, \dots, e_n\}$  and consider  $e_1 + e_2$  which is on the boundary of the symmetric cone of  $\mathcal{V}$ , but not an extreme vector); so,  $K$  and the symmetric cone

of  $\mathcal{V}$  are not isomorphic. This shows that in every simple algebra of rank  $n \geq 3$ , there is a proper (irreducible) spectral cone that is not isomorphic to the corresponding symmetric cone.

## References

- [1] M. Baes, *Convexity and differentiability properties of spectral functions in Euclidean Jordan algebras*, Linear Algebra Appl., 422 (2007) 664-700.
- [2] J. Borwein and A.S. Lewis, *Convex Analysis and Nonlinear Optimization*, Springer-Verlag, New York, 2006.
- [3] V. Chandrasekaran, P.A. Parrilo, and A.S. Willsky, *Convex graph invariants*, SIAM Review, 54 (2012) 513-541.
- [4] A. Daniilidis, A. Lewis, J. Malick, and H. Sendov, *Prox-regularity of spectral functions and spectral sets*, J. Convex Anal., 15 (2008) 547-560.
- [5] A. Daniilidis, D. Drusvyatskiy, and A.S. Lewis, *Orthogonal invariance and identifiability*, SIAM J. Matrix. Anal., 35 (2014) 580-598.
- [6] J.J. Duistermaat and J.A.C. Kolk, *Lie Groups*, Springer-Verlag, Berlin, 2000.
- [7] K. Fan, *On a theorem of Weyl concerning eigenvalues of linear transformations I*, Proc. Nat. Acad. Sci. U.S.A., 35 (1949) 652-655.
- [8] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford, 1994.
- [9] M.S. Gowda and J. Jeong, *Commutation principles in Euclidean Jordan algebras and normal decomposition systems*, SIAM J. Optim., 27 (2017) 1390-1402.
- [10] B.C. Hall, *Lie Groups, Lie Algebras, and Representations*, Springer-Verlag Graduate Texts in Mathematics, New York, 2003.
- [11] R. Hauser and O. Güler, *Self-scaled barrier functions on symmetric cones and their classification*, Foundations of Comput. Math., 2 (2002) 121-143.
- [12] R. Henrion and A. Seeger, *Inradius and circumradius of various convex cones arising in applications*, Set Valued Analysis, 18 (2010) 483-511.
- [13] A. Iusem and A. Seeger, *Angular analysis of two classes of non-polyhedral convex cones: the point of view of optimization theory*, Computational & Appl. Math., 26 (2007) 191-214.
- [14] J. Jeong, *Spectral sets and functions on Euclidean Jordan algebras*, PhD Thesis, University of Maryland, Baltimore County, 2017.

- [15] J. Jeong and M.S. Gowda, *Spectral cones in Euclidean Jordan algebras*, Linear Algebra Appl., 509 (2016) 286-305.
- [16] J. Jeong and M.S. Gowda, *Spectral sets and functions in Euclidean Jordan algebras*, Linear Algebra Appl., 518 (2017) 31-56.
- [17] A. S. Lewis, *Group invariance and convex matrix analysis*, SIAM J. Matrix Anal., 17 (1996) 927-949.
- [18] A. S. Lewis, *Convex analysis on the Hermitian matrices*, SIAM J. Optim., 6 (1996) 164-177.
- [19] A.S. Lewis and H.S. Sendov, *Twice differentiable spectral functions*, SIAM J. Matrix Anal., 23 (2001) 368-386.
- [20] H. Ramirez, A. Seeger, and D. Sossa, *Commutation principle for variational problems on Euclidean Jordan algebras*, SIAM J. Optim., 23 (2013) 687-694.
- [21] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, 2006.
- [22] A. Seeger, *Convex analysis of spectrally defined matrix functions*, SIAM J. on Optim., 7 (1997) 679-696.
- [23] D. Sossa, *Euclidean Jordan algebras and variational problems under conic constraints*, PhD Thesis, University of Chile, 2014.
- [24] D. Sun and J. Sun, *Löwner's operator and spectral functions in Euclidean Jordan algebras*, Math. Operations Res., 33 (2008) 421-445.
- [25] T.-Y. Tam, *An extension of a result of Lewis*, Electronic J. Linear Algebra, 5 (1999) 1-10.