

Access to this work was provided by the University of Maryland, Baltimore County (UMBC) ScholarWorks@UMBC digital repository on the Maryland Shared Open Access (MD-SOAR) platform.

Please provide feedback

Please support the ScholarWorks@UMBC repository by emailing [scholarworks-group@umbc.edu](mailto:scholarworks-group@umbc.edu) and telling us what having access to this work means to you and why it's important to you. Thank you.

# A generalized model of flocking with steering

Guy A. Djokam<sup>1</sup> and Muruhan Rathinam<sup>2</sup>

<sup>1</sup>Department of Mathematics and Statistics, University of Maryland  
Baltimore County

<sup>2</sup>Department of Mathematics and Statistics, University of Maryland  
Baltimore County

## Abstract

We introduce and analyze a model for the dynamics of flocking and steering of a finite number of agents. In this model, each agent's acceleration consists of flocking and steering components. The flocking component is a generalization of many of the existing models and allows for the incorporation of many real world features such as acceleration bounds, partial masking effects and orientation bias. The steering component is also integral to capture real world phenomena. We provide rigorous sufficient conditions under which the agents flock and steer together. We also provide a formal singular perturbation study of the situation where flocking happens much faster than steering. We end our work by providing some numerical simulations to illustrate our theoretical results.

## 1 Introduction

The emergence of phenomena such as flocking of birds, schooling of fish and swarming of bacteria have attracted considerable attention by mathematicians, scientists and engineers in the recent years. See [2, 5, 6, 7, 13, 14], and references therein. Studying these phenomena not only help us understand the natural world, but also help us better engineer systems such as unmanned aerial vehicles. In [18], Viscek and his team introduced a novel discrete time dynamics to investigate the emergence of self ordered motion. In Viscek's model, all agent have the same absolute velocity and at each step, they adjust their orientation based on their neighbors orientation. Inspired by this model, Cucker and Smale proposed the celebrated continuous time model [6], which led to many other subsequent studies. The Cucker-Smale (CS) model is:

$$\begin{aligned}\frac{dx_i}{dt} &= v_i, \\ \frac{dv_i}{dt} &= \frac{\alpha}{N} \sum_{j=1}^N a_{ij}(v_j - v_i),\end{aligned}\tag{1.1}$$

where  $N$  is the number of agents,  $x_i$  and  $v_i$  are the position and velocity of agent  $i$ , and the influence  $a_{ij}$  of agent  $j$  on agent  $i$  is assumed to be symmetric ( $a_{ij} = a_{ji}$ ) and is a function of the Euclidean distance  $\|x_i - x_j\|$  between  $i$  and  $j$ , so that  $a_{ij} = \phi(\|x_i - x_j\|)$ . The function  $\phi$  was chosen to be  $\phi(r) = \frac{K}{(a^2 + r^2)^\beta}$ , so that it was positive and non increasing.

Cucker and Smale defined *flocking* by the condition that

$$\sup_{t \geq 0} \|x_i(t) - x_j(t)\| < \infty$$

and that

$$\lim_{t \rightarrow \infty} \|v_i(t) - v_j(t)\| = 0$$

for every pair  $(i, j)$  of agents. The analysis of the CS model is based on the parameter  $\beta$  and it is shown [6] that if  $\beta < 1/2$  there is unconditional flocking and if  $\beta \geq 1/2$  then flocking depends on initial conditions. While the symmetric property of the influence functions led to ease of mathematical analysis, it is not realistic to assume symmetry.

Motivated by the CS model, many variants have been extensively studied in the literature. For instance, in [5, 15] the authors propose models to address collision avoidance and in [8] the authors study a modified CS model with nonlinear velocity couplings. A stochastic version of the CS model with multiplicative white noise is studied in [1, 9]. In [16, 3, 13] authors study model with hierarchical leader. An elegant analysis of flocking via the use of a system of differential inequalities coupled with a Lyapunov function was introduced in [10]. In [13], Motsch and Tadmor present a more general model where the symmetry assumption on the influence functions is dropped. The Motsch and Tadmor (MT) model is given by

$$\begin{aligned} \frac{dx_i}{dt} &= v_i \\ \frac{dv_i}{dt} &= \alpha(\bar{v}_i - v_i) \end{aligned} \tag{1.2}$$

where  $\bar{v}_i = \sum_{j=1}^N a_{ij} v_j$  is a convex combination of the influences of all agents  $j$  on agent  $i$  so that  $\sum_j a_{ij} = 1$  and  $a_{ij} \geq 0$ . In this model,  $\alpha > 0$  is a constant while  $a_{ij}$  are taken to be some function of the pairwise distances of the following form:

$$a_{ij}(x) = \frac{\phi(\|x_i - x_j\|)}{\sum_k \phi(\|x_i - x_k\|)}$$

where  $\phi$  is a nonnegative function of distance. This form of  $a_{ij}$  lead to lack of symmetry ( $a_{ij} \neq a_{ji}$ ) and necessitated Motsch and Tadmor to introduce some new ideas into the analysis of flocking; in particular the concept of *maximal action* by a skew-symmetric matrix and the notion of an *active set*.

Our study is based on a finite number of agents where each agent follows a similar rule though parameters appearing in these rules may vary from agent to agent. The notion of the presence of leader agents is an important concept and has been investigated in [13, 16]. It is important to mention the development of continuum models which arise as limiting models when the number of agents approaches infinity. These models are based on partial differential equations that describe the evolution of the density of the agents that formed

the system. See [4, 11, 13] and reference therein. It must be noted that flocking models usually are concerned with a number of agents moving in the physical space and Newton's laws dictate that such systems have a second order dynamics so that it is the accelerations of agents that are usually controlled. Models of first order self-organized systems commonly arise in other applications such as opinion dynamics models or flocking situations where one may reasonably assume that agents can directly control their velocities. See [12, 17] for instance.

In this manuscript, we further generalize the MT model in ways that are inspired by the ability to account for acceleration bounds, masking effects as well as orientation bias. We endeavor to keep the model as general and flexible as possible while ensuring flocking behavior. Moreover, despite these generalizations, we believe that many real world phenomena may not be captured by a model that only incorporates flocking mechanisms without what we call *steering*. By steering, we mean additional acceleration by each agent which accounts for their individual responses to other external influences such as the need to compensate friction and gravity, pursuit of targets and evasion of danger.

The paper is organized as follows. In section 2 we motivate our generalized flocking model via the need for acceleration bounds, the presence of masking effects and orientation bias. We introduce the open loop and closed loop aspects of the flocking model. Once the flocking part of the model is described, we show that in the presence of friction the velocities of all agents asymptotically approach zero. This and other considerations motivate us to the introduction of the steering forces. We also briefly discuss existence and uniqueness of solutions. In section 3 we provide an analysis of our model and prove some sufficient conditions on flocking. Section 4 investigate the leading order behavior of the flocking and steering model via a formal singular perturbation approach when flocking is much faster than steering. Numerical simulations are provided in Section 5 that illustrate our analysis.

## 2 The generalized flocking and steering model

We first discuss the generalization of the flocking model and then include steering. We observe that the Motsch-Tadmor model has two aspects. First is the velocity alignment aspect which is given by:  $\dot{v}_i = \alpha(\bar{v}_i - v_i)$  where  $\alpha > 0$  is a constant and  $\bar{v}_i = \sum_{j=1}^N a_{ij}v_j$ , is a (time dependent) convex combination of  $v_1, \dots, v_N$ . Regardless of the nature of this combination, in the velocity space, the acceleration of agent  $i$  is always pointed towards a point in the convex hull of all the velocities. The second aspect of the model involves how  $a_{ij}$  depend on the positions  $x_1, \dots, x_N$ . We note that throughout this paper  $\|z\|$  stands for the Euclidean norm of a vector  $z \in \mathbb{R}^d$ .

### 2.1 Apriori acceleration bounds

We start with the reasonable assumption that the magnitude of the acceleration  $\|\dot{v}_i\|$  of any agent  $i$  may not exceed a certain predetermined value, say  $A > 0$ . It is readily observed that in the Motsch-Tadmor model of (1.2), the acceleration of agent  $i$  is always given by  $\alpha(\bar{v}_i - v_i)$  and since  $\alpha > 0$  is independent of  $t$  and  $i$ , this does not readily allow for the condition  $\alpha\|\bar{v}_i - v_i\| \leq A$  to be satisfied. Simply relaxing the model to allow for  $\alpha$  to depend

on  $i$  and  $t$ , readily provides for the condition on acceleration bound to be

$$\alpha_i(t) \leq \frac{A}{\|\bar{v}_i(t) - v_i(t)\|},$$

which can always be satisfied since agent  $i$  choose a time varying value for  $\alpha_i(t)$ . Thus, one may regard  $\alpha_i(t)$  as a scalar control input from agent  $i$ . The only condition on each agent  $i$  is that the agent accelerates in a direction parallel to  $\bar{v}_i - v_i$  and pointing in the same sense so that  $\alpha_i(t) > 0$ . A simple feedback law that each agent  $i$  can implement may take the form

$$\alpha_i(t) = \xi_i(\bar{v}_i(t) - v_i(t)) \quad (2.1)$$

where  $\xi_i : \mathbb{R}^{Nd} \rightarrow [0, \infty)$ . Then the condition on acceleration bound becomes  $\xi_i(u) \leq A/\|u\|$ . Motivated by this discussion, we state the following assumption.

**Assumption 1** *For  $i = 1, \dots, N$ , the functions  $\xi_i : \mathbb{R}^{Nd} \rightarrow (0, \infty)$  are  $C^1$  (continuously differentiable), strictly positive and there exists  $A > 0$  such that*

$$\xi_i(u) \leq A/\|u\|, \text{ for } u \neq 0, i = 1, \dots, N. \quad (2.2)$$

We note that the  $C^1$  assumption helps ensure existence uniqueness of solutions. A simple example of  $\xi_i$  is given by

$$\xi_i(u) = \frac{A}{\sqrt{a^2 + \|u\|^2}} \quad i = 1, \dots, N, \quad (2.3)$$

where  $a > 0$  is some constant.

## 2.2 Masking effect and orientation bias

In the CS model, the influence of agent  $j$  on  $i$  is given by the form  $a_{ij} = \phi(\|x_i - x_j\|)$  whereas in the MT model it is given by

$$a_{ij} = \phi(\|x_i - x_j\|) / \sum_k \phi(\|x_i - x_k\|)$$

where  $\phi : [0, \infty) \rightarrow [0, \infty)$ . This form assumes that the influence of  $j$  on  $i$  is a function of all the pairwise distances. This specific form is not general enough to model masking effects. In order to explain this, we refer to Figure 1. In the position space, if a third agent  $l$  is present in the line segment joining agents  $i$  and  $j$ , then the influence of  $j$  on  $i$  may be lesser than if there were no agents present. This motivates a very general form of position dependence for  $a_{ij}$ . Additionally, the effect of agent  $j$  on agent  $i$  will depend on the orientation of the field of view of agent  $i$ . It is natural to consider the orientation of agent  $i$  as the unit vector  $v_i/\|v_i\|$ . However, this is undefined when  $v_i = 0$ . To avoid singularities, we consider agent  $i$ 's orientation  $u_i$  to be a  $C^1$  function of  $v_i$ , so that  $u_i = \sigma_i(v_i)$  where  $\sigma_i : \mathbb{R}^d \rightarrow \bar{B}^d$  where  $\bar{B}^d$  is the closed unit ball in  $\mathbb{R}^d$ . An example of  $\sigma_i$  is given by

$$\sigma_i(u) = \frac{u}{\sqrt{\|u\|^2 + b_i^2}}.$$

where  $b_i$  is a nonzero real number. These two observations suggest the following form for  $a_{ij}$ :

$$a_{ij} = \phi_{ij}(x; \sigma_i(v_i)) \quad (2.4)$$

where  $x = (x_1, \dots, x_N) \in \mathbb{R}^{Nd}$ ,  $\sigma_i : \mathbb{R}^d \rightarrow \bar{B}^d$  and  $\phi_{ij} : \mathbb{R}^{Nd} \times \mathbb{R}^d \rightarrow [0, \infty)$ . We note that  $\bar{B}^d$  is the unit ball in  $\mathbb{R}^d$ . Thus the influence of agent  $j$  on agent  $i$  can be a nuanced function of the positions of all the agents as well as the velocity of agent  $i$ . We state our assumptions on  $\phi_{ij}$ .

**Assumption 2** For  $1 \leq i, j \leq N$ ,  $\phi_{ij} : \mathbb{R}^{Nd} \times \mathbb{R}^d \rightarrow (0, \infty)$  are  $C^1$  and strictly positive. Moreover,  $\phi_{ij}$  are shift invariant in position:

$$\phi_{ij}(x_1 + y, x_2 + y, \dots, x_N + y; u) = \phi_{ij}(x_1, x_2, \dots, x_N; u) \quad (2.5)$$

$\forall x \in \mathbb{R}^{Nd}, \forall y \in \mathbb{R}^d, \forall u \in \bar{B}^d$ . Additionally,  $\sigma_i : \mathbb{R}^d \rightarrow \bar{B}^d$  are  $C^1$ .

We note that the shift invariance assumption is reasonable since the influence of agent  $j$  on agent  $i$  must only depend on the relative positions of all the agents, but not on their absolute positions. As before, the  $C^1$  assumption helps ensure existence uniqueness results. The strict positivity assumptions on  $\phi_{ij}$  are utilized in our flocking results and are a statement of lack of complete masking. That is, each agent has a nontrivial influence on every other agent regardless of the relative configuration.

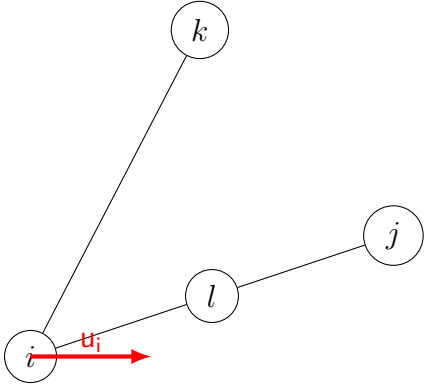


Figure 1: *Masking effect and orientation bias.* The agents  $j$  and  $k$  are equidistant from agent  $i$ . Nevertheless, agent  $l$  contributes to masking effect which diminishes agent  $j$ 's influence on agent  $i$ . On the other hand, agent  $i$  is moving to the right and in agent  $i$ 's field of view agent  $j$  is in a more prominent position than agent  $k$ , which diminishes agent  $k$ 's influence on agent  $i$ .

## 2.3 The open loop and closed loop models

It is instructive to consider our general model as forming two layers. The first layer, is the “open loop” model given by

$$\begin{aligned} \dot{x}_i &= v_i, \\ \dot{v}_i &= \alpha_i(\bar{v}_i - v_i), \\ \bar{v}_i &= \sum_{j=1}^N a_{ij}v_j, \\ a_{ij} &\geq 0, \quad \sum_{j=1}^N a_{ij} = 1, \quad \alpha_i \geq 0 \end{aligned} \tag{2.6}$$

where  $\alpha_i$  and  $a_{ij}$  are considered to be given functions of  $t$ , which can be regarded as control inputs from agent  $i$ . The second layer of our model specifies how  $\alpha_i$  and  $a_{ij}$  are chosen as functions of positions and velocities, thus “closing the loop”. The closed loop model thus contains the equations

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= \alpha_i(\bar{v}_i - v_i) \\ \bar{v}_i &= \sum_{j=1}^N \phi_{ij}(x; u_i)v_j \\ u_i &= \sigma_i(v_i) \\ \alpha_i &= \xi_i(\bar{v}_i - v_i) \end{aligned} \tag{2.7}$$

for  $i = 1, \dots, N$ , where  $\xi_i$  and  $\phi_{ij}$  satisfy Assumptions 1 and 2 respectively.

## 2.4 Inclusion of friction

In the real physical world, forces such as aerodynamic friction are present. We will consider a form of friction which is proportional to some power of the velocity. We have the following (open loop) system:

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= \alpha_i(t)(\bar{v}_i - v_i) - c_i\|v_i\|^r v_i, \end{aligned} \tag{2.8}$$

where  $r \geq 0$ . The following lemma shows very trivial asymptotic behavior.

**Lemma 1** *Suppose  $\{x_i(t), v_i(t)\}_{i=1}^N$  is a  $C^1$  solution of the system (2.8). Then for each  $i$*

$$\lim_{t \rightarrow \infty} v_i(t) = 0.$$

**Remark 1** *We note that the Lemmas 5 and 6 given in the appendix will be frequently used in the proofs of the results in this paper.*

**Proof** We define an energy of the system by  $E = \max_{1 \leq j \leq N} E_j$  where  $E_j = \frac{1}{2} \|v_j\|^2$ . Then by Lemmas 5 and 6  $E(t)$  is absolutely continuous and  $dE/dt(t) = dE_i/dt(t)$  for almost all  $t$  where  $i = i(t)$  is an index of the maximum. Thus, for almost all  $t$ ,

$$\begin{aligned} \frac{dE}{dt} &= \langle v_i, \dot{v}_i \rangle = \langle v_i, \alpha_i(\bar{v}_i - v_i) - c_i \|v_i\|^r v_i \rangle = -c_i \|v_i\|^{r+2} + \alpha_i \langle \bar{v}_i, v_i \rangle - \alpha_i \|v_i\|^2 \\ &= -c_i \|v_i\|^{r+2} + \alpha_i \sum_j a_{ij} \langle v_i, v_j \rangle - \alpha_i \|v_i\|^2 \\ &\leq -c_i \|v_i\|^{r+2} - \alpha_i \|v_i\|^2 + \alpha_i \|v_i\| \sum_j a_{ij} \|v_j\| \leq -c_i \|v_i\|^{r+2} \end{aligned}$$

where we have used the Cauchy-Schwartz inequality and the fact that  $\|v_i\| \geq \|v_j\|$  for all  $j$ . We also note that the index  $i$  in general varies with  $t$ . Letting  $\underline{c} = \min_i c_i$ , we have

$$\frac{dE(t)}{dt} \leq -\underline{c} \|v_i\|^{r+2} \leq -2^{\frac{r}{2}+1} \underline{c} (E(t))^{\frac{r}{2}+1}. \quad (2.9)$$

Multiplying both side by  $(E(t))^{-\frac{r}{2}-1}$ , we have

$$\begin{aligned} (E(t))^{-\frac{r}{2}-1} \frac{dE(t)}{dt} &\leq -2^{\frac{r}{2}+1} \underline{c} \\ -\frac{2}{r} \frac{dE(t)^{-\frac{r}{2}}}{dt} &\leq -2^{\frac{r}{2}+1} \underline{c} \end{aligned}$$

integrating the last inequality from 0 to  $t$  after some algebra manipulation, we have

$$(E(t))^{-\frac{r}{2}} - (E(0))^{-\frac{r}{2}} \geq 2^{\frac{r}{2}} r \underline{c} t$$

which implies that

$$E(t) \leq \frac{1}{((E(0))^{-\frac{r}{2}} + 2^{\frac{r}{2}} r \underline{c} t)^{\frac{2}{r}}}.$$

Thus  $E(t) \rightarrow 0$  as  $t \rightarrow \infty$ . ■

Thus the addition of the friction shows that the asymptotic velocities go to zero. This observation and other real world considerations show that in addition to pure flocking, there should be a “steering” component to each agent’s acceleration.

## 2.5 Steering

In reality a group of agents may want to follow a desired trajectory in addition to staying together as a flock. This necessitates an extra “steering” term. Thus, each agent  $i$  may have an extra acceleration  $\beta_i(t)$  which contributes to steering. This steering term can also act to cancel other external forces such as friction and gravity. We interpret  $\beta_i(t)$  in the following as the steering component in excess of friction and gravity.



This leads to the system

$$\begin{aligned}
\dot{x}_i &= v_i, \\
\dot{v}_i &= \alpha_i(\bar{v}_i - v_i) + \beta_i, \\
\bar{v}_i &= \sum_{j=1}^N a_{ij}v_j, \\
a_{ij} &\geq 0 \quad \sum_{j=1}^N a_{ij} = 1, \quad \alpha_i \geq 0
\end{aligned} \tag{2.10}$$

for the open loop with steering and

$$\begin{aligned}
\dot{x}_i &= v_i \\
\dot{v}_i &= \alpha_i(\bar{v}_i - v_i) + \beta_i \\
\bar{v}_i &= \sum_{j=1}^N \phi_{ij}(x; u_i)v_j \\
u_i &= \sigma_i(v_i) \\
\alpha_i &= \xi_i(\bar{v}_i - v_i)
\end{aligned} \tag{2.11}$$

for the closed loop with steering.

**Assumption 3** *The steering functions  $\beta_i : [0, \infty) \rightarrow \mathbb{R}^d$  for  $i = 1, \dots, N$  are continuous.*

## 2.6 Existence and uniqueness

We briefly discuss existence and uniqueness of solutions of the open loop and closed loop models (2.10) and (2.11). The open loop model is linear and non-autonomous and hence it is adequate to assume that  $\alpha_i(t)$ ,  $a_{ij}(t)$  and  $\beta_i(t)$  are all continuous in time. The closed loop model is of the form

$$\dot{z} = F(z) + \beta(t)$$

where  $z = (x_1, \dots, x_N, v_1, \dots, v_N) \in \mathbb{R}^{2Nd}$  and  $F$  is  $C^1$  by our assumptions on  $\phi_{ij}$  and  $\xi_i$ . Again if we assume  $\beta_i(t)$  to be continuous in  $t$  then for any given initial condition for  $z(0)$ , we are assured of a unique solution in an open maximal interval of time containing 0.

In order to discuss flocking behavior, it is important to ensure that the forward maximal interval of existence is  $[0, \infty)$ . When the steering is open-loop with Assumption 3 it is shown in Lemma 7 that the forward maximal interval is infinite. When steering is considered to be closed-loop, that is some function of position and velocity, then a different analysis is needed.

## 3 Analysis of flocking

### 3.1 Mathematical preliminaries

First we define some relevant concepts and state some useful lemmas. Given the positions  $x_i(t)$  and velocities  $v_i(t)$  (where  $i = 1, \dots, N$ ) of agents, we denote by  $d_X(t)$  and  $d_V(t)$  the

diameters in position and velocity spaces  $\mathbb{R}^{Nd}$ :

$$\begin{aligned} d_X(t) &= \max_{i,j} \|x_j(t) - x_i(t)\| \\ d_V(t) &= \max_{i,j} \|v_j(t) - v_i(t)\|. \end{aligned} \quad (3.1)$$

The system  $\{x_i(t), v_i(t)\}$   $i = 1, \dots, N$  is said to *converge to a flock* if the following two conditions hold:

$$\sup_{t \geq 0} d_X(t) < \infty, \quad \lim_{t \rightarrow \infty} d_V(t) = 0. \quad (3.2)$$

We define  $d_\beta(t)$ , the diameter in the “steering space” by

$$d_\beta(t) = \max_{i,j} \|\beta_j(t) - \beta_i(t)\|. \quad (3.3)$$

The flocking analysis in this paper uses the notion of *active sets* developed in [13]. Recall that  $a_{ij}(t)$  denotes the influence of agent  $j$  on agent  $i$  at time  $t$  and that  $a_{ij}(t) \geq 0$  and  $\sum_{j=1}^N a_{ij}(t) = 1$ . Given  $\theta > 0$ , it is instructive to consider the set of all agents who influence a given agent  $1 \leq p \leq N$  by an amount greater than or equal to  $\theta$ . This is known as the *active set*  $\Lambda_p(\theta)$  for agent  $p$ :

$$\Lambda_p(\theta) = \{j \mid a_{pj} \geq \theta\}. \quad (3.4)$$

For a pair of agents  $p$  and  $q$ , the common active set  $\Lambda_{pq}(\theta)$  is simply the intersection  $\Lambda_p(\theta) \cap \Lambda_q(\theta)$ . The *global active set*  $\Lambda(\theta)$  is the intersection of all the active sets:

$$\Lambda(\theta) = \bigcap_p \Lambda_p(\theta). \quad (3.5)$$

The following lemma from [13] is critical.

**Lemma 2** [13] *Let  $S$  be an antisymmetric matrix,  $S_{ij} = -S_{ji}$  with  $|S_{ij}| \leq M$ . Let  $u, w \in \mathbb{R}^n$  be two given vectors with positive entries,  $u_i, w_i \geq 0$  and let  $\bar{U}, \bar{W}$  denoted their respective sums,  $\bar{U} = \sum_i u_i$  and  $\bar{W} = \sum_j w_j$ . Fix  $\theta > 0$  and let  $\lambda(\theta)$  denoted the number of “active entries” of  $u$  and  $w$  at the level  $\theta$ , in sense that,*

$$\lambda(\theta) = |\Lambda(\theta)|$$

$$\Lambda(\theta) = \{j \mid u_j \geq \theta \bar{U} \text{ and } w_j \geq \theta \bar{W}\}.$$

*Then for every  $\theta > 0$ , we have*

$$|\langle Su, w \rangle| \leq M \bar{U} \bar{W} (1 - \lambda^2(\theta) \theta^2)$$

For our analysis, in addition to Lemma 2, we need the following simple lemma about the convex hull of a finite set of points in  $\mathbb{R}^d$ .

**Lemma 3** *Let  $\{v_i\}_{i=1}^N$  be a set of vectors in  $\mathbb{R}^d$  and let  $\Omega$  be their convex hull. If  $v_p$  and  $v_q$  delimit the diameter of the convex hull, (that is  $v_p$  and  $v_q$  are furthest apart), then for each  $v \in \Omega$*

$$\langle v_p - v_q, v - v_q \rangle \geq 0.$$

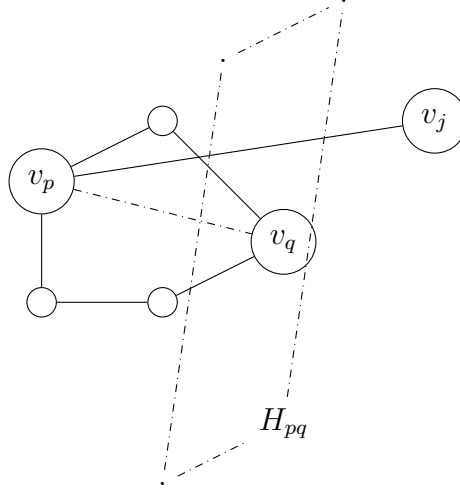


Figure 2: *Illustration of the lemma*

**Proof** Let the diameter of  $\Omega$  equal  $\|v_p - v_q\|$ . We first show that

$$\langle v_p - v_q, v_i - v_q \rangle \geq 0 \quad \forall i.$$

Let  $H_{pq}$  be the hyper plane passing through  $v_q$  and is perpendicular to  $v_p - v_q$ . (See Figure 2). Suppose there is some  $i$  such that  $\langle v_p - v_q, v_i - v_q \rangle < 0$ . This shows that  $v_i$  and  $v_p$  will be on opposite sides of the hyper plane  $H_{pq}$ , implying that  $\|v_i - v_p\| > \|v_q - v_p\|$ , a contradiction. Given any  $v \in \Omega$ , there exist  $a_i \geq 0$  for  $i = 1, \dots, n$  such that  $\sum_{i=1}^N a_i = 1$  and  $v = \sum_{j=1}^N a_j v_j$ . Hence

$$\langle v_p - v_q, v - v_q \rangle = \langle v_p - v_q, \sum_{i=1}^N a_i (v_i - v_q) \rangle \geq 0.$$

■

### 3.2 Analysis of the open loop

We shall suppose that Assumption 3 holds.

**Theorem 1** *Let  $(x(t), v(t)) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$  be a  $C^1$  solution of the open loop (2.10). At time  $t$ , let  $d_V(t) = \|v_p(t) - v_q(t)\|$ . Fix an arbitrary  $\theta > 0$  and let  $\lambda_{pq}(\theta)$  be the number of agents in the common active set  $\Lambda_{pq}(\theta)$  associated with the influence matrix  $a_{ij}(t)$  of the system. Let  $\alpha_0(t) = \min_i \alpha_i(t)$ . Then for almost all  $t$ , the diameters of the system,  $d_X(t)$ ,  $d_V(t)$  and  $d_\beta(t)$  satisfy :*

$$\begin{aligned} \frac{d}{dt} d_X(t) &\leq d_V(t) \\ \frac{d}{dt} d_V(t) &\leq -\alpha_0 \lambda_{pq}^2(\theta) \theta^2 d_V(t) + d_\beta(t). \end{aligned} \tag{3.6}$$

**Remark 2** In Theorem (1), we note that  $p$  and  $q$  are functions of  $t$ , and so is  $\lambda_{pq}(\theta)$ . This theorem is a generalization of Theorem 3.4 of [13] where  $\alpha_i(t)$  was independent of  $i$  and  $t$ . One needs Lemma 3 to handle the extra terms that appear in our analysis.

**Proof** By Lemma 5  $d_X(t)$  is absolutely continuous. We choose  $i = i(t)$  and  $j = j(t)$  such that  $d_X(t) = \|x_i(t) - x_j(t)\|$  for all  $t$ . Using Lemma 6 we obtain

$$\left| \frac{d}{dt}(d_X(t))^2 \right| = \left| \frac{d}{dt} \|x_i - x_j\|^2 \right| = 2|\langle x_i - x_j, v_i - v_j \rangle|.$$

Hence

$$\left| 2\|x_i - x_j\| \frac{d}{dt} \|x_i - x_j\| \right| = 2|\langle x_i - x_j, v_i - v_j \rangle| \leq 2\|x_i - x_j\| \|v_i - v_j\|.$$

This yields that for almost all  $t$

$$\frac{d}{dt} d_X(t) \leq \|v_i - v_j\| \leq d_V(t).$$

Note that if for some  $t > 0$ ,  $d_X(t) = 0$  and  $d_X(t)$  is differentiable, then  $\frac{d}{dt} d_X(t) = 0$ .

For the second inequality, we again proceed by using Lemmas 5 and 6. Let  $p = p(t)$  and  $q = q(t)$  be such that  $d_V(t) = \|v_p - v_q\|$  for all  $t$ . Then (for almost all  $t$ )

$$\begin{aligned} \frac{d}{dt}(d_V(t))^2 &= \frac{d}{dt}(\|v_p - v_q\|^2) = 2\langle v_p - v_q, \dot{v}_p - \dot{v}_q \rangle \\ &= 2\langle v_p - v_q, \alpha_p(\bar{v}_p - v_p) - \alpha_q(\bar{v}_q - v_q) \rangle + 2\langle v_p - v_q, \beta_p - \beta_q \rangle \\ &= 2\alpha_p \langle v_p - v_q, \bar{v}_p - v_p \rangle - 2\alpha_q \langle v_p - v_q, \bar{v}_q - v_q \rangle + 2\langle v_p - v_q, \beta_p - \beta_q \rangle. \end{aligned}$$

We proceed by assuming WLOG that  $\alpha_p \leq \alpha_q$  and write

$$\begin{aligned} \frac{d}{dt}(\|v_p - v_q\|^2) &= 2\alpha_p \langle v_p - v_q, \bar{v}_p - \bar{v}_q \rangle - 2\alpha_p \|v_p - v_q\|^2 \\ &\quad - 2(\alpha_q - \alpha_p) \langle v_p - v_q, \bar{v}_q - v_q \rangle + 2\langle v_p - v_q, \beta_p - \beta_q \rangle. \end{aligned}$$

Using Lemma (3), Cauchy-Schwartz inequality and the fact that  $\|\beta_p - \beta_q\| \leq d_\beta$ , we have:

$$\frac{d}{dt}(\|v_p - v_q\|^2) \leq 2\alpha_p \langle v_p - v_q, \bar{v}_p - \bar{v}_q \rangle - 2\alpha_p \|v_p - v_q\|^2 + 2\|v_p - v_q\| d_\beta.$$

Moreover

$$\begin{aligned} \bar{v}_p - \bar{v}_q &= \sum_{j=1}^N a_{pj} v_j - \bar{v}_q = \sum_{j=1}^N a_{pj} (v_j - \bar{v}_q) \\ &= \sum_{j=1}^N a_{pj} (v_j - \sum_{i=1}^N a_{qi} v_i) = \sum_{i,j} a_{pj} a_{qi} (v_j - v_i). \end{aligned}$$

Hence

$$\frac{d}{dt}(\|v_p - v_q\|^2) \leq 2\alpha_p \sum_{i,j} a_{pj} a_{qi} \langle v_p - v_q, v_j - v_i \rangle - 2\alpha_p \|v_p - v_q\|^2 + 2d_\beta \|v_p - v_q\|.$$

Now we use Lemma (2) with  $u_i = a_{pi}$ ,  $w_i = a_{qi}$ , and the anti symmetric matrix

$$S_{ij} = \langle v_p - v_q, v_i - v_j \rangle.$$

Since  $|S_{ij}| \leq d_V^2$ , we have

$$\left| \sum_{i,j}^N a_{pj} a_{qi} \langle v_p - v_q, v_j - v_i \rangle \right| \leq d_V^2 (1 - \lambda_{pq}^2(\theta) \theta^2)$$

Therefore we have

$$\frac{d}{dt}(\|v_p - v_q\|^2) \leq 2\alpha_p d_V^2 (1 - \lambda_{pq}^2(\theta) \theta^2) - 2\alpha_p \|v_p - v_q\|^2 + 2d_\beta \|v_p - v_q\|$$

Noting that  $v_p$  and  $v_q$  are such that  $\|v_p(t) - v_q(t)\| = d_V(t)$  and that  $\alpha_p(t) \geq \alpha_0(t)$  by definition, we have

$$\frac{d}{dt}(d_V(t)^2) \leq -2\alpha_0 d_V^2 \lambda_{pq}^2(\theta) \theta^2 + 2d_\beta d_V.$$

An argument similar to the one used in deriving the first inequality proves (1). ■ The

following corollary is immediate.

**Corollary 1** *If  $\lambda(\theta)$  is the number of elements in the global active set  $\Lambda(\theta)$  and if  $\underline{\alpha}$  denotes the infimum of  $\alpha_i(t)$  over  $i$  and  $t \geq 0$  then*

$$\frac{d}{dt} d_X(t) \leq d_V(t). \tag{3.7a}$$

$$\frac{d}{dt} d_V(t) \leq -\underline{\alpha} \lambda(\theta) \theta^2 d_V(t) + d_\beta(t). \tag{3.7b}$$

### 3.3 Analysis of the closed loop

We shall suppose that Assumptions 1, 2 and 3 hold. As observed in Section 2.6 these Assumptions guarantee existence and uniqueness of a solution to the closed-loop equations on the time interval  $[0, \infty)$ . Moreover, this solution is  $C^1$  in time  $t$ . We note that, if all steering terms  $\beta_i$  are equal for all  $t$ , then  $d_\beta(t) = 0$  and the system of inequalities given by (3.7a) and (3.7b) show that the diameter  $d_V$  is decreasing in time. Even in this case, in order to show flocking, one needs stronger inequalities. To that end, we shall modify the ideas from Ha et al [10] and also from Motsch and Tadmor [13] in order to prove the flocking results. We define the function  $\psi : [0, \infty) \rightarrow (0, \infty)$  by

$$\psi(r) = \min_{1 \leq i, j \leq N} \min \{ \phi_{ij}(x; u) \mid \|x_l - x_k\| \leq r, u \in \bar{B}^d \text{ and } 1 \leq l, k \leq N \}. \tag{3.8}$$

In order to see that the minimum exists, we observe that by shift invariance (Assumption 2),

$$\begin{aligned} & \{ \phi_{ij}(x; u) \mid \|x_l - x_k\| \leq r, u \in \bar{B}^d \} \\ &= \{ \phi_{ij}(x; u) \mid x_1 = 0, \|x_l - x_k\| \leq r, u \in \bar{B}^d \} \end{aligned}$$

and that

$$\{(x, u) \in \mathbb{R}^{Nd} \times \bar{B}^d \mid x_1 = 0, \|x_l - x_k\| \leq r, u \in \bar{B}^d\}$$

is a compact set and that  $\phi_{ij}$  are continuous. Since  $\phi_{ij}$  are strictly positive by Assumption 2, it follows that  $\psi$  is strictly positive. Moreover, it is also clear that  $\psi$  is a decreasing (non-increasing) function. Since  $\psi$  is decreasing, it is also positive and measurable, and hence  $\int_0^{r_0} \psi(r) dr < \infty$  and  $\int_0^\infty \psi(r) dr \leq \infty$  are well-defined.

**Lemma 4** *Let  $\underline{\alpha}$  be the infimum of  $\alpha_i(t)$  over  $i$  and  $t \geq 0$ . Suppose that*

$$\int_0^\infty d_\beta(t) dt < \infty.$$

*Then  $\underline{\alpha} > 0$ .*

**Proof** Let  $M$  be defined by

$$M = d_V(0) + \int_0^\infty d_\beta(t) dt.$$

Then from (3.7b) it follows that  $d_V(t) \leq M$  for all  $t \geq 0$ . Hence, for all  $t \geq 0$  and for all  $1 \leq i \leq N$ ,  $\|\bar{v}_i(t) - v_i(t)\| \leq d_V(t) \leq M$ , where we have used the fact that  $\bar{v}_i$  is in the convex hull of all velocities  $v_j$ . Now

$$\begin{aligned} \underline{\alpha} &= \inf\{ \xi_i(\bar{v}_i(t) - v_i(t)) \mid t \geq 0, 1 \leq i \leq N \}, \\ &\geq \min\{ \xi_i(u) \mid 0 \leq \|u\| \leq M, 1 \leq i \leq N \} > 0, \end{aligned}$$

where we have used the fact that  $\xi_i$  are continuous by Assumption 1. ■

**Theorem 2** *Consider the closed loop system (2.11). Suppose  $\psi$  is defined by (3.8) and that*

$$\int_0^\infty d_\beta(t) dt < \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} d_\beta(t) = 0.$$

*Further suppose that the initial diameters satisfy*

$$d_V(0) + \int_0^\infty d_\beta(t) dt < \underline{\alpha} N^2 \int_{d_X(0)}^\infty \psi(s) ds. \quad (3.9)$$

*Then the solution  $(x(t), v(t))$  flocks. In particular, if  $\int_0^\infty \psi(s) ds = \infty$ , then the condition on initial diameters is always satisfied.*

**Proof** At any given time, by choosing  $\theta(t) = \sqrt{\psi(d_X(t))}$ , one readily obtains that the number of elements in the global active set is  $N$ , and hence the inequality

$$\frac{d}{dt} d_V(t) \leq -\underline{\alpha} N^2 \psi(d_X(t)) d_V(t) + d_\beta(t).$$

We define the energy functional  $\mathcal{E} : \mathbb{R}^{Nd} \times \mathbb{R}^{Nd} \rightarrow \mathbb{R}$

$$\mathcal{E}(d_X(t), d_V(t)) = d_V(t) + \underline{\alpha} N^2 \int_0^{d_X(t)} \psi(s) ds, \quad (3.10)$$

The time derivative of the energy functional satisfies

$$\dot{\mathcal{E}} = \dot{d}_V + \underline{\alpha} N^2 d_V \psi(d_X) \leq d_\beta.$$

Hence

$$\mathcal{E}(d_V(t), d_X(t)) - \mathcal{E}(d_V(0), d_X(0)) \leq \int_0^t d_\beta(s) ds,$$

Which implies that

$$d_V(t) - d_V(0) \leq -\underline{\alpha} N^2 \int_0^{d_X(t)} \psi(s) ds + \underline{\alpha} N^2 \int_0^{d_X(0)} \psi(s) ds + \int_0^t d_\beta(s) ds.$$

We deduce that

$$d_V(t) - d_V(0) \leq \underline{\alpha} N^2 \int_{d_X(t)}^{d_X(0)} \psi(s) ds + \int_0^t d_\beta(s) ds. \quad (3.11)$$

By the assumption (3.9), there exists  $d_*$  (independent of  $t$ ) such that

$$\int_0^\infty d_\beta(t) dt + d_V(0) \leq \underline{\alpha} N^2 \int_{d_X(0)}^{d_*} \psi(s) ds. \quad (3.12)$$

Replacing this inequality in (3.11), we obtain that

$$d_V(t) \leq \underline{\alpha} N^2 \int_{d_X(t)}^{d_X(0)} \psi(s) ds + \underline{\alpha} N^2 \int_{d_X(0)}^{d_*} \psi(s) ds \leq \underline{\alpha} N^2 \int_{d_X(t)}^{d_*} \psi(s) ds.$$

Since  $d_V(t) \geq 0$ , we have that the diameter in the position space is uniformly bounded. That is,  $d_X(t) \leq d_*$  for all  $t \geq 0$ . Defining  $\psi_* = \psi(d_*)$ , we note that  $\psi(s) \geq \psi_*$  for  $s \in [0, d_*]$ . Using the inequality

$$\frac{d}{dt} d_V(t) \leq -\underline{\alpha} N^2 \psi(d_X(t)) d_V + d_\beta,$$

we have that

$$\frac{d}{dt} d_V(t) \leq -\underline{\alpha} N^2 \psi_* d_V + d_\beta.$$

Which implies that:

$$d_V(t) \leq e^{-\underline{\alpha} N^2 \psi_* t} d_V(0) + \int_0^t e^{-\underline{\alpha} N^2 \psi_* (t-s)} d_\beta(s) ds$$

Now let us show that the velocity diameter goes to zero asymptotically. The first term above goes to zero asymptotically in time. The second term can be written as

$$\frac{\int_0^t e^{\underline{\alpha} N^2 \psi_* s} d_\beta(s) ds}{e^{\underline{\alpha} N^2 \psi_* t}}.$$

There are two cases. If

$$\lim_{t \rightarrow \infty} \int_0^t e^{\underline{\alpha} N^2 \psi_* s} d_\beta(s) ds < \infty$$

then this second term clearly limits to zero. On the other hand, the limit above is infinity and hence an application of L'Hospital's rule and the hypothesis that  $\lim_{t \rightarrow \infty} d_\beta(t) = 0$  shows that

$$\lim_{t \rightarrow \infty} \frac{\int_0^t e^{\alpha N^2 \psi_* s} d_\beta(s) ds}{e^{\alpha N^2 \psi_* t}} = \lim_{t \rightarrow \infty} \frac{e^{\alpha N^2 \psi_* t} d_\beta(t)}{e^{\alpha N^2 \psi_* t}} = \lim_{t \rightarrow \infty} d_\beta(t) = 0.$$

■

## 4 Study of fast flocking with slow steering via singular perturbation approach

We consider the model given by (2.11) and investigate the scenario where flocking is much faster than steering. In the singular perturbation approach, we capture this by the introduction of a small parameter  $\epsilon$ . For ease of analysis, we ignore the orientation bias and assume that  $a_{ij} = \phi_{ij}(x)$ . This leads us to the family of equations

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= \frac{\alpha_i}{\epsilon} (\bar{v}_i - v_i) + \beta_i \\ \bar{v}_i &= \sum_{j=1}^N \phi_{ij}(x) v_j \\ \alpha_i &= \xi_i (\bar{v}_i - v_i) \quad \forall i = 1, \dots, N. \end{aligned} \tag{4.1}$$

Here,  $0 < \epsilon \ll 1$  is a parameter that allows the model to flock rapidly.

Let  $x_i(t, \epsilon)$  and  $v_i(t, \epsilon)$  for all  $i = 1, \dots, N$  be the solution of our new model (4.1). We expand these solutions and some related variables of the model in a power series in  $\epsilon$ :

$$\begin{aligned} x_i(t, \epsilon) &= x_{i,0}(t) + \epsilon x_{i,1}(t) + \dots \\ v_i(t, \epsilon) &= v_{i,0}(t) + \epsilon v_{i,1}(t) + \dots \\ \alpha_i(t, \epsilon) &= \alpha_{i,0}(t) + \epsilon \alpha_{i,1}(t) + \dots \\ \bar{v}_i(t, \epsilon) &= \bar{v}_{i,0}(t) + \epsilon \bar{v}_{i,1}(t) + \dots \\ \beta_i(t, \epsilon) &= \beta_{i,0}(t) + \epsilon \beta_{i,1}(t) + \dots \end{aligned} \tag{4.2}$$

### 4.1 Leading order behavior

We shall use  $x_0(t)$  to denote

$$(x_{1,0}(t), \dots, x_{N,0}(t)),$$

and likewise  $v_0(t)$  and  $\beta_0(t)$ . We are interested in characterizing the leading order terms  $x_0(t)$  and  $v_0(t)$ . In what follows, we frequently omit showing the dependence on time for brevity. Substituting the expansions (4.2) into (4.1) we obtain

$$\begin{aligned} \dot{x}_{i,0} + \epsilon \dot{x}_{i,1} + \dots &= v_{i,0} + \epsilon v_{i,1} + \dots \\ \dot{v}_{i,0} + \epsilon \dot{v}_{i,1} + \dots &= \frac{1}{\epsilon} (\alpha_{i,0} + \epsilon \alpha_{i,1} + \dots) (\bar{v}_{i,0} - v_{i,0}) \\ &\quad + \epsilon (\bar{v}_{i,1} - v_{i,1}) + \dots + \beta_{i,0}(t) + \epsilon \beta_{i,1}(t) + \dots \end{aligned} \tag{4.3}$$



Furthermore we obtain

$$\begin{aligned}\bar{v}_{i,0} &= \sum_{j=1}^N \phi_{ij}(x_0) v_{j,0} \\ \bar{v}_{i,1} &= \sum_{j=1}^N \phi_{ij}(x_0) v_{j,1} + \sum_{j=1}^N \left\{ \sum_{l=1}^N \sum_{k=1}^d \frac{\partial \phi_{ij}}{\partial x_l^k}(x_0) x_{l,1}^k \right\} v_{j,0}.\end{aligned}\tag{4.4}$$

We note that  $x_i^k$  and  $v_i^k$  are the  $k$ th components of the  $i$ th agent's position and velocity. Also  $x_{i,0}^k$  and  $x_{i,1}^k$  denote the leading order and the next order terms of  $x_i^k$  and likewise for  $v_{i,0}^k$  and  $v_{i,1}^k$ . Balancing the terms of order  $\epsilon^{-1}$  in (4.3), we obtain that

$$\alpha_{i,0}(t)(\bar{v}_{i,0}(t) - v_{i,0}(t)) = 0.\tag{4.5}$$

This means that  $\alpha_{i,0} = 0$  or  $\bar{v}_{i,0} - v_{i,0} = 0$ . since  $\underline{\alpha} = \min \alpha_i > 0$ , we have that  $\bar{v}_{i,0} = v_{i,0}$ . Therefore

$$\sum_{j=1}^N \phi_{ij}(x_0) v_{j,0} = v_{i,0}$$

and hence

$$\sum_{j=1}^N \phi_{ij}(x_0) v_{j,0}^k = v_{i,0}^k$$

where we use the superscript to denote the  $k$ th component of the velocity. Fixing a component  $1 \leq k \leq d$  and writing the previous equation for all agents, we obtain

$$P(t) v_0^k(t) = v_0^k(t),\tag{4.6}$$

where the matrix  $P$  is given by:

$$P = \begin{bmatrix} \phi_{11}(x_0) & \cdots & \phi_{1N}(x_0) \\ \vdots & \ddots & \vdots \\ \phi_{N1}(x_0) & \cdots & \phi_{NN}(x_0) \end{bmatrix},\tag{4.7}$$

and  $v^k = (v_{1,0}^k, \dots, v_{N,0}^k) \quad \forall k = 1, \dots, d$ . Since  $P_{ij} = \phi_{ij} > 0$  and

$$\sum_{j=1}^N P_{ij} = 1,$$

the matrix  $P$  is a stochastic matrix. Since  $P_{ij} > 0$  for all  $i, j$ ,  $P$  has eigenvector  $e = (1, \dots, 1)^t$  corresponding to the eigenvalue 1 of multiplicity one. Thus for each  $k = 1, \dots, d$ , (4.6) has a unique solution for  $v^k$  which is a multiple of  $e = (1, \dots, 1)^t$ . This shows that  $v_{i,0}(t)$  are all equal for  $i = 1, \dots, N$ , indicating flocking. We shall denote this flocking velocity by  $v^f(t)$ .

Balancing the terms of order  $\epsilon^0$  in (4.3) gives the system

$$\begin{aligned}\dot{x}_{i,0}(t) &= v_{i,0}(t), \\ \dot{v}_{i,0}(t) &= \alpha_{i,1}(\bar{v}_{i,0}(t) - v_{i,0}(t)) + \alpha_{i,0}(\bar{v}_{i,1}(t) - v_{i,1}(t)) + \beta_{i,0}(t).\end{aligned}\tag{4.8}$$

Since  $v_{i,0} = v^f$  for all  $i$ , it follows that  $\bar{v}_{i,0} = v^f$  for all  $i$ , and hence, from (4.4) we obtain that

$$\bar{v}_{i,1} = \sum_{j=1}^N \phi_{ij}(x_0) v_{j,1} + \sum_{j=1}^N \left\{ \sum_{l=1}^N \sum_{k=1}^d \frac{\partial \phi_{ij}}{\partial x_l^k}(x_0) x_{l,1}^k \right\} v^f.$$

We change the order of the summation in the second term and use the condition  $\sum_{j=1}^N \phi_{i,j}(x) = 1$  to obtain that

$$\sum_{j=1}^N \left\{ \sum_{l=1}^N \sum_{k=1}^d \frac{\partial}{\partial x_l^k} \phi_{ij}(x_0) x_{l,1}^k \right\} v^f = \sum_{l=1}^N \left\{ \sum_{k=1}^d x_{l,1}^k \frac{\partial}{\partial x_l^k} \left( \sum_{j=1}^d \phi_{i,j}(x_0) \right) \right\} v^f = 0.$$

Thus

$$\bar{v}_{i,1} = \sum_{j=1}^N \phi_{ij}(x_0) v_{j,1}.$$

Substituting these results in equation (4.8), we have that for each  $i$

$$\dot{v}^f = \alpha_{i,0}(\bar{v}_{i,1} - v_{i,1}) + \beta_{i,0}. \quad (4.9)$$

From the first equation of (4.8) we have that

$$\dot{x}_{i,0} = v_{i,0} = v^f.$$

This implies that for each  $i$

$$x_{i,0}(t) = x_{i,0}(0) + \int_0^t v^f(s) ds. \quad (4.10)$$

Hence for all  $i$  and  $j$

$$x_{i,0}(t) - x_{j,0}(t) = x_{i,0}(0) - x_{j,0}(0). \quad (4.11)$$

It follows from (4.11) that the leading order relative positions of agents do not change with time. Hence by the shift invariance assumption on  $\phi_{ij}$ , it follows that  $\phi_{ij}(x_0(t))$  is independent of  $t$ . We denote by  $\bar{a}_{ij}$ :

$$\bar{a}_{ij} = \phi_{ij}(x_0) \quad \forall i, j.$$

Since  $\bar{v}_{i,0} = v_{i,0}$ , it follows that  $\alpha_{i,0} = \xi_i(\bar{v}_{i,0} - v_{i,0}) = \xi_i(0) > 0$ . Hence, for each  $i$ ,

$$\dot{v}^f = \xi_i(0) \left( \sum_{j=1}^N \bar{a}_{ij} v_{j,1} - v_{i,1} \right) + \beta_{i,0}. \quad (4.12)$$

Taking the  $k$ th component in equation (4.12) we have that

$$\dot{v}^{f,k} = \xi_i(0) \left( \sum_{j=1}^N \bar{a}_{ij} v_{j,1}^k - v_{i,1}^k \right) + \beta_{i,0}^k, \quad (4.13)$$

for  $k = 1, \dots, d$ . We define for  $1 \leq i, j \leq N$

$$\begin{aligned} q_{ij} &= \xi_i(0) \bar{a}_{ij} \quad \forall i \neq j, \\ q_{ii} &= \xi_i(0) \bar{a}_{ii} - \xi_i(0). \end{aligned}$$

The matrix  $Q = [q_{ij}]$  is a transition rate matrix of a continuous time Markov chain. Moreover, since  $q_{ij} = \xi_i(0) \bar{a}_{ij} > 0$  for all  $i \neq j$ , the matrix  $Q$  corresponds to an ergodic Markov chain in continuous time. Thus there exists a unique vector  $(\pi_i)_{i=1}^N$  such that  $\sum_{i=1}^N \pi_i = 1$  and

$$\sum_{i=1}^N \pi_i q_{ij} = 0.$$

With the introduction of matrix  $Q$ , (4.13) may be written as

$$\dot{v}^{f,k} = \sum_{j=1}^N q_{ij} v_{j,1}^k + \beta_{i,0}^k.$$

Multiplying by  $\pi_i$  and summing over  $i = 1 \dots N$ , and using properties of  $q_{ij}$  and  $\pi_i$  we obtain that

$$\dot{v}^{f,k} = \sum_{i=1}^N \pi_i \beta_{i,0}^k. \quad (4.14)$$

Hence the flocking velocity  $v^f(t)$  evolves according to the equation

$$\dot{v}^f = \sum_{i=1}^N \pi_i \beta_{i,0}. \quad (4.15)$$

In general, one may expect the steering terms  $\beta_i$  to depend on  $x_i, v_i$  and possible  $t$ , so that

$$\beta_i(t) = \eta_i(x_i(t), v_i(t), t) \quad (4.16)$$

where we suppose  $\eta_i : \mathbb{R}^d \times \mathbb{R}^d \times [0, \infty) \rightarrow \mathbb{R}^d$  is  $C^1$  in its arguments. Then, it follows that the evolution equation for  $v^f$  is given by

$$\dot{v}^f(t) = \sum_{j=1}^N \pi_i(x_0(t)) \eta_i(x_{0,i}(t), v^f(t), t), \quad (4.17)$$

where  $x_{i,0}(t)$  are given by

$$x_0(t) = x(0) + \int_0^t v^f(s) ds. \quad (4.18)$$

Here  $x(0) = (x_1(0), \dots, x_N(0))$  is the initial position of the agents and we observe that  $\pi_i(x_0(t))$  is constant in time since  $\phi_{ij}(x_0(t))$  is constant in time. We may summarize the leading order time evolution by the system of ODEs

$$\begin{aligned} \dot{x}_0(t) &= v^f(t), \\ \dot{v}^f(t) &= \sum_{j=1}^N \pi_i(x_0(t)) \eta_i(x_{0,i}(t), v^f(t), t). \end{aligned} \quad (4.19)$$

This is a  $(N + 1)d$  dimensional system and the leading order velocities are given by  $v_{i,0}(t) = v^f(t)$ . We observe that in order to obtain a unique solution, we need an initial condition for  $v^f(0)$  which may not be the true initial velocities  $v_i(0)$  of the agents. Intuitively, one expects a rapid initial transient layer during which flocking occurs and the agents reach the flocking velocity  $v^f(0)$ .

In the next subsection, we scale time to investigate this transient layer.

## 4.2 Initial transient layer

The given problem has initial condition,  $x(0) = (x_1(0), \dots, x_N(0))$  and  $v(0) = (v_1(0), \dots, v_N(0))$ .

We zoom into the transient layer at  $t = 0$  by introducing the variable  $\tau = t/\epsilon$ . We define  $X$  and  $V$  by

$$X(\tau, \epsilon) = x(t, \epsilon) = x(\epsilon\tau, \epsilon) \quad \text{and} \quad V(\tau, \epsilon) = v(t, \epsilon) = v(\epsilon\tau, \epsilon).$$

Differentiating with respect to  $\tau$ , we have that

$$\frac{1}{\epsilon} \frac{dX_i(\tau, \epsilon)}{d\tau} = \frac{dx_i(t, \epsilon)}{dt}$$

and

$$\frac{1}{\epsilon} \frac{dV_i(\tau, \epsilon)}{d\tau} = \frac{dv_i(t, \epsilon)}{dt}$$

With the change of variable we have the following system of differential equations:

$$\begin{aligned} X'_i &= \epsilon V_i, \\ V'_i &= \alpha_i(\bar{V}_i - V_i) + \epsilon \beta_i, \end{aligned} \tag{4.20}$$

where the prime denotes differentiation with respect to  $\tau$ . The initial conditions to impose are

$$\begin{aligned} X_i(0) &= x_i(0), \\ V_i(0) &= v_i(0). \end{aligned} \tag{4.21}$$

As before, we assume an  $\epsilon$ -expansion for  $X_i$  and  $V_i$  of the following form:

$$\begin{aligned} X_i(\tau, \epsilon) &= X_{i,0}(\tau, \epsilon) + \epsilon X_{i,1}(\tau, \epsilon) + \dots \\ V_i(\tau, \epsilon) &= V_{i,0}(\tau, \epsilon) + \epsilon V_{i,1}(\tau, \epsilon) + \dots \end{aligned} \tag{4.22}$$

Substituting this expansion in (4.20) we obtain

$$\begin{aligned} X'_{i,0} + \epsilon X'_{i,1} + \dots &= \epsilon(V_{i,0} + \epsilon V_{i,1} + \dots), \\ V'_{i,0} + \epsilon V'_{i,1} + \dots &= (\alpha_{i,0} + \epsilon \alpha_{i,1} + \dots)((\bar{V}_{i,0} - V_{i,0}) + \epsilon(\bar{V}_{i,1} - V_{i,1}) \dots), \\ &\quad + \epsilon(\beta_{i,0} + \epsilon \beta_{i,1} + \dots). \end{aligned}$$

Balancing the  $\epsilon^0$  terms, we find that

$$\begin{aligned} X'_{i,0} &= 0, \\ V'_{i,0} &= \alpha_{i,0}(\bar{V}_{i,0} - V_{i,0}). \end{aligned} \tag{4.23}$$

It follows that  $X_{i,0}(\tau) = X_i(0) = x_i(0)$ . This means that during the initial transient the leading order positions do not change in time  $\tau$ .

The model (4.23) is similar to (2.11) without the steering terms, except that the positions  $X_{i,0}$  are constant. Hence the influence matrix  $a_{ij} = \phi_{ij}(X_0)$  is constant and strictly positive. Defining

$$d_X(\tau) = \max_{i,j} \|X_{i,0}(\tau) - X_{j,0}(\tau)\|, \quad d_V(\tau) = \max_{i,j} \|V_{i,0}(\tau) - V_{j,0}(\tau)\|,$$

to be the diameters in the position and the velocity spaces respectively, we see that the assumptions of Lemma (4) and Theorem (2) are satisfied since the diameter in the steering space is zero. Thus Theorem (2) can be invoked to conclude that  $d_V(\tau) \rightarrow 0$  as  $\tau \rightarrow \infty$ .

Now let us find  $\lim_{\tau \rightarrow \infty} V_{i,0}(\tau)$ . The second equation of (4.23) is

$$V'_{i,0} = \alpha_{i,0}(\bar{V}_{i,0} - V_{i,0}) = \alpha_{i,0} \left( \sum_{j=1}^N \phi_{ij}(X_0) V_{j,0} - V_{i,0} \right) = \sum_{j=1}^N q_{ij} V_{j,0}.$$

Where  $Q = (q_{ij})$  is the same matrix that we have used in (4.13). Taking the  $k$ th components and letting  $Z_i^k = V_{i,0}^k$  and  $Z^k = (Z_1^k, \dots, Z_N^k)$  we have  $Z'^k = Q Z^k$ . That is

$$Z_i'^k = \sum_{j=1}^N q_{ij} Z_j^k.$$

Multiplying by  $\pi_i$  and sum it from 1 to  $N$ , we have

$$\sum_{i=1}^N \pi_i Z_i'^k = \sum_{i=1}^N \sum_{j=1}^N \pi_i q_{ij} Z_j^k = \sum_{j=1}^N \left( \sum_{i=1}^N \pi_i q_{ij} \right) Z_j^k = 0.$$

This implies that for  $t \geq 0$ ,

$$\sum_{i=1}^N \pi_i Z_i^k(t) = \sum_{i=1}^N \pi_i Z_i^k(0). \quad (4.24)$$

However, all the eigenvalues of  $Q$  except for one zero eigenvalue have negative real parts. Thus  $Z^k(t) \rightarrow \bar{Z}^k$  where  $\bar{Z}^k$  is a multiple of  $(1, \dots, 1)^t$ . That is  $\bar{Z}_k = c_k (1, \dots, 1)^t$ . To find  $c_k$ , we take limits in (4.24):

$$\lim_{\tau \rightarrow \infty} \sum_{i=1}^N \pi_i Z_i^k(t) = c_k = \sum_{i=1}^N \pi_i Z_i^k(0) = \sum_{i=1}^N \pi_i V_{i,0}^k(0).$$

We deduce that

$$\lim_{\tau \rightarrow \infty} V_{i,0}(\tau) = V^f(0) = (c_1, \dots, c_d) = \left( \sum_{i=1}^N \pi_i V_{i,0}^1(0), \dots, \sum_{i=1}^N \pi_i V_{i,0}^d(0) \right). \quad (4.25)$$

## 5 Numerical Examples

In this section, we present some numerical simulations to illustrate our theoretical analysis. We consider the collection of  $N = 7$  agents in two dimensions. We shall choose the initial positions and initial velocities randomly inside square regions  $[0, 8] \times [0, 8]$  in position and  $[0, 3] \times [0, 3]$  in velocity spaces respectively.

We assume all agents wish to follow the same trajectory

$$y(t) = (100 + 10 \sin(0.1t), 10 + 10 \cos(0.1t))^t$$

in the position space. We assume each agent  $i$  implements a feedback law for steering according to

$$\beta_i(t) = \gamma_1(\dot{y}(t) - v_i(t)) + \gamma_2(y(t) - x_i(t)),$$

Where  $\gamma_1$  and  $\gamma_2$  are two parameters. In these MATLAB simulations, we took  $\gamma_1 = 2$  and  $\gamma_2 = 0.1$ . We computed the solutions of the full model (2.11) for  $\epsilon = 0.1$ ,  $\epsilon = 0.01$  and  $\epsilon = 0.001$ .

Additionally, we also computed the solution of the reduced model (4.19) obtained via the singular perturbation theory. In order to compute the correct initial flocking velocity  $v^f(0)$  to be used in conjunction with (4.19), we use the equation (4.25).

Finally, we computed the leading order approximation and compared it to the simulation when  $\epsilon = 0.1$ ,  $\epsilon = 0.01$  and  $\epsilon = 0.001$ . For all  $\epsilon$  we used the same randomly chosen initial conditions which we provide here. Initial positions were

$$\begin{aligned} x_1(0) &= (6.8897, 7.1568)^t, & x_2(0) &= (1.6819, 4.4079)^t, & x_3(0) &= (4.0103, 5.8168)^t, \\ x_4(0) &= (6.3834, 6.7922)^t, & x_5(0) &= (2.4842, 6.1173)^t, & x_6(0) &= (5.8959, 1.4635)^t, \\ x_7(0) &= (1.0710, 4.4853)^t, \end{aligned}$$

and the initial velocities were

$$\begin{aligned} v_1(0) &= (2.8792, 1.0212)^t, & v_2(0) &= (1.7558, 0.6714)^t, & v_3(0) &= (2.2538, 0.7653)^t, \\ v_4(0) &= (1.5179, 2.0972)^t, & v_5(0) &= (2.6727, 2.8779)^t, & v_6(0) &= (1.6416, 0.4159)^t, \\ v_7(0) &= (0.4479, 0.7725)^t. \end{aligned}$$

In these simulations, we have used the following functions.

$$\begin{aligned} \phi_{ij}(x, u) &= \frac{\phi(r_{ij})}{\sum_k \phi(r_{ik})} \text{ where, } r_{ij} = \|x_j - x_i\| \\ \phi(r) &= \frac{1}{(1 + r^2)^{0.3}} \\ \alpha_i(t) = \xi_i(u_i) &= \frac{10}{(0.1 + \|u_i\|^2)^{0.5}} \text{ where, } u_i = \bar{v}_i - v_i. \end{aligned}$$

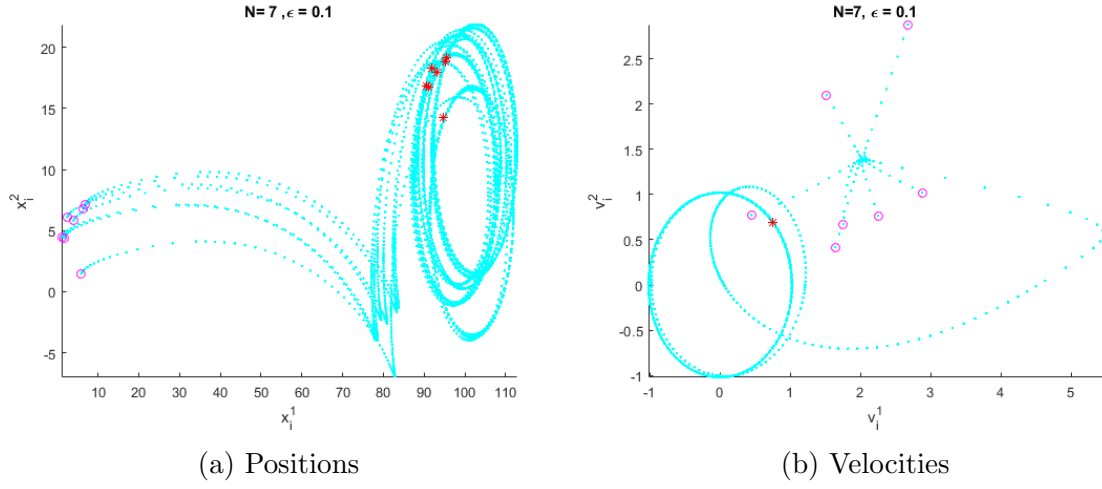


Figure 3: Trajectories in position and velocity spaces. Cyan circles represent initial values and red stars the final values. Case  $\epsilon = 0.1$ .

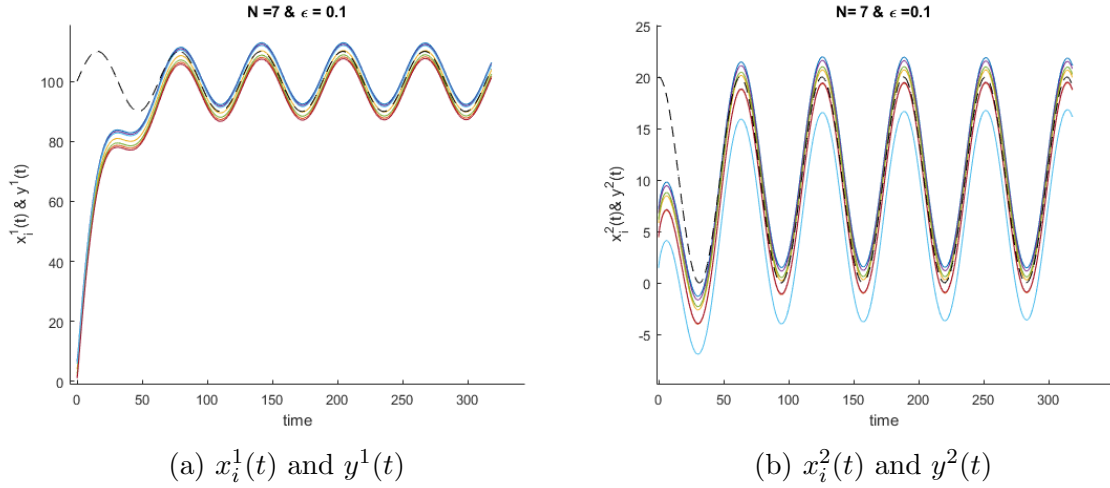


Figure 4: Positions against time. Case  $\epsilon = 0.1$ . Target trajectory in dash black.

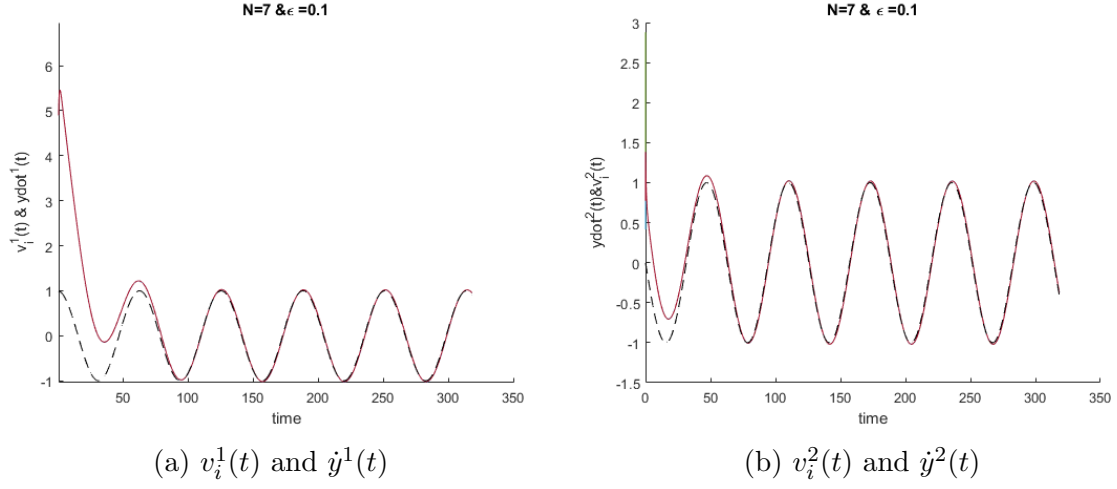


Figure 5: Velocities against time. Case  $\epsilon = 0.1$ . Target velocity in dash black

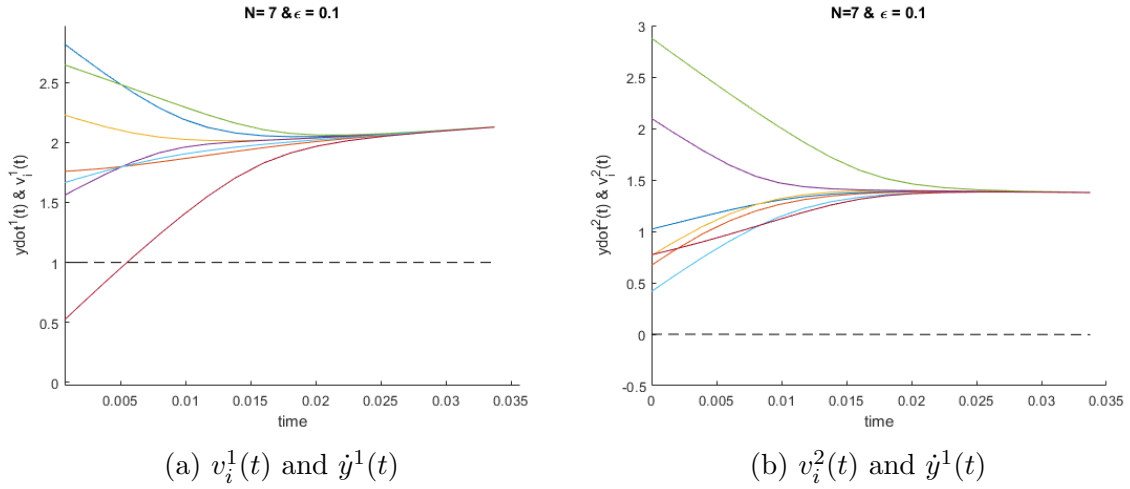


Figure 6: Velocities against time for  $t$  close to zero. Case  $\epsilon = 0.1$  short representation. Target velocity in dash black



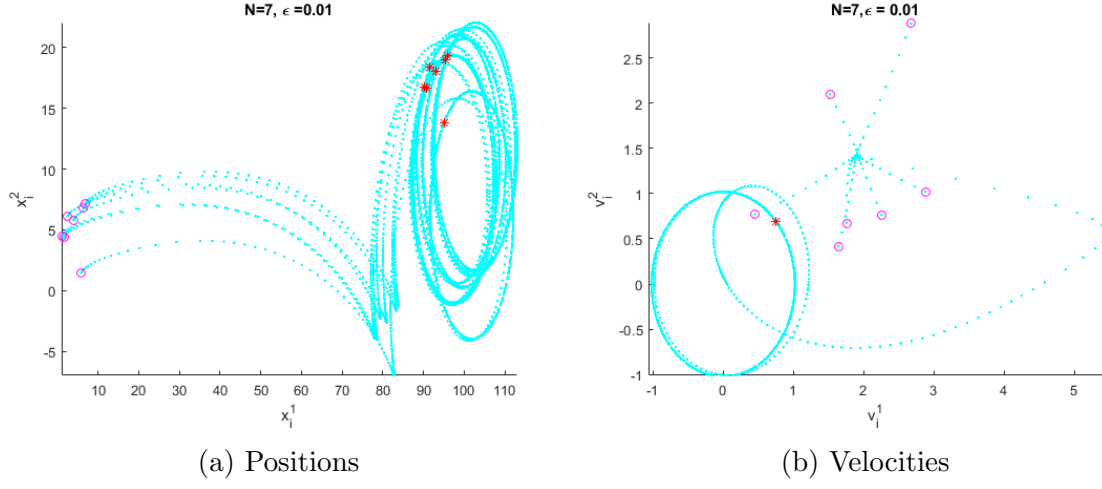


Figure 7: Trajectories in position and velocity spaces. Cyan circles represent initial values and red stars the final values.. Case  $\epsilon = 0.01$ .

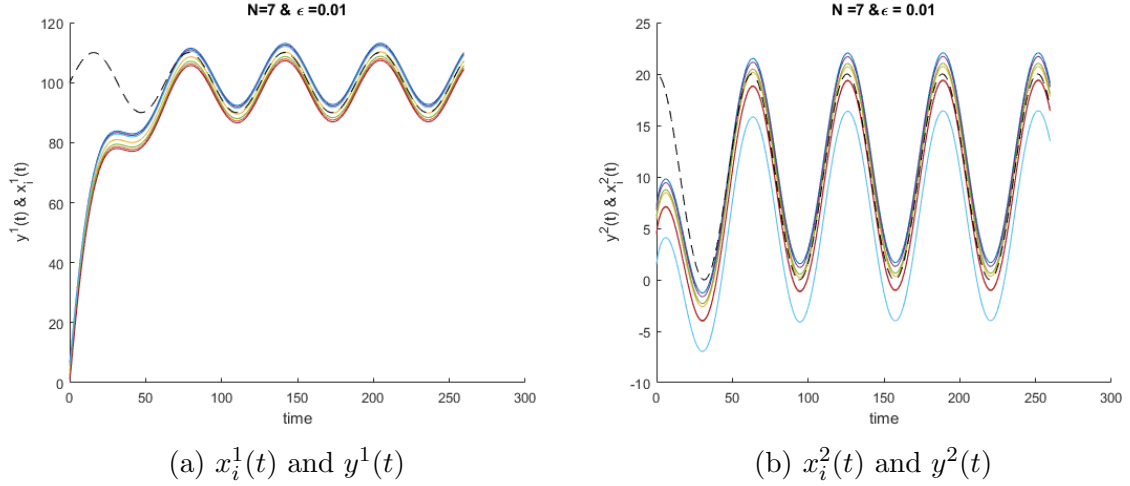
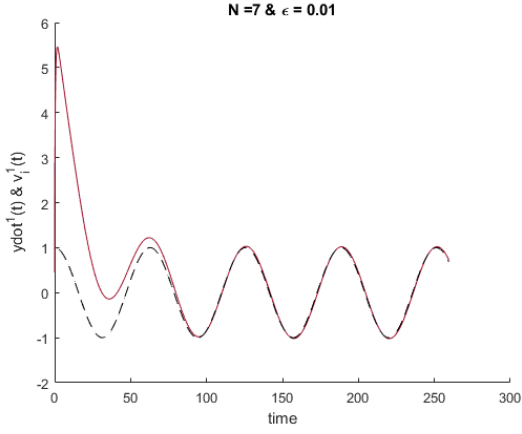
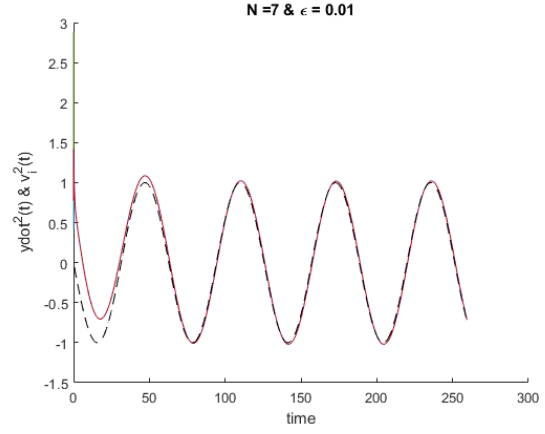


Figure 8: Trajectories against time. Case  $\epsilon = 0.01$ . Target trajectory in dash black

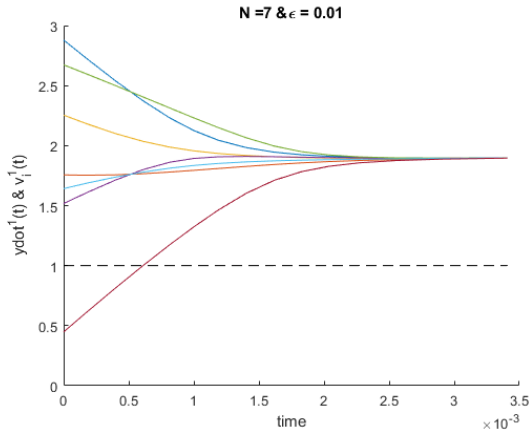


(a)  $v_i^1(t)$  and  $\dot{y}^1(t)$

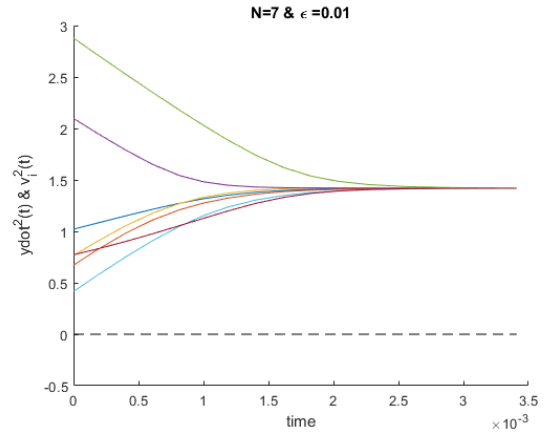


(b)  $v_i^2(t)$  and  $\dot{y}^2(t)$

Figure 9: Velocities against time. Case  $\epsilon = 0.01$ . Target velocity in dash black



(a)  $v_i^1(t)$  and  $\dot{y}^1(t)$



(b)  $v_i^2(t)$  and  $\dot{y}^1(t)$

Figure 10: Velocities against time for  $t$  close to zero. Case  $\epsilon = 0.01$ . Target velocity in dash black

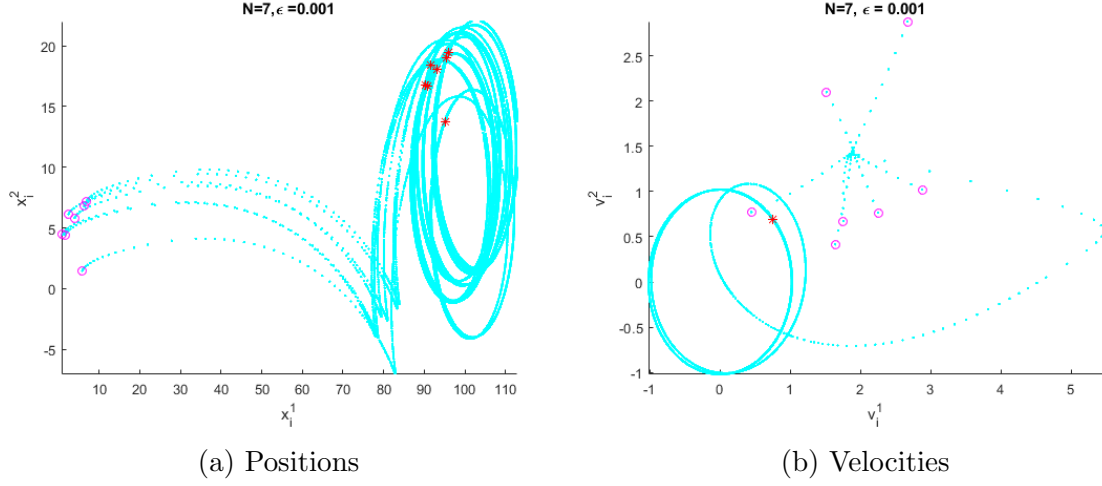


Figure 11: Trajectories in position and velocity spaces. Cyan circles represent initial values and red stars the final values.. Case  $\epsilon = 0.001$ .

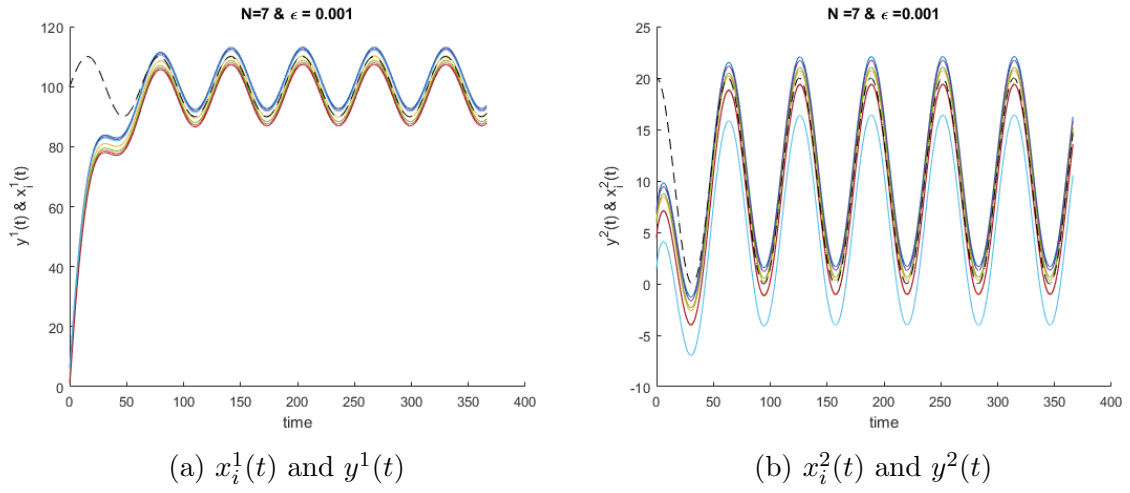
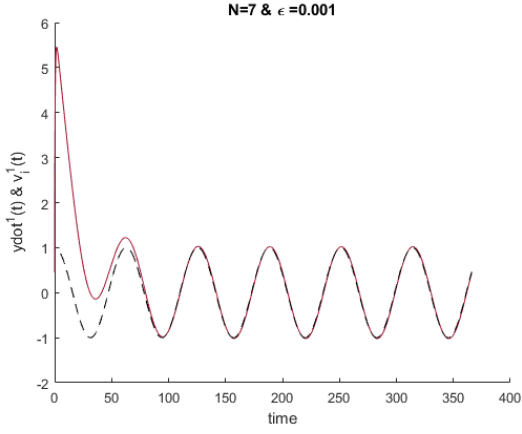
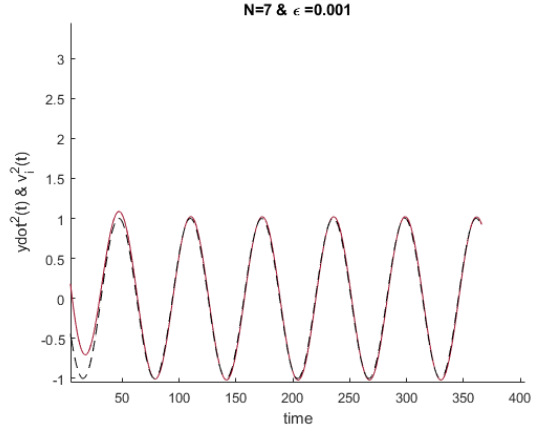


Figure 12: Positions against time. Case  $\epsilon = 0.001$ . Target trajectory in dash black

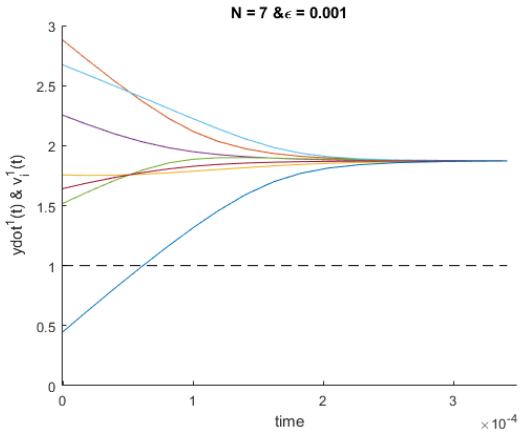


(a)  $v_i^1(t)$  and  $\dot{y}^1(t)$

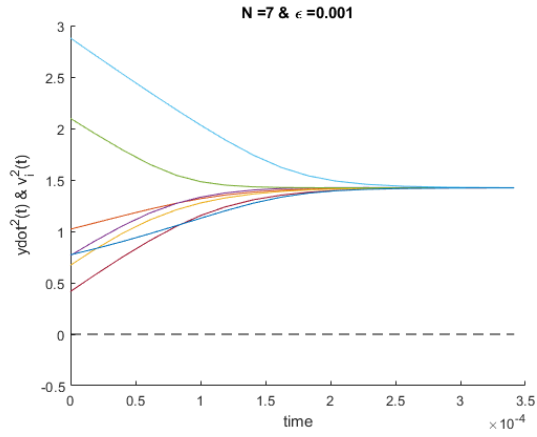


(b)  $v_i^2(t)$  and  $\dot{y}^2(t)$

Figure 13: Velocities against time. Case  $\epsilon = 0.001$ . Target velocity in dash black



(a)  $v_i^1(t)$  and  $\dot{y}^1(t)$



(b)  $v_i^2(t)$  and  $\dot{y}^1(t)$

Figure 14: Velocities against time for  $t$  close to zero. Case  $\epsilon = 0.001$ . Target velocity in dash black

In the position and velocity spaces, the small cyan circles represent initial values of our simulations and the stars represent the final time values. Figures 3, 7 and 11 show that our model flocks. In fact, Figures 3b, 7b and 11b show that all agents flock to a common velocity  $V^f(0)$  and then stay together and steer towards the target velocity. Our simulations also show that after some time the trajectory of each agent is similar to the target trajectory which is a circle. Furthermore, Figures 4, 8 and 12 which show the plots of trajectories components against time we see that after some time, all the components follow the target trajectory components (in dashed black). Figures 5, 9 and 13 is the plots of components velocities against time. In conjunction with figures 6, 10 and 14 we read that velocities components converge very fast to a common velocity and then in the long run, align with the target velocity (in dashed black).

### Comparison of the leading order and the cases $\epsilon = 0.1$ , $\epsilon = 0.01$ and $\epsilon = 0.001$

We graphically compare the positions and the velocities of the leading order approximation model (4.19) and the actual model (4.1) for different values of  $\epsilon$ . In Figures 15, 16 and 17, the approximate model solution is in green color while the actual model solution is in blue. The small red circles and red stars represent the initial and final positions for the actual model. Likewise, the small yellow circles and yellow stars represent the initial and final position for the approximate model. Graphically, we see that the singular perturbation theory shows good agreement with the exact model.

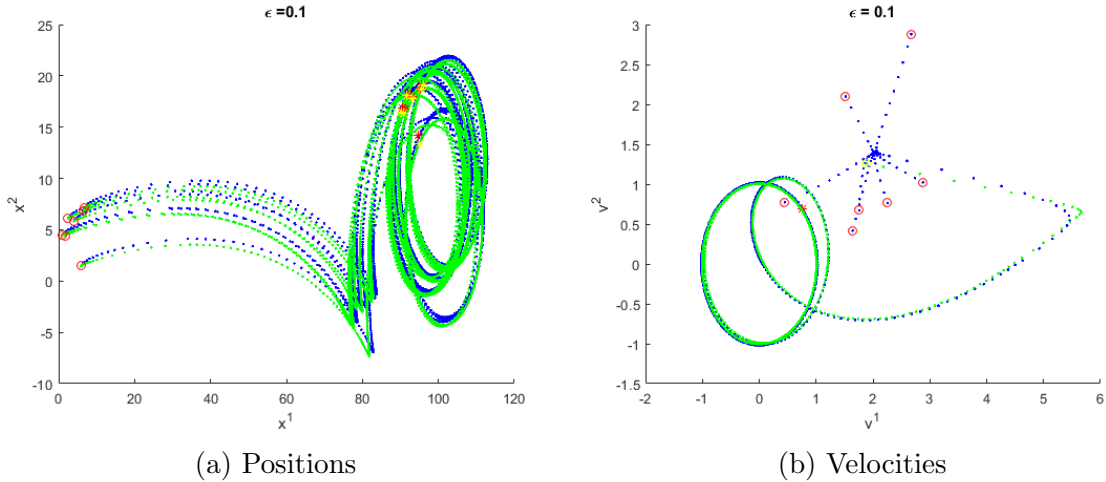


Figure 15: Comparison between the reduced model (in green) and exact solution (in blue) for  $\epsilon = 0.1$

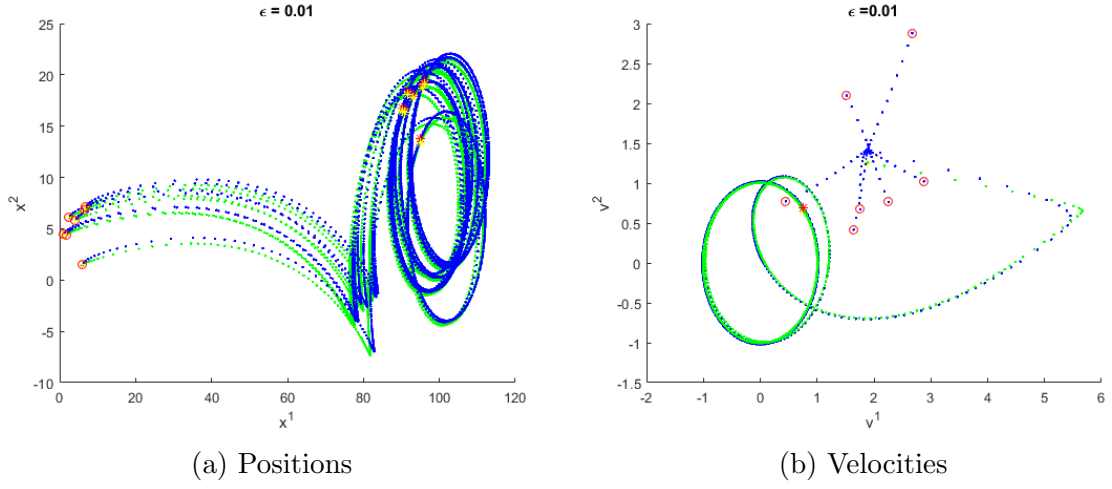


Figure 16: Comparison between the reduced model (green) and exact solution (in blue) for  $\epsilon = 0.01$

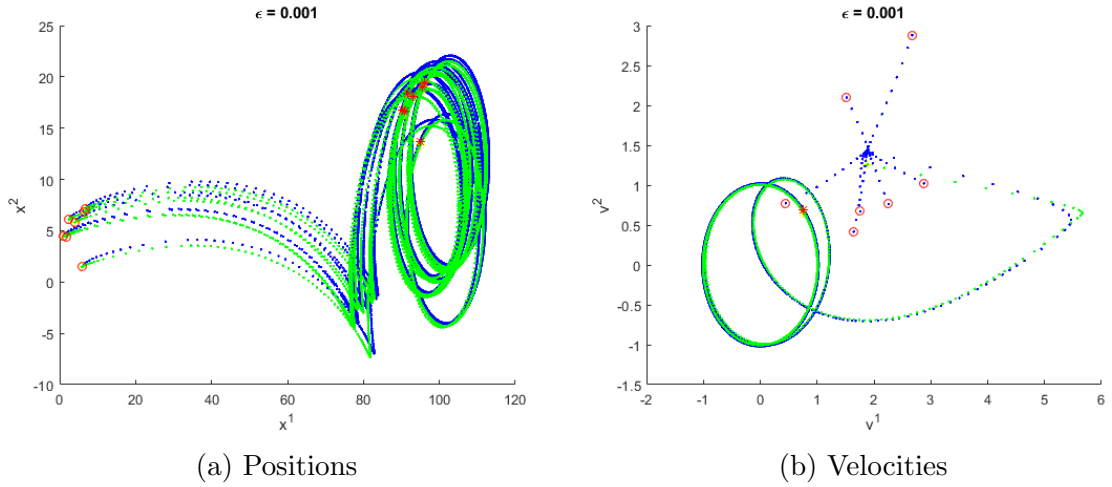


Figure 17: Comparison between the reduced model (in green) and exact solution (in blue) for  $\epsilon = 0.001$

## 6 Concluding remarks

We introduced and analyzed a generalized model of flocking with steering. In our model, the acceleration of each agent has flocking and steering components. The flocking component is a generalization of many existing models and takes into account real world factors such as apriori bound on acceleration, masking effects and orientation bias. We proved that the generalized model with steering flocks under certain sufficient conditions which naturally include assumptions on the steering components  $\beta_i(t)$  of the accelerations of the agents. We also studied the case where flocking is much faster than steering using formal singular perturbation theory and showed that the leading order behavior is one where the agents flock together with velocity  $v^f$  which evolves in time, see 4.15. Our simulations showed that the leading order approximation was very similar to the real solution for small values of  $\epsilon$  a scale parameter indicating the magnitude difference between flocking and steering accelerations. While this supports our formal derivation via singular perturbation theory, in future we would like to derive rigorous results that support the formal theory.

We also observe that the influence functions  $\phi_{ij}$  were assumed to be nonvanishing for all  $i, j \in 1, \dots, N$  in our flocking results. This implies that the communication graph formed by the agents is strongly connected. In the case of the robotic systems, this will be computationally expensive. Even in the case of biological agents, all to all communication among agents may not be a reasonable assumption. This raises the question whether one could relax the strict positivity condition and still obtain flocking results.

Our flocking results assumed that the steering components  $\beta_i(t)$  of the agents were asymptotically in agreement ( $\beta_i(t) - \beta_j(t) \rightarrow 0$  as  $t \rightarrow \infty$ ). A related natural question is if the agents form subgroups within which this condition holds but fails across these subgroups, then can we obtain clusters of agents such that agents within each cluster flock together.

## A Useful lemmas

**Lemma 5** *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  be locally Lipschitz and  $u : [0, T] \rightarrow \mathbb{R}$  be absolutely continuous. Then  $F \circ u : [0, T] \rightarrow \mathbb{R}$  is absolutely continuous.*

**Lemma 6** *The function  $f_i : \mathbb{R} \rightarrow \mathbb{R}$  be absolutely continuous on  $[0, T]$  for  $i = 1, \dots, n$  and let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by*

$$f(t) = \max\{f_i(t) \mid i = 1 \dots n\}.$$

*Suppose  $i_* : \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $f_{i_*(t)}(t) \geq f_j(t)$  for all  $t$  and  $j = 1, \dots, n$ . Then  $f$  is absolutely continuous and  $f'(t) = f'_{i_*(t)}(t)$  for almost all  $t$ .*

**Lemma 7** *The forward maximal interval of existence of the model 2.10 is  $[0, \infty)$  where we assume that  $\alpha_i, \beta_i, a_{ij}$  are all continuous functions on  $[0, \infty)$ .*

**Proof** Let us suppose that the forward maximal interval of existence is the interval  $[0, T^*)$ , with  $T^* < \infty$ . We define the energy of the system by  $E(t) = \max_i E_i(t) = \max_i \frac{1}{2} \|v_i(t)\|^2$ .

Then

$$\begin{aligned}
\frac{dE(t)}{dt} &= \langle v_i, \dot{v}_i \rangle = \langle v_i, \alpha_i(\bar{v}_i - v_i) + \beta_i \rangle \\
&= \alpha_i \langle v_i, \bar{v}_i \rangle - \alpha_i \langle v_i, v_i \rangle + \alpha_i \langle v_i, \beta_i \rangle \\
&\leq \alpha_i \|v_i\| \left( \sum_j a_{ij} \|v_j\| \right) - \alpha_i \|v_i\|^2 + \alpha_i \|v_i\| \|\beta_i\| \\
&\leq \alpha_i \|v_i\| \|\beta_i\|
\end{aligned}$$

We have used the Cauchy-Schwartz inequality and the conditions  $\|v_j\| \leq \|v_i\|$  and  $\sum_j a_{ij} = 1$ . We rearrange this inequality to

$$\frac{dE(t)}{dt} \leq \alpha_i (2E(t))^{\frac{1}{2}} \|\beta_i\| \leq \bar{\alpha} (2E(t))^{\frac{1}{2}} \|\beta_i\|.$$

Where  $\bar{\alpha}$  is the maximum of  $\alpha_i(t)$  over  $i$  and  $t \in [0, T^*]$ . Multiplying this inequality by  $(2E(t))^{-\frac{1}{2}}$  we may obtain

$$\frac{dE^{\frac{1}{2}}(t)}{dt} \leq 2^{\frac{1}{2}} \bar{\alpha} \|\beta_i\|.$$

Let  $M_\beta > 0$  satisfy  $\|\beta_i(t)\| \leq M_\beta$  for all  $i$  and  $t \in [0, T^*]$ . We obtain

$$\left( E^{\frac{1}{2}}(t) - E^{\frac{1}{2}}(0) \right) \leq 2^{-\frac{1}{2}} \bar{\alpha} M_\beta T^*.$$

And we deduce that

$$\|v_i\| \leq \left( E^{\frac{1}{2}}(0) + 2^{-\frac{1}{2}} \bar{\alpha} M_\beta T^* \right) < \infty.$$

We then deduce the upper bound of the vector position  $x_i(t)$  as

$$\|x_i(t)\| \leq \left( E^{\frac{1}{2}}(0) + 2^{-\frac{1}{2}} \bar{\alpha} M_\beta T^* \right) T^* + \|x_i(0)\|.$$

Since the solution remains in a compact set for  $t \in [0, T^*)$  we obtain a contradiction. ■

## References

- [1] Shin Mi Ahn and Seung-Yeal Ha. “Stochastic flocking dynamics of the Cucker–Smale model with multiplicative white noises”. In: *Journal of Mathematical Physics* 51.10 (2010), p. 103301.
- [2] Shin Mi Ahn et al. “On collision-avoiding initial configurations to Cucker-Smale type flocking models”. In: *Communications in Mathematical Sciences* 10.2 (2012), pp. 625–643.
- [3] Matteo Aureli and Maurizio Porfiri. “Coordination of self-propelled particles through external leadership”. In: *EPL (Europhysics Letters)* 92.4 (2010), p. 40004.



- [4] José A Canizo, José A Carrillo, and Jesús Rosado. “A well-posedness theory in measures for some kinetic models of collective motion”. In: *Mathematical Models and Methods in Applied Sciences* 21.03 (2011), pp. 515–539.
- [5] Felipe Cucker and Jiu-Gang Dong. “Avoiding collisions in flocks”. In: *IEEE Transactions on Automatic Control* 55.5 (2010), pp. 1238–1243.
- [6] Felipe Cucker, Steve Smale, et al. “Emergent behavior in flocks”. In: *IEEE Transactions on automatic control* 52.5 (2007), pp. 852–862.
- [7] Felipe Cucker, Steve Smale, and Ding-Xuan Zhou. “Modeling language evolution”. In: *Foundations of Computational Mathematics* 4.3 (2004), pp. 315–343.
- [8] Seung-Yeal Ha, Taeyoung Ha, and Jong-Ho Kim. “Emergent behavior of a Cucker-Smale type particle model with nonlinear velocity couplings”. In: *IEEE Transactions on Automatic Control* 55.7 (2010), pp. 1679–1683.
- [9] Seung-Yeal Ha, Kiseop Lee, Doron Levy, et al. “Emergence of time-asymptotic flocking in a stochastic Cucker-Smale system”. In: *Communications in Mathematical Sciences* 7.2 (2009), pp. 453–469.
- [10] Seung-Yeal Ha, Jian-Guo Liu, et al. “A simple proof of the Cucker-Smale flocking dynamics and mean-field limit”. In: *Communications in Mathematical Sciences* 7.2 (2009), pp. 297–325.
- [11] Jan Haskovec. “Flocking dynamics and mean-field limit in the Cucker-Smale-type model with topological interactions”. In: *Physica D: Nonlinear Phenomena* 261 (2013), pp. 42–51.
- [12] Julien M Hendrickx and John N Tsitsiklis. “Convergence of type-symmetric and cut-balanced consensus seeking systems”. In: *IEEE Transactions on Automatic Control* 58.1 (2013), pp. 214–218.
- [13] Sebastien Motsch and Eitan Tadmor. “A new model for self-organized dynamics and its flocking behavior”. In: *Journal of Statistical Physics* 144.5 (2011), p. 923.
- [14] Reza Olfati-Saber. “Flocking for multi-agent dynamic systems: Algorithms and theory”. In: *IEEE Transactions on automatic control* 51.3 (2006), pp. 401–420.
- [15] Jaemann Park, H Jin Kim, and Seung-Yeal Ha. “Cucker-Smale flocking with inter-particle bonding forces”. In: *IEEE Transactions on Automatic Control* 55.11 (2010), pp. 2617–2623.
- [16] Jackie Shen. “Cucker-Smale flocking under hierarchical leadership”. In: *SIAM Journal on Applied Mathematics* 68.3 (2008), pp. 694–719.
- [17] Serap Tay Stamoulas and Muruhan Rathinam. “Convergence, stability, and robustness of multidimensional opinion dynamics in continuous time”. In: *SIAM Journal on Control and Optimization* 56.3 (2018), pp. 1938–1967.
- [18] Tamás Vicsek et al. “Novel type of phase transition in a system of self-driven particles”. In: *Physical review letters* 75.6 (1995), p. 1226.