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# Hitting a prime in 2.43 dice rolls (on average)

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## Abstract

What is the number of rolls of fair 6-sided dice until the first time the total sum of all rolls is a prime? We compute the expectation and the variance of this random variable up to an additive error of less than  $10^{-4}$ , showing that the expectation is 2.4284.. and the variance is 6.2427... This is a solution of a puzzle suggested a few years ago by DasGupta in the Bulletin of the IMS, where the published solution is incomplete. The proof is simple, combining a basic dynamic programming algorithm with a quick Matlab computation and basic facts about the distribution of primes.

*Keywords:* *dynamic-programming, prime number theorem, stopping time*

*MSC2020:* *60G40, 90C39, 11A41*

The following puzzle appears in the Bulletin of the Institute of Mathematical Statistics (DasGupta, 2017): Let  $X_1, X_2, \dots$  be independent uniform random variables on the integers  $1, 2, \dots, 6$ , and define  $S_n = X_1 + \dots + X_n$  for  $n = 1, 2, \dots$ . Denote by  $\tau$  the discrete time in which  $S_n$  first hits the set of prime numbers  $P$ :

$$\tau = \min \{n \geq 1 : S_n \in P\}.$$

The contributing Editor (DasGupta, 2017) provides a lower bound of 2.3479 for the expectation  $E(\tau)$  and mentions the following heuristic approximation for it:  $E(\tau) \approx 7.6$ . He also adds that it is not known whether or not  $\tau$  has finite variance.

In this note we compute the value of  $E(\tau)$  up to an additive error of less than  $10^{-7}$ , showing that it is much closer to the lower bound mentioned above than to 7.6. We also show that the variance is finite and compute its value up to an additive error of less than  $10^{-4}$ . It will be clear from the discussion that it is not difficult to get better approximation for both

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quantities by increasing the amount of computation performed. The proof is very simple, and the note is written mainly in order to illustrate the power of the combination of a simple dynamic programming algorithm combined with a quick computer-aided computation and basic facts about the distribution of primes in the study of problems of this type.

Before describing the rigorous argument, we present, in Table 1 below, the outcomes of Monte-Carlo simulations of the process.

number of repetitions	$mean(\tau)$	$variance(\tau)$	$\max(\tau)$
$10^6$	2.4316	6.2735	49
$2 * 10^6$	2.4274	6.2572	67
$3 * 10^6$	2.4305	6.2372	70
$5 * 10^6$	2.4287	6.2418	64
$10^7$	2.4286	6.2463	65

Table 1: Monte-Carlo simulations

We proceed with a rigorous computation of  $E(\tau)$  and  $Var(\tau)$  up to an additive error smaller than  $1/10,000$ . Not surprisingly, this computation shows that the simulations supply accurate values.

Note first that

$$E(\tau) = \sum_{k \geq 1} P(\tau \geq k), \quad E(\tau^2) = \sum_{k \geq 1} (2k - 1)P(\tau \geq k). \quad (1)$$

We next apply dynamic programming to compute  $p(k) \equiv P(\tau \geq k)$  exactly for every  $k$  up to 1000, and then provide an upper bound for the sums of the remaining terms.

### Dynamic-Programming Algorithm

For each integer  $k \geq 1$  and each non-prime  $n$  satisfying  $k \leq n \leq 6k$ , let  $p(k, n)$  denote the probability that  $X_1 + \dots + X_k = n$  and that for every  $i < k$ ,  $X_1 + \dots + X_i$  is non prime. Fix a parameter  $K$  (in our computation we later take  $K = 1000$ .) By the definition of  $p(k, n)$  and the rule of total probability we have the following dynamic programming algorithm for computing  $p(k, n)$  precisely for all  $1 \leq k \leq K$  and  $k \leq n \leq 6k$ .

1.  $p(1, 1) = p(1, 4) = p(1, 6) = 1/6$ .
2. For  $k = 2, \dots, K$  and any non-prime  $n$  between  $k$  and  $6k$

$$p(k, n) = \frac{1}{6} \sum_i p(k-1, n-i),$$

where the sum ranges over all  $i$  between 1 and 6 so that  $n - i$  is non-prime.

Denote by  $E_K$  and  $Var_K$  the estimators (lower bounds) of  $E(\tau)$  and  $Var(\tau)$  based on the values of  $p(k)$  for the first  $K$  values, which we obtain as follows:

$$E(\tau) = E_K + R_K, \quad E(\tau^2) = E_K^{(2)} + R_K^{(2)},$$

where  $E_K = \sum_{k=1}^K p(k)$ ,  $R_K = \sum_{k \geq K+1} p(k)$ ,  $E_K^{(2)} = \sum_{k=1}^K (2k-1)p(k)$ ,  $R_K^{(2)} = \sum_{k \geq K+1} (2k-1)p(k)$ . We also have

$$Var(\tau) = Var_K + RV_K,$$

where  $Var_K = E_K^{(2)} - (E_K)^2$ ,  $RV_K = R_K^{(2)} - 2E_K R_K - (R_K)^2$ .

Applying the dynamic-programming algorithm in Matlab, with an execution time of less than 5 seconds, we obtain

$$E_{1000} = 2.428497913693504, \quad Var_{1000} = 6.242778668279075.$$

### Bounding the remainders

It remains to show that each of the sums of the remaining terms is bounded by  $10^{-4}$ . To this end, we prove the following simple result by induction on  $k$ .

**Proposition 1.** *For every  $k$  and every non-prime  $n$ ,*

$$p(k, n) < \frac{1}{3} \left( \frac{5}{6} \right)^{\pi(n)}, \quad (2)$$

where  $\pi(n)$  is the number of primes smaller than  $n$ .

*Proof.* Note first that (2) holds for  $k = 1$ , as  $1/6 = p(1, 6) < (1/3)(5/6)^3$ ,  $1/6 = p(1, 4) < (1/3)(5/6)^2$  and  $1/6 = p(1, 1) < (1/3)(5/6)^0$ , with room to spare. Assuming the inequality holds for  $k - 1$  (and every relevant  $n$ ) we prove it for  $k$ . Suppose there are  $q$  primes in the set  $\{n - 6, \dots, n - 1\}$ , then  $\pi(n - i) \geq \pi(n) - q$  for all non-prime  $n - i$  in this set. Thus, by the induction hypothesis

$$p(k, n) \leq \frac{1}{6}(6 - q) \frac{1}{3} \left( \frac{5}{6} \right)^{\pi(n) - q} \leq \left( \frac{5}{6} \right)^q \frac{1}{3} \left( \frac{5}{6} \right)^{\pi(n) - q} = \frac{1}{3} \left( \frac{5}{6} \right)^{\pi(n)}.$$

□

By the prime number theorem (cf., e.g., Hardy and Wright (2008)), for every  $n > 1000$   $\pi(n) > 0.9 \frac{n}{\ln n}$  (again, with room to spare). Therefore, by the above estimate we get:

**Corollary 1.** For every  $k > 1000$  and every non-prime  $n (n \geq k)$ ,

$$p(k, n) < \frac{1}{3} \left( \frac{5}{6} \right)^{0.9 \frac{n}{\ln n}}. \quad (3)$$

Using the above, we can now bound the sums of the remaining terms.

$$\begin{aligned} R_{1000} &\equiv \sum_{k > 1000} P(\tau \geq k) = \sum_{k > 1000} \sum_{\{n: k \leq n \leq 6k\}} p(k, n) \\ &< \sum_{k > 1000} \sum_{\{n: k \leq n \leq 6k\}} \frac{1}{3} \left( \frac{5}{6} \right)^{0.9n/\ln n} = \sum_{n \geq 1001} \sum_{k=\max(1001, n/6)}^n \frac{1}{3} \left( \frac{5}{6} \right)^{0.9n/\ln n} \\ &< \sum_{n \geq 1000} (n - 1000) \frac{1}{3} \left( \frac{5}{6} \right)^{0.9n/\ln n}, \end{aligned}$$

where the first inequality is obtained from Corollary 1.

Define  $f(n) = (n - 1000) \frac{1}{3} \left( \frac{5}{6} \right)^{0.9n/\ln n}$ , where  $n$  is an integer  $\geq 1000$ . It is easy to check that for  $n \geq 1000$  the function  $f(n)$  has a unique maximum at  $n = 1050$ . (To see it, it suffices to compute  $f(n)$  precisely for all  $1000 \leq n \leq 1100$  and observe that for  $n > 1100$   $f(n)$  is far smaller than  $f(1050)$ .) It is also easy to check that for any  $n \geq 1050$   $f(n + 13 \ln n)/f(n) < 1/2$ . Therefore,

$$\begin{aligned} R_{1000} &< \sum_{n \geq 1000} (n - 1000) \frac{1}{3} \left( \frac{5}{6} \right)^{0.9n/\ln n} = \sum_{n \geq 1000} f(n) \\ &< 50f(1050) + 2(13 \ln 1050)f(1050) < 7 \cdot 10^{-8}. \end{aligned} \quad (4)$$

Similarly

$$\begin{aligned} R_{1000}^{(2)} &\equiv \sum_{k > 1000} (2k - 1)P(\tau \geq k) = \sum_{k > 1000} \sum_{\{n: k \leq n \leq 6k\}} (2k - 1)p(k, n) \\ &< \sum_{k > 1000} \sum_{\{n: k \leq n \leq 6k\}} (2k - 1) \left( \frac{5}{6} \right)^{0.9n/\ln n} \leq \sum_{n \geq 1001} \frac{1}{3} \left( \frac{5}{6} \right)^{0.9n/\ln n} \sum_{k=\max(1001, n/6)}^n (2k - 1) \\ &\leq \sum_{n \geq 1000} \frac{1}{3} \left( \frac{5}{6} \right)^{0.9n/\ln n} (n^2 - 1000^2), \end{aligned}$$

where the first inequality is obtained from Corollary 1.

Denote by  $g(n) = (n^2 - 1000^2) \frac{1}{3} \left( \frac{5}{6} \right)^{0.9n/\ln n}$ , where  $n$  is an integer  $\geq 1000$ . For  $n \geq 1000$  the function  $g(n)$  has an unique maximum at  $n = 1051$  and also for any  $n \geq 1051$   $g(n + 13 \ln n)/g(n) < 1/2$ . Therefore,