BANZHAF VOTING POWER, RANDOM ELECTIONS, AND THE ELECTORAL COLLEGE WINNER'S ADVANTAGE

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June 2009
Revised December 2009
Final Revision August 2011

Forthcoming in *Electoral Studies*

Abstract

In a recent article, Riggs, Hobbs, and Riggs (2009) aim to measure the 'Electoral College winner's advantage' — in particular, the extent to which the winner's electoral vote margin of victory is magnified as a result of (i) the 'two electoral vote add-on' given to each state and (ii) the 'winner-take-all' mode of casting state electoral votes. Their results are based on two sets of one million simulated two-candidate elections. This note has two purposes. The first is to demonstrate that RHR's simulation estimates can be calculated precisely using the theory of voting power measurement. The second is to correct several flaws in RHR's analysis, the most substantial of which pertains to the effect of the two electoral vote add-on, which actually has a negative effect on the winner's advantage.

Key words: Electoral College, winner's advantage, Banzhaf voting power

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1. Introduction

In a recent article, Riggs, Hobbs, and Riggs (2009; henceforth RHR) aim to measure the 'Electoral College winner's advantage' — in particular, the extent to which the winner's margin of victory with respect to electoral votes is magnified as a result of (i) the 'addition of two electoral votes to each state regardless of population size' and (ii) the 'winner-take-all' mode of casting state electoral votes (based on the 'general ticket' system for electing presidential electors) that has been the almost universal state practice since the 1830s. RHR begin with the common observation that the Electoral College creates the 'illusion of a popular mandate,' in that the winner's electoral vote percent almost always exceeds his popular vote percent. RHR attempt to measure the fundamental Electoral College winner's advantage in an *a priori* fashion — that is, independent of demographic characteristics, historical voting trends, polling or survey data, turnout, actual election results, and other empirical contingencies — and, more specifically, to measure the extent to which it is enhanced by the two-electoral vote bonus for all states and by the winner-take-all mode of casting electoral votes.

RHR address these questions by analyzing two sets of 'one million random simulation elections,' in which the electoral votes of each state (or district) were assigned with equal likelihood to one of two candidates. For each election, the electoral votes for each candidate were added up and the winning candidate determined. With this data at hand, RHR count up the 'wins' for each state — that is, the number of times each state votes for the winning candidate — under each electoral college variant. Dividing the number of 'wins' for each state by one million gives its 'win rate' — that is, the proportion of times each state votes for the winner. Even a (hypothetical) state with zero electoral votes will have a 'win rate' of .5 (plus or minus a small sampling error); the smallest states with three electoral votes will have 'win rates' somewhat greater than .5 and 'win rates' will increase with the number of electoral votes a state has.

Given each state's 'win rate,' RHR then add up the electoral votes of all the states weighted by their 'win rates,' to determine the expected electoral vote for the winning candidate. On this basis, RHR conclude that "the net effect of the Electoral College is to give the winning candidate an average 29.45 electoral vote advantage per election due to the winner-take-all methodology [as opposed to election by districts]. This winner's advantage includes an average 0.42 electoral vote advantage given to the winner per election due to the two electoral vote add-on."

This note has two purposes. The first is to point out that the 'win rates' derived by RHR from random election simulations can be determined on the basis of the theory of voting power measurement, with (in principle) perfect accuracy and without resort to simulation. The second is to replicate RHR's analysis using these perfectly accurate 'win rates' and to identify and correct several flaws in RHR's analysis. The most substantial of these pertains the effect on the winner's advantage of the two electoral vote add-on. Another pertains to their treatment of electoral vote ties. I also refine their conclusion pertaining to the effect of the winner-take-all mode of casting electoral votes on the winner's advantage. Following RHR, I use the 1990 apportionment of electoral votes, ignore the fact that Nebraska and Maine do not actually use the 'winner-take-all' rule, and refer to the District of Columbia as if it were a state. Since this note makes frequent reference to RHR, it should be read with their article at hand.

2. Banzhaf Voting Power

The work of Felsenthal and Machover (1998, 2005) provides the most authoritative presentation of the theory voting power measurement. They argue persuasively that the *absolute Banzhaf measure* of voting power (Banzhaf 1965, 1968) is the appropriate tool for analyzing ordinary voting situations, including the U.S. Electoral College. Before applying it to the issue at hand, I will concisely sketch out the essentials of Banzhaf voting power measurement, using the example of the 51-state Electoral College.

(1) There are a great many different ways in which 51 states can 'choose up sides' — or, more formally, can be partitioned into complementary subsets — with respect to which of two presidential candidates wins their electoral votes. Call such an alignment of states a *bipartition*.

(2) In any bipartition of states, either one side is winning (with 270 or more electoral votes) or there is a tie (each side has 269 electoral votes).

(3) A state is *critical* in a bipartition if the side to which it belongs is winning but would no longer be winning if the state switched sides.

(4) The *Banzhaf score of a state* is the number of bipartitions in which it is critical.

(5) The *absolute Banzhaf voting power Bz* of a state is its Banzhaf score divided by the total number of bipartitions.[1]

While this ratio may seem to lack theoretical justification, it has direct and intuitive meaning in terms of *a priori* probability. If we know nothing about a voting situation other than its formal rules, our *a priori* expectation must be that everyone votes randomly, i.e., as if independently flipping fair coins. In this *random voting model*, every bipartition of states into complementary sets, in which every state in one set votes for one candidate and every state in the other set votes for the other candidate, has an equal probability of occurring.

(6) Therefore, the absolute Banzhaf voting power of a state is the *a priori* probability that the state's electoral vote is critical and determines which candidate wins in a random election.

The relevance of the Banzhaf voting power measure to the RHR analysis of the winner's advantage in the Electoral College follows from two facts. First, RHR's simulated elections are random elections in which the states are voters. Second, a simple theorem (Felsenthal and Machover, p. 45) relates a state's Banzhaf voting power *Bz* to its *probability of success* — that is, its 'win rate' in random elections. Since a state is successful whenever it casts a decisive vote (the probability of

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[1] Calculating Banzhaf power values can be burdensome and, given an even modestly large number of voters, exact calculation may exceed the capabilities of the most powerful computers. But for 51 voters, so-called 'generating functions' allow exact calculations. The website *Computer Algorithms for Voting Power Analysis* created by Dennis Leech and Robert Leech calculates Bahzhaf power values displayed to six decimal places. I have used its *ipgenf* algorithm to make the calculations used here.

which is Bz) and (like a powerless state) is also successful half the time it does not cast a decisive vote (the probability of which is $1 - Bz$), *probability of success* $= Bz + (1 - Bz)/2$, which simplifies to the following.

(7) The *probability of success* (or 'win rate') of a state is equal to .5 plus half of its Banzhaf power.

3. The Winner's Expected Electoral Vote with the Two Electoral Vote Add-On

In their Table 1, RHR report the number of 'wins' for each state in the one million random elections both with and without the two electoral vote add-on. To save space, my similar tables show data for a small sample of states only.[2] Table 1(a) duplicates the first three columns of RHR's Table 1, showing the name of the state, its number of electoral votes, and its number of 'wins' with the add-on in one million elections. Column 4 converts each state's wins into its 'win rate.' Column 5 multiplies each state's number of electoral votes by its 'win rate' to show the expected number of electoral votes for the winner from each state. The total in Column 5 (307.6858) thus gives the winner's expected electoral vote based on the simulated random elections. This matches the sum reported in RHR's text on p. 355.

Column 6 of my Table 1(a) shows the Banzhaf voting power of each state (as calculated by *ipgenf* at the Leech website and accurate to six decimal places), and Column 7 shows its probability of success given by the formula above. Each number in Column 7 should match the corresponding numbers in Column 4, except that RHR's numbers are subject to a small amount of sampling error (as indicated by the fact that states with the same number of electoral votes, e.g., Alaska, Delaware and D.C., have slightly different 'win rates'). However, it is evident that there is a further discrepancy between the two columns that cannot be attributed to sampling error, in that each entry in Column 7 is (almost) always (slightly) greater than the corresponding entry in Column 4. This consistent bias is carried over into Column 8, which gives each state's number of electoral votes times its probability of success, the sum (309.7391) of which (i.e., the winner's expected electoral vote based on Banzhaf values) is distinctly larger than the sum of Column 5.

This discrepancy arises because the RHR calculations do not distinguish between elections in which a state votes for the losing candidate and elections in which a state votes for either of two tied candidates. RHR report that there were 7,630 tied elections. Following what the Banzhaf measure in effect does, Column 9 gives each state 'half credit' for voting for a tied candidate — that is, it adds 3815 elections to each number in Column 3. Column 10 show each state's win rate adjusted to take account of ties multiplied by its number of electoral votes. The total of Column 10 (309.7382) matches the total in Column 8 essentially perfectly (the sampling error in Column 10 showing up only in the sixth digit).

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[2] Versions of this and subsequent tables showing all states are available on my website at http://userpages.umbc.edu/~nmiller/RESEARCH/ECWA.TABLES.pdf .

4. The Winner's Expected Electoral Vote Without the Two-Vote Add-On

Table 1(b) duplicates Table 1(a) for the modified Electoral College in which each state has as many electoral votes as House seats. (Column 3 duplicates data shown in RHH's Table 1 without the two electoral vote add-on.) RHR do not report how many electoral vote ties occurred in their simulated elections when electoral votes were modified in this manner. However, the expected number of electoral vote ties in one million random elections can be calculated from Banzhaf formulas and is equal to 8,376.[3] Table 1(b) uses this expectation to produce the numbers in Columns 9 and 10, and the sum in the latter again matches the sum in Column 8 essentially perfectly. The 254.7585 sum of Column 8 of Table 1(b) gives the expected number of electoral votes for the winning candidate in a random election under an Electoral College modified to remove the two vote add-on. RHR do not report, and perhaps did not calculate, the corresponding 252.9412 sum of Column 5.

When we compare the expected number of electoral votes for the winning candidate under the two Electoral College variants, we must take account of the fact that the modified variant has 102 fewer electoral votes in play than the existing system. (It is not clear whether RHR did this.) We certainly cannot compare the winner's advantage under the two systems in terms of the winner's expected electoral votes (309.7391 vs. 254.7585); we must look instead at the winner's advantage relative to the number of electoral votes in play in each variant. Under the existing Electoral College, the winner's expected electoral vote of 309.7391 represents 57.57% of the 538 electoral votes, while under the modified Electoral College, the winner's expected electoral vote of 254.7585 represents 58.43% of the 436 electoral votes. Alternatively, we can rescale the number of electoral votes without the two-vote bonus so that they add up to 538 by multiplying by 254.76 by 538/436, giving 314.36. In any event, the 'winner's advantage' deriving from the two vote add-on is actually a *disadvantage* (−4.62 electoral votes or −0.86 percentage points), contrary to RHR's claim that the add-on adds 'an average of 0.42 electoral votes to the winner total electoral vote count.'[4]

Instead of directly comparing the winner's expected electoral vote under each Electoral College variant in the way we have just done, RHR in their Table 1 focus on the *difference* between the number of wins for each state under the Electoral College with and without the add-on. They then multiply this difference for each state by its number of electoral votes (including the add-on) and

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[3] The same calculation for Table 1(a) gives 7,621 expected ties under the existing Electoral College, compared with the 7,630 found in the RHR simulation. The failure of RHR to give states half credit for voting for a tied candidate (together with sampling error) explains why, under the Electoral College without the two vote add-on, several of the smallest states win less than half of the time, even though a state with zero electoral votes (and zero Banzhaf power) can be expected to win precisely half the time.

[4] This disadvantage is to be expected, as the effect of giving every state a bonus of some fixed number of electoral votes is to make voting weights more equal across the states, thereby making Banzhaf power (and success) more equal and reducing the winner's expected (relative) electoral vote. For example, giving every state an additional 10 electoral votes reduces the winner's expected electoral vote (when rescaled so that the total electoral vote remains 538) to about 302.2. In the limit, when all states have equal voting weight, the winner's expected electoral vote is about 299.2.

divide by one million to get the 'average advantage' to the winner in each state in each election (the last column of their Table 1). Finally, RHR add up these averages over all 51 states to get +0.418140 (not shown in their Table 1 but reported as +0.42 in the text), which they claim represents 'the intrinsic advantage for the winner as a result of the two electoral vote add-on methodology.' Of course, like their other results, this is subject to the problem of electoral vote ties and incorporates the small sampling error inherent in the simulation data. Columns 1-5 in Table 2 shows parallel calculations based on exact Banzhaf values, producing a sum of +0.2275, rather than about +0.42, electoral votes. But, so far as I can see, neither of these numbers represents a quantity of interest. What RHR evidently aim to do is to take the magnitude of the winner's advantage given by the Electoral College without the two-vote bonus as a baseline and then determine how much the two-vote bonus adds to this winner's advantage and, moreover, to determine how much each state contributes to this addition.

Supposing that this is RHR's goal, the remaining columns in Table 2 carry out the appropriate calculations. On the one hand, the winning candidate gains (or loses) the difference for each state times its electoral votes *without* the two-vote bonus, as shown in Column 6. On the other hand, the winning candidate gains two electoral votes from every state he carries, as shown in Column 7. The sum of these two quantities is the total gain the winning candidate earns in each state in one million random elections as a result of the two electoral vote add-on, as shown in Column 8. Dividing this sum by one million gives the average gain in each state shown in Column 9, which sums to 54.98, precisely equal to the difference between 309.74 vs. 254.76, i.e., the difference between the previously calculated expected electoral votes for the winning candidate under each system. What Table 2 adds to the previous calculations is how much individual states contribute to this gain. Given that the total electoral vote is increased by 2 for each state, we would expect the winning candidate to gain a bit more than one electoral vote from each state, which is true both on average (the mean of Column 9 is 1.078) and for every individual state except California.[5]

5. The Winner's Expected Electoral Vote Without Winner-Take-All

To assess the effect of the winner-take-all feature of the Electoral College on the winner's advantage, RHR ran an additional one million simulated elections with 538 districts each casting one electoral vote. RHR do not replicate their Table 1 for this set of elections, as their simulation undoubtedly showed the unsurprising result that (apart from sampling error) all districts voted for the winner the same number of times. The Banzhaf power of each district is .034293, so its probability

[5] California has an especially dominant position with respect to House seats, which is substantially reduced by giving every state the two additional electoral votes. California's consequent loss of Banzhaf power, and therefore of success, more than balances off its gain of two electoral votes. The proper interpretation of the numbers in this last column is that (for example) Illinois contributes on average an additional 1.1721 electoral votes to the winning candidate in each random election, not that 'Illinois gains an average of 1.1721 electoral votes per election' as stated by RHR on p. 354. RHR's statistic that in about 2.55% of random elections the two Electoral College variants produced different winners cannot (so far as I can see) be derived from Banzhaf calculations so, in this respect at least, RHR's simulations produce a finding that cannot otherwise be obtained.

of success is .5171465. Therefore the expected number of electoral votes won by the winning candidate is equal to .5171465 times 538 or 278.2248, which very closely matches the figure 278.24 that RHR report.[6] With respect the Banzhaf calculations, the winner gains an average advantage of about 309.7391 − 278.2248 = 31.5143 electoral votes from the existing system relative to the district system. (RHR's estimate is about 29.45, because of the downward bias in their first estimate.)

This 31.5149 electoral vote gain can be partitioned into two components, reflecting two distinct points of difference between a system with 538 districts and the existing Electoral College system: first, in the former the units cast equal electoral votes, while in the latter they cast unequal electoral votes; and second, in the former there are 538 'electoral entities' (RHR's term), while in the latter there are only 51. A third calculation — for 51 entities each with an equal number of 538/51 ≈ 10.55 electoral votes — indicates that the second difference makes a considerably greater contribution to the increase in the winner's advantage than the first. The Banzhaf power of each of 51 equally weighted entities is .112275, so the probability of success of each is .5561 and the expected electoral vote for the winner is about 299.2 (as previously reported in footnote 4). That is to say, reducing the number of equally weighted electoral entities from 538 to 51 itself augments the winner's advantage by about 21 electoral votes, while replacing 51 equally weighted districts with 51 states casting unequal electoral votes on a winner-take-all basis augments the winner's advantage by only about 10.5 additional electoral votes.

6. Concluding Remarks

While the results presented by RHR and modified here may be of interest, it should be noted that they do not bear directly on 'the illusion of a popular mandate' conferred by the Electoral College on the winner — that is, the fact that in actual Presidential elections the winning candidate typically wins a much larger proportion of the electoral votes than of the popular vote — to which RHR refer in their introductory remarks. This is because in the state-level random elections that RHR simulated, and likewise in my own state-level Banzhaf calculations, there is no popular vote with which to compare the electoral vote (as RHR themselves note on p. 356). It is possible to simulate *two-tier* random elections in which individual voters decide how to vote by independently flipping fair coins and these random votes are is then aggregated by electoral entity (e.g., Miller, 2009). It remains true that the entities are equally likely to give their electoral votes to either candidate (as in the RHR simulations), but we now have an underlying popular vote with the winner's electoral vote may be compared. The problem is that this underlying popular vote is very strange.[7] In essentially every random election, the popular vote is a virtual tie both nationally and within each state. For example, given a national electorate of 130,000,000 voters, the winning candidate's expected popular vote

[6] Many more ties (RHR report 34,059) occur with 538 equally weighted districts than with 51 unequally weighted districts, and evidently RHR did make an adjustment for ties in reaching this result.

[7] This strangeness is the basis of a common criticism (see Margolis, 1983; Gelman et al., 2004) of the concept of *a priori* voting power based on the random voting model.

margin over the loser is less than 10,000 votes and his expected popular vote percent is about 50.0035%.[8]

Thus, if we compare the winner's expected vote margin with respect to the electoral vote with the winner's expected margin with respect to the popular vote in two-tier random elections, the 'illusion of a popular mandate' is extraordinarily large — and extraordinarily unrealistic — under *any* mode of apportioning or casting electoral votes. Indeed, given incredible closeness of the popular vote in almost all random elections, it is perhaps surprising that they can produce the degree of variation in electoral vote margins displayed in RHR's Figure 1. As we saw, in random elections the victor wins on average about 310 electoral votes, while in actual post-World War II elections the victor has won on average about 389 electoral votes. There is a substantial discrepancy between these two statistics, but it is nothing like the discrepancy between the winner's average popular vote of about 50.0035% in random elections and about the winner's average of about 54.3% in actual elections (based on the two-party vote).

[8] Put otherwise, a typical random election in Florida would be only slightly more lopsided than the actual 2000 election — the winner would have an expected 997 vote margin over the loser, in contrast to Bush's 537 vote margin over Gore.

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