THE BUTTERFLY EFFECT UNDER STV

Nicholas R. Miller
Department of Political Science
University of Maryland Baltimore County (UMBC)
Baltimore MD 21250

410-455-2187
nmiller@umbc.edu

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Abstract

This note presents an example of the sometimes chaotic character of the Single Transferable Vote that is both somewhat simpler, and even more striking, than previous examples, and it offers several comments about the practical and theoretical implications of this feature of STV.

Keywords: STV, chaos, butterfly effect

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The 'Butterfly Effect,' or more technically the 'sensitive dependence on initial conditions,' is the essence of chaos.[1]

More than twenty years ago, Michael Dummett (1984, p. 280) observed that the Single Transferable Vote (STV) method of election can operate in an arbitrary fashion 'in which a small change in the ballot papers returned by a few voters will make a radical alteration in the overall outcome.' Dummett returned even more emphatically to this point in his more recent book (1997, p.142, emphasis added):

> The assessment process of STV . . . may, however, be said to be *quasi-chaotic*, in that small changes at the initial stage may be magnified into huge changes at later stages, because they cause different candidates to be eliminated, and that in turn may result in a big variation in the allocation of votes at subsequent stages, owing to the differing redistributions of votes from one candidate and from another.

Dummett (1997, pp. 143-149) also provided an example involving eight candidates contesting four seats before an electorate of 99,995 voters. More recently, Geller (2005, p. 267) picked up on Dummett's example, also invoked the concept of chaos, and specifically referred to the 'Butterfly Effect.' [2]

In this brief note, I present an example of the Butterfly Effect under STV that is somewhat simpler, and even more striking, than Dummett's example. I also offer comments about its practical and theoretical implications.

1. An Example

Consider the following example. Seven candidates (*A*, *B*, *C*, *D*, *E*, *F*, and *G*) are competing before an electorate of 1001 voters for three seats. The quota for election is therefore 251 (and the residual 248 votes will be "wasted").

Voter preferences on the morning of the election are given by Ballot Profile 1 shown in Table 1A. As shown in Table 1B, the vote transfer process under Ballot Profile 1 results in the election of *C*, *F*, and *G*.

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[1] Michael Cross, http://www.cmp.caltech.edu/~mcc/chaos_new/Lorenz.html .

[2] To remedy the quasi-chaotic character of STV, Dummett recommends what he calls the "Quota/Borda system"(which combines elements of proportional representation with Borda scores), and Geller recommends STV with Borda elimination.

But before ballots are cast, a butterfly flaps its wings — or, more accurately, two butterflies flap their wings.[3] Before getting to their polling places, two voters slightly modify their preferences with respect to *A* and *B* (both losing candidates under Ballot Profile 1). While both voters initially ranked *A* first and *B* second, they now both rank *B* first and *A* second, but they make no other changes their rankings, and none of the other 999 voters makes any change whatsoever in his or her preferences. The resulting Ballot Profile is shown in Table 2A (with column 2* excluded).

What would we expect the electoral consequence of this slight change in the ballot profile to be? Most likely, that it would have no effect at all on the winning vs. losing status of any of the candidates. But if it were to produce a change, we would most likely expect that *B* (now ranked higher by two voters) would convert from losing to winning status and one of the previously winning candidates *C*, *F*, or *G* would convert to losing status to make room for *B* among the winners.

In fact, this change in the ballot profile does *not* convert *B* from losing to winning status. But, in all other respects, it has a maximally profound impact — that is, *it converts the winning vs. losing status of every other candidate*. (See Table 2B.) *C*, *F*, and *G* now all lose and *D* and *E* now win, despite the fact that no voter has changed his or her ranking of any of these candidates. Moreover, *A* also converts from losing to winning status, despite the fact that the only voters who have changed their preferences moved *A* down in their rankings. The example therefore also illustrates STV's by now well-known 'monotonicity' problem. Note further that the two voters who changed their ballots continue to prefer the now losing *F* and *G* to the now winning *D* and *E*. Indeed, it can be checked that both voters can push *A* down to the very bottom of their ballot ranking (as shown in column 2*, producing Ballot Profile 2*) without affecting the sequence of vote transfers in Table 2B (since *A* is elected before their ballots transfer), in which event they prefer *all* the old winners to *all* the new winners elected as a result of their own slight ballot changes.

The key feature of the example is that candidates *B* and *F* are virtually tied with the fewest first preferences; the flapping (or not) of butterfly wings determines who gets eliminated first. Before the two voters change their preferences, *B* is eliminated and *F* picks up most of the ballots transferred from *B*, thereby surviving second-round elimination also, which sets up a cascade of transfers that leads to the election of *C*, *E*, and *F*. But after the two voters change their preferences, *F* is eliminated at the outset, *F*'s ballots are transferred to (and elect) *A*, and a quite different cascade of transfers leads to the election of a completely different set of winners.

It should be noted that this example is in no way affected by any of the practical problems associated with STV. In particular, (i) no voters cast incomplete ballot rankings (the truncated rankings displayed in Tables 1A and 2A can be completed in any manner without affecting the winning or losing status of any candidates), (ii) accordingly the issue of recalculating the quota does not arise, and (iii) candidates are never tied, so it is always clear who will be eliminated. Moreover, the example completely sidesteps the (variously resolved) issue of how the surplus ballots of elected candidates should be transferred because (it can be checked), when a candidate is elected and surplus ballots transfer, *all* of the candidate's ballots transfer in the same way.

[3] Two butterflies are needed to preclude ties.

The size of the electorate can be increased (or decreased) by adding (or subtracting) any fixed number *k* of voters to (or from) each column in Tables 1A and 1B without affecting the outcome. It is also reasonably evident that additional candidates can be added to the field and incorporated into voter preferences in a manner that preserves that chaotic impact of just two voters changing their preferences.

2. Implications

This example also demonstrates that STV is, in principle, highly susceptible to strategic voting. Suppose that Table 2B, with the modification that the two-voter bloc ranks candidate *A* last (column 2*) rather than second (column 2), is the sincere Ballot Profile 2*. As noted above, this modification does not affect the ballot transfers shown in Table 2B, so candidates *A*, *D*, and *E* are the sincere winners. It therefore follows from this revised example (and with the sequence of the two tables reversed) that (i) the two voters can strategically raise their least preferred candidate *A* to the top of their rankings and thereby defeat *all* the sincere winners *A*, *D*, and *E* and replace them with *C*, *F*, and *G* and that (ii) they prefer all candidates in the second set to all in the first. However, such vulnerability in principle to strategic considerations may yet be deemed invulnerability in practice, because the highly beneficial effect of such a strategic ballot is essentially impossible for the voters to foresee and, in any event, is highly dependent on the exact ballot profile they confront, concerning which they are likely to have at best incomplete knowledge. That is, the quasi-chaotic and non-transparent character of STV may render it effectively strategyproof (cf. Bartholi and Orlin, 1991).

While it is relatively easy to construct ballot profiles illustrating the 'quasi-chaotic' character of STV (or its monotonicity problem), it is not entirely clear what to make of them. It seems evident than such profiles occur occasionally in the real world of STV elections. However, even when they do occur, instances of quasi-chaos (or monotonicity failure) under STV do not directly reveal themselves in the manner of, for example, the 'reversal of winners' problem to which districted plurality electoral systems are subject (exemplified by the 2000 U.S. Presidential election and the 1951 U.K. general election). They certainly are not apparent from election results as normally published (e.g., the first line, perhaps together with the last line, of Tables 1B or 2B) or even from more detailed tabulations showing the sequence of transferred votes (e.g., the complete Tables 1B or 2B). Rather it is necessary to inspect full ballot profiles (e.g., Tables 1A or 2A). Moreover, both quasi-chaos and monotonicity failure pertain, not just to the actual ballot profile, but to the relationship between the actual and possible counterfactual ballot profiles (e.g., between Tables 1A *and* 2A).[4]

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[4] It is unsurprising, therefore, that Bradley (1995), in his 'hands-on assessment of STV' based on his service as the Chief Electoral Officer for Northern Ireland, reports that 'the experience of the use of STV in Northern Ireland over the past 22 years, involving a range of election types and sizes, *reveals no evidence* to support *in practice* the lack of monotonicity' (first emphasis added).

However, one can imagine one circumstance in which the quasi-chaotic character of STV would be revealed — namely, an election dispute that produces a recount of STV ballots. Suppose the ballots are initially read in the manner of Table 1A and counted in the manner of Tables 1B, but that candidate *B* then requests a recount. As a result of the recount, it is determined that in fact two ballots were misread in a manner unfavorable to *B* and that when this error is corrected the ballot profile is that shown in Table 2A and the recount proceeds as in Table 2B. It is reasonable to expect that the results of the recount would provoke considerable consternation.

Dummett (1997, p. 142) says that the 'assessment process of STV cannot be called chaotic in the strict sense, because its 'initial conditions' — the ballot papers submitted by the voters — allow of a precise description and yield a determinate outcome.' The first point is on-the-mark, because voting systems deal in discrete variables (vote counts) whose initial conditions can (in principle) be measured with total precision. In contrast, initial conditions in (strictly) chaotic systems pertain to continuous variables that can never be measured with total precision. (Moreover, such systems evolve continuously through time, in contrast to the discrete rounds of counting and transfers under STV.) Dummett's second point is off-the-mark, however, because chaotic systems are in fact mathematically deterministic — that is, they (like STV vote counting) are well defined and contain no random elements.

Geller (2005) notes one other important contrast between STV and (strictly) chaotic systems. That latter always, or at least typically, exhibit sensitive dependence on initial conditions, while STV pretty clearly exhibits such dependence only atypically. How atypical this dependence is remains an important open question.

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Table 1A — Ballot Profile 1

144	125		160	145	153	126	148
144	27	98	160	145	153	126	148
A	*B*	*B*	*C*	*D*	*E*	*F*	*G*
B	*C*	*F*	*G*	*G*	*C*	*A*	*F*
C	*G*	*A*	*F*	*F*		*B*	*D*
G		*D*		*A*		*C*	*A*
		E		*E*			*E*

Table 1B — Vote Transfers under Ballot Profile 1

	A	*B*	*C*	*D*	*E*	*F*	*G*
(1)	144	[125]	160	145	153	126	148
(2)	[144]	—	187	145	153	224	148
(3)	—	—	331	145	153	224	148
(4)	—	—	**251**	[145]	153	224	228
(5)	—	—	**251**	—	153	224	373
(6)	—	—	**251**	—	153	346	**251**
(7)	—	—	**251**	—	248	**251**	**251**

(1) First preferences on all ballots are tallied. No candidate meets quota, so the weakest candidate is eliminated. By a single ballot, *B* has the fewest votes and is eliminated.

(2) Following the second preferences indicated, 27 of *B*'s ballots transfer to *C* and 98 to *F*. It remains true that no candidate meets quota, so the next weakest candidate is eliminated. Votes having now transferred from *B* to *F*, *A* is the weakest remaining candidate and is eliminated

(3) Given that candidate *B* has been eliminated, all of *A*'s 144 ballots transfer to *C*, who now meets quota and is elected.

(4) Candidate *C*'s surplus ballots transfer on the basis of lower preferences. The highest ranked remaining candidate on all 331 of *C*'s (original and transferred) ballots is *G*, so all 80 surplus ballots transfer to *G*, but *G* still does not meet quota. The weakest candidate *D* is eliminated.

(5) Following the second preference on all of *D*'s ballots, all of *D*'s 145 ballots transfer to *G*, who now meets quota and is elected.

(6) Candidate *G*'s surplus ballots transfer on the basis of lower preferences. The highest ranked remaining candidate on all 373 of *G*'s (original and transferred) ballots is *F*, so all 122 surplus ballots transfer to *F*, who now meets quota and is elected.

(7) Three candidates having been elected, the vote-counting process terminates with candidate *E* (after a final transfer of surplus ballots from *F*) holding the residual 248 "wasted votes."

Table 2A — Ballot Profiles 2 and 2*

142	127				160	145	153	126	148
142	2	2*	27	98	160	145	153	126	148
A	*B*	*B*	*B*	*B*	*C*	*D*	*E*	*F*	*G*
B	*A*	*C*	*C*	*F*	*G*	*G*	*C*	*A*	*F*
C	*C*	*G*	*G*	*A*	*F*	*F*		*B*	*D*
G	*G*	*F*		*D*		*A*		*C*	*A*
		D		*E*		*E*			*E*
		E							
		A							

Table 2B — Vote Transfers under Ballot Profiles 2 and 2*

	A	*B*	*C*	*D*	*E*	*F*	*G*
(1)	142	127	160	145	153	[126]	148
(2)	<u>268</u>	127	160	145	153	—	148
(3)	**251**	[144]	160	145	153	—	148
(4)	**251**	—	206	243	153	—	[148]
(5)	**251**	—	206	<u>391</u>	153	—	—
(6)	**251**	—	206	**251**	<u>293</u>	—	—
(7)	**251**	---	248	**251**	**251**	—	—

(1) First preferences on all ballots are tallied. No candidate meets quota, so the weakest candidate is eliminated. By a single ballot, *F* has the fewest votes and is eliminated.

(2) Following the second preferences indicated, all of *F*'s 126 ballots transfer to *A*, who now meets quota and is elected.

(3) The highest ranked remaining candidate on all 268 of *A*'s (original and transferred) ballots is *B*, so all 17 surplus ballots transfer to *B*. Despite this transfer, *B* remains the weakest remaining candidate and is eliminated.

(4) Given that *A* has been elected and *F* has been eliminated, the highest ranked remaining candidate is C on 46 of *B*'s (original and transferred) ballots and is *D* on 98 of B's (original) ballots, so these votes transfer accordingly. Despite these transfers, no additional candidate meets quota, so the weakest remaining candidate *G* eliminated.

(5) Given that candidate *F* has been eliminated, *G*'s ballots are all transferred on the basis of third preferences to candidate *D*, who now meets quota and is elected.

(6) Given that candidate *A* has been elected and candidates *B*, *F*, and *G* eliminated, the highest available preference on all of *D*'s (original and transferred) ballots is *E*, so the entire surplus of 140 votes transfers to *E*, who accordingly meets quota and is elected.

7) Three candidates having been elected, the vote-counting process terminates, with candidate *C* (after a final transfer of surplus ballots from *E*) holding the residual 248 "wasted votes."