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ABSTRACT

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Efficient multigrid methods for optimal control of partial differential
equations

by

Mona Hajghassem

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0.1 Introduction

This thesis is focused on efficient methods for solving optimal control problems constrained by partial differential equations (PDEs). Research in this area started with the work of Lions in the 1960s [1]. Several research studies have been devoted to this subject in recent years [2–5]. One important application of PDE-constrained optimization problems is parameter identification. Many quantitative phenomena in various fields such as engineering, physics, and biological science can be modeled using PDEs. All involved parameters are assumed to be known when running simulation based on a PDE model. However, identifying accurate quantitative values for all parameters is challenging, and this may lead to erroneous predictions. One way of obtaining better estimates for the parameters is by formulating a PDE-constrained optimization problem, in which the parameters to be identified are the optimization variables, and the cost functional measures the discrepancy between the measurements and predictions. Other applications of PDE-constrained optimization include medical applications (optical tomography, radiation therapy) [6,7], economical (rate, pricing) [8–10], and engineering (shape optimization, optimal design of cooling process, minimize harmful byproducts of chemical reactions) [11].

In spite of the remarkable developments in the high performance, parallel computing technologies, which allow us to conduct numerical computations at unprecedented resolutions, solving large-scale PDE-constrained optimization problems still requires significant advances in algorithmic development. For example, the implementation of the four-dimensional variational data assimilation (4D-Var) requires solving a time dependent PDE with thousands of time steps to obtain the value of the cost functional. Due to significant computer processing requirements for solving such problem, and given existing and emerging computing capabilities, development of novel efficient large-scale optimization algorithms is still necessary [12,13].

The multigrid paradigm is that by using multiple discretizations of the same problem (for example, multiple grids/meshes or polynomial degrees) one can speed up the numerical solution process to the point where it becomes optimal in a certain sense. Multigrid originated in solving linear systems representing discretizations of elliptic equations, and since then has been extended to many classes of PDEs, including the Stokes and Navier-Stokes equations. Multigrid methods for PDEs are different from the multigrid methods for the problems of interest. Multigrid methods have been used for PDE-constrained optimization problems beginning with the work of Hackbusch [14,15]. The methods proposed in this project are related to those developed by Rieder [16], Hanke and Vogel [17], Akcelik *et al.* [18], Biros and Dogan [19], and Draganescu and Dupont [20]. Related methods are found in the works of King [21] and Kaltenbacher [22], both works being applicable to more generic inverse problems, while [14,16,17] are for more specific integral equations, and [18,19] are for PDE-constrained optimization problems. The results in this thesis push the boundaries of the applicability of the multigrid strategy, by showing that it can be applied efficiently to two classes of problems: distributed optimal control of linear parabolic equations, and boundary control of elliptic equations. Ultimately, it is shown that the algorithms developed and analyzed lead to a solution

process whose computational cost, at high resolution, is a relatively small multiple of the cost of solving the PDE itself.

0.2 General formulation and description of main contribution

Consider a general form of a PDE-constrained optimization problem

$$\min J(y, u) \tag{1}$$

$$\text{subject to } e(y, u) = 0, \tag{2}$$

$$y \in Y_{ad} \subseteq Y, u \in U_{ad} \subseteq U,$$

where $e(y, u) = 0$ is a well-posed PDE in the sense that for each valid u there exists a unique $y = y(u)$ for which $e(y, u) = 0$, and y depends continuously on u . We refer to y as the state and u as the control. If $Y_{ad} \neq Y$, respectively $U_{ad} \neq U$ we say that (1)-(2) has state constraints, respectively control constraints. The cost functional is oftentimes quadratic and of tracking type and includes regularization. An example of a cost functional is as follows:

$$J(y, u) = \frac{1}{2} \|y - y_d\|_{L^2(\Omega)} + \frac{\beta}{2} \|u\|_{L^2(\Omega)}, \tag{3}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain and β is the regularization parameter. PDEs can be linear or nonlinear elliptic (Poisson, elasticity, Stokes) [23–25], parabolic (heat equation) [26, 27], fluid flows, magneto hydrodynamics (MHD) [28, 29], etc. The controls can be boundary values, initial values, or forcing terms (distributed). Current research directions include existence, uniqueness, and regularity of optimal controls, discretization and error estimation, and solvers. This thesis is concerned with the latter.

The primary focus in solving PDE-constrained optimization problems is the KKT system (named after the Karush-Kuhn-Tucker optimality conditions). This can be a linear or a nonlinear, possibly non-smooth large-scale system of equations. In the absence of inequality constraints, the KKT is a smooth nonlinear system. In the particular case of linear-quadratic problems (PDE is linear, cost functional is quadratic) the KKT is a linear system. The main technique for solving PDE constrained optimization problems is by solving the associated KKT systems. There are two main directions in solving a KKT system numerically. One method is to discretize the PDE constraints, then solve the resulting discrete optimization problem (first-discretize-then-optimize). The second method is to discretize the continuous system representing the first order optimality conditions (the KKT system), then discretize it appropriately (first-optimize-then-discretize). The first approach is more attractive computationally than the second approach, since one is faced with solving an actual finite dimensional optimization problem, while the second one is more convenient for convergence analysis, but the system one has to solve may not exactly represent the first order optimality conditions of a discrete optimization problem. Regardless of whether the first or the second method is chosen, one has

to solve numerically a sparse, potentially nonlinear and non-smooth system (the discrete KKT system) whose linearization is an indefinite operator. Two further approaches are common in both methods: (i) solve the linear KKT system using a multigrid iteration or preconditioned MINRES; or (ii) eliminate the variables causing indefiniteness, and solve the resulting reduced system. The advantages of (i) is that the system is sparse, but it is difficult to solve and precondition due to its indefiniteness; this is the approach found in the work of Wathen and collaborators (e.g., [30, 31]), and of Borzi and collaborators. In the approach (ii), the matrix of the system is usually dense, however, it is positive definite. Since for (ii) each matrix-vector multiplication usually involves solving the equivalent of two linearized PDE solves, this approach is viable only if very good preconditioners are available. The question of when each approach is more advantageous than the other remains open. In this thesis we work with the reduced system. The thesis is centered around devising preconditioners rooted in the multigrid approach developed in Draganescu and Dupont [20] for an inverse problem.

This work is primarily concerned with linear-quadratic problems where the cost-function is of tracking type. For the first problem discussed in Chapter 1, the PDE constraint is a linear parabolic equation and the control is distributed, while for the second problem discussed in Chapters 2 and 3, the PDE constraint is an elliptic equation and the control is on the boundary. We impose no inequality constraints. Both problems are formulated in reduced form and lead to the problem of solving large-scale linear systems. While these linear systems can be solved using Krylov space methods, a critical aspect that needs to be addressed is preconditioning. Systems are never formed (this is the so-called matrix-free approach). Each matrix-vector multiplication (Mat-vec) involves two PDE solves, making it very expensive. Therefore, we need highly-efficient preconditioners. Systems are similar in character (they resemble an integral equation of the second kind) and similar to the system arising in the distributed optimal control of elliptic equations. The design of the preconditioners follows a recipe that proved to be successful in the distributed elliptic case. However, for the first problem (distributed optimal control problem constrained by a linear parabolic equation), standard space-time finite element discretizations (e.g., Crank-Nicolson discretization) lead to suboptimal results. For the boundary control of elliptic equations we noticed numerically an important distinction between Dirichlet and Neumann boundary control. We observed optimal order results for Neumann boundary control problem and suboptimal results for Dirichlet boundary control problem. The main contribution of this thesis for the first problem is to point out a discretization that leads to optimal order preconditioners, and for the second problem is to provide analysis.

Chapter 1: Multigrid preconditioning for space-time distributed optimal control of parabolic equations

1.1 Introduction and literature review

In this chapter, we devise efficient methods for solving large-scale optimization problems constrained by linear parabolic equations. There are real life problems such as data assimilation for fluid flows with application to atmospheric and ocean modeling which lead to a large-scale optimization problems constrained by time-dependent PDEs. Hence, finding an efficient method for solving time-dependent PDE constrained optimization problems help solve such real-life problems. A particular challenge specific to time-dependent PDE-constrained optimization problems is related to the PDE solving technique itself; namely, when solving large-scale time-dependent PDEs, at any stage we access a limited number of snap-shots in time because of memory limitations. Since most time-stepping methods require the access to a few past states when computing the next state, this is enough to solve the time dependent PDEs numerically. However for large scale optimization problems constrained by time dependent PDEs, accessing to the entire system may not be possible at the same time. Hence, there may be advantages to treat such optimiza-

tion problems in reduced form, by eliminating states and Lagrange multipliers from the KKT system. Gradients with respect to the remaining independent variables (the controls) are computed by solving the adjoint equation, a task that is nontrivial in itself due to the need to access the entire space-time state of the PDE during the computation of the gradient. In particular, this task has given rise to an entire set of algorithms revolving around checkpointing, where certain computations are repeated to avoid expensive data accessing and storage. While secant methods such as L-BFGS rely only on gradients for approximating the action of the Hessian or its inverse, for large-scale problems they tend to converge too slowly, and this is mainly due to the fact that the number of gradients needed for approximating the action of the (very large) Hessian with sufficient accuracy would have to be too large. For certain classes of problems including those arising from 4D-variational data assimilation (4D-Var), other suboptimal approximations of the Hessian are used which are only based on first-order information [12].

In this work we develop optimal order preconditioners, which, for the optimization problems under scrutiny lead to a low number of linear iterations in the solution process. In fact, given that the equations involved in the reduced system are integral in character (the solution operator of the PDE, which is of integral type, is explicitly involved in the reduced KKT system), the number of multigrid iterations decrease with increasing resolution or with decreasing mesh size, unlike the case of multigrid methods for differential equations, where the number of iterations is expected to be bounded with respect to mesh size.

In this chapter we focus on the distributed control of linear parabolic equations,

that is, the space-time control appears as a forcing term of parabolic equation.

It is notable that standard discretization of the parabolic equation does not give rise to optimal order preconditioners. In fact, as numerical results show, we see a drop in the approximation order by one half both for backward Euler and Crank-Nicolson. However, if we use a non-standard discretization based on discontinuous-in-time-continuous-in-space finite elements as presented by Leykekhman and Vexler in [32], it turns out that the preconditioner is also of optimal order. In addition to recognizing the impact of the discretization on the design of preconditioners, the main contribution of this work is the analysis certifying the optimality of the preconditioner.

1.2 Problem description

Consider a convex polygonal or polyhedral open bounded set $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and Y, U be subspaces of

$$L^2((0, T), L^2(\Omega)) = \{v(x, t) : \Omega \times (0, T) \rightarrow \mathbb{R}; \int_0^T \|v(\cdot, t)\|_{L^2(\Omega)}^2 dt < \infty\},$$

with norm

$$\|v\|_{L^2((0, T), L^2(\Omega))} = \left(\int_0^T \|v(\cdot, t)\|_{L^2(\Omega)}^2 dt \right)^{\frac{1}{2}},$$

where $T > 0$ fixed. Let $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. The problem studied in this chapter is:

$$\min_{(y,u) \in (Y,U)} J(y, u) := \frac{1}{2} \int_Q |y(x, t) - y_d(x, t)|^2 dxdt + \frac{\beta}{2} \int_Q |u(x, t)|^2 dxdt \quad (1.1)$$

$$\text{subject to : } \begin{cases} y_t - \Delta y = u & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(x, 0) = 0 & \text{in } \Omega. \end{cases} \quad (1.2)$$

We call (1.1) a tracking-type cost functional. We refer to y as the state and u as the control. A significant literature [33–38] is devoted to questions regarding existence, uniqueness, and regularity of solutions to the optimal control problem (1.1)-(1.2), as well as its discretization and error estimation. In this section we briefly discuss the precise formulation of the optimization problem under consideration. We focus on heat equation for the sake of clarity, however we expect the results to hold for more general parabolic equations. Furthermore, we recall theoretical results on existence, uniqueness, and regularity of optimal solutions as well as optimality conditions.

1.2.1 The control-to-state map and reduced problem

As stated above, in our attempt to extend ideas from distributed optimal control of elliptic problems to develop multigrid preconditioners for the reduced Hessian, the preliminary numerical results show a rather surprising behavior: when

using standard discretization such as Crank Nicolson or backward Euler, we observe a degrading by precisely half and order of the expected optimality of the preconditioners. Following [32] we then used a continuous-in-space-discontinuous-in-time Galerkin discretization and we obtained the desired optimality.

To set up a weak formulation of the state equation, we introduce the following notation: We denote V to be $H_0^1(\Omega)$ and $\tilde{V} = H^1(\Omega)$. For a time interval $I = (0, T)$ we introduce the state and control spaces

$$Y := \{v | v \in L^2(I, V) \text{ and } \partial_t v \in L^2(I, V^*)\},$$

$$U := L^2(I, L^2(\Omega)).$$

We use the following notations for the inner product and norms on $L^2(\Omega)$:

$$(\cdot, \cdot) := (\cdot, \cdot)_U, \text{ and } \|\cdot\| = \|\cdot\|_U.$$

In this setting, a standard weak formulation of the state equation for a given control $u \in U$ is: find a state $y \in Y$ satisfying

$$\begin{cases} (\partial_t y, \phi) + (\nabla y, \nabla \phi) = (u, \phi) \quad \forall \phi \in V \\ y(\cdot, 0) = 0. \end{cases} \quad (1.3)$$

For this formulation of the state equation, we recall the following result on existence and regularity that appears as Theorem 3.1 in [39]:

Theorem 1.2.1 *For fixed control $u \in U$, there exists a continuous linear operator*

$\mathcal{S} : U \rightarrow Y$ such that $y = \mathcal{S}u$ solves (1.3) and the following stability estimates hold.

$$\|\mathcal{S}u\| \leq C\|u\|, \quad (1.4)$$

$$\|\partial_t y\| + \|\Delta y\| \leq C\|u\|, \quad (1.5)$$

where the first inequality just expresses continuity, and the second expresses maximal parabolic regularity.

The following theorem appears as Theorem. 3.16 in [2] proves the solvability of the optimal control problem (1.1)-(1.2).

Theorem 1.2.2 *For given $y_d \in U$ and $\beta > 0$ the optimal control problem admits a unique solution $(\bar{u}, \bar{y}) \in U \times Y$. The optimal control \bar{u} possesses the regularity*

$$\bar{u} \in L^2(I, H^2(\Omega)) \cap H^1(I, L^2(\Omega)).$$

Moreover, the optimal solution satisfies the KKT system associated with the optimization problem (1.1)-(1.2)

$$\begin{aligned} y_t - \Delta y &= u \quad \text{in } Q & -z_t - \Delta z &= y - y_d \quad \text{in } Q \\ y &= 0 \quad \text{on } \Sigma & z &= 0 \quad \text{on } \Sigma \\ y(x, 0) &= 0 \quad \text{in } \Omega & z(x, T) &= 0 \quad \text{in } \Omega. \end{aligned} \quad (1.6)$$

As stated at the beginning of this section, our approach relies on reduced system rather than solving the KKT system (1.6). The existence result for the

state equation in Theorem 1.2.1 ensures the existence of a control-to-state mapping $u \rightarrow y = \mathcal{S}u$ given by (1.3). By means of this mapping we introduce the reduced cost functional $J : U \rightarrow \mathbb{R}$:

$$\hat{J}(u) := J(u, \mathcal{S}u).$$

The reduced optimal control problem can then be equivalently reformulated as

$$\min_{u \in U} \hat{J}(u). \tag{1.7}$$

The first order necessary optimality condition for (1.7) reads

$$\hat{J}'(\bar{u})(\delta u) = (\nabla \hat{J}(\bar{u}), \delta u) = 0, \quad \forall \delta u \in U. \tag{1.8}$$

Due to the linear-quadratic structure of the optimal control problem this condition is also sufficient for optimality. Since the reduced cost functional is

$$\hat{J}(u) := \frac{1}{2} \|\mathcal{S}u - y_d\|^2 + \frac{\beta}{2} \|u\|^2,$$

the Fréchet derivative of it can be expressed as

$$\hat{J}'(u)(\delta u) = (\beta u + \mathcal{S}^*(\mathcal{S}u - y_d), \delta u) \tag{1.9}$$

where $\mathcal{S}^* : U \rightarrow U$ is the adjoint of \mathcal{S} , that is

$$(\mathcal{S}^*u, v) = (u, \mathcal{S}v) \quad \forall u, v \in U. \tag{1.10}$$

Therefore, by (1.9),

$$\nabla \hat{J}(u) = \beta u + \mathcal{S}^*(\mathcal{S}u - y_d). \quad (1.11)$$

Theorem 1.2.3 *For $v \in U$, we have $z = \mathcal{S}^*v$ if and only if z satisfies*

$$\begin{cases} -(\phi, \partial_t z) + (\nabla \phi, \nabla z) = (\phi, v), \quad \forall \phi \in V, \\ z(\cdot, T) = 0. \end{cases} \quad (1.12)$$

Proof. Let z be the solution of (1.12). Recall (1.3) that

$$(\partial_t y, \phi)_{L^2(\Omega)} + (\nabla y, \nabla \phi)_{L^2(\Omega)} = (u, \phi)_{L^2(\Omega)} \quad \forall \phi \in V.$$

Replace ϕ in (1.12) with y in (1.3), then we obtain

$$-(y, \partial_t z)_{L^2(\Omega)} + (\nabla y, \nabla z)_{L^2(\Omega)} = (y, v)_{L^2(\Omega)}.$$

Since $y(\cdot, 0) = 0$ and $z(\cdot, T) = 0$,

$$\begin{aligned} 0 &= \int_0^T \frac{\partial}{\partial t} (y, z)_{L^2(\Omega)} dt \\ &= \int_0^T ((\partial_t y, z)_{L^2(\Omega)} + (y, \partial_t z)_{L^2(\Omega)}) dt \\ &= \int_0^T ((\nabla y, \nabla z)_{L^2(\Omega)} - (y, v)_{L^2(\Omega)} + (u, z)_{L^2(\Omega)} - (\nabla y, \nabla z)_{L^2(\Omega)}) dt \\ &= \int_0^T (-(y, v)_{L^2(\Omega)} + (u, z)_{L^2(\Omega)}), \\ &= -(y, z) + (u, z), \end{aligned}$$

hence

$$(\mathcal{S}u, v) = (u, \mathcal{S}^*v).$$

■

We note that by defining $\tilde{z}(x, t) = z(x, T - t)$, where z satisfies (1.12), \tilde{z} satisfies

$$\begin{cases} (\phi, \partial_t \tilde{z}(\cdot, T - t)) + (\nabla \phi, \nabla \tilde{z}(\cdot, T - t)) = (\phi, v(\cdot, T - t)), \forall \phi \in V, \\ \tilde{z}(\cdot, 0) = 0, \end{cases} \quad (1.13)$$

which is a parabolic equation with the right hand side \tilde{v} with $\tilde{v}(x, t) = v(\cdot, T - t)$.

Therefore, by Theorem 1.2.1

$$\|\partial_t \tilde{z}\| + \|\Delta \tilde{z}\| \leq C \|\tilde{v}\|, \quad (1.14)$$

which is equivalent to

$$\|\partial_t z\| + \|\Delta z\| \leq C \|v\|. \quad (1.15)$$

The Hessian operator at $u \in U$ is given, in general, by the following equality involving the second variation

$$(\mathcal{H}(u)v_1, v_2) = \hat{J}''(u)(v_1, v_2), \quad \forall v_1, v_2 \in U.$$

Since \hat{J} is quadratic, we have

$$\hat{J}''(u)(v_1, v_2) = ((\mathcal{S}^* \mathcal{S} + \beta I)v_1, v_2), \quad \forall v_1, v_2 \in U.$$

It follows that

$$\mathcal{H} = \mathcal{S}^* \mathcal{S} + \beta I. \quad (1.16)$$

We note that the Hessian \mathcal{H} is independent of u and is a symmetric positive definite operator because

$$(\mathcal{H}v, v) \geq \beta(v, v) \quad \forall v \in U.$$

The optimality condition (1.8) is equivalent, due to (1.11), to

$$\mathcal{H}\bar{u} = (\mathcal{S}^* \mathcal{S} + \beta I)\bar{u} = \mathcal{S}^* y_d, \quad \text{with } \bar{y} = \mathcal{S}\bar{u}. \quad (1.17)$$

Hence, the reduced problem has a unique solution \bar{u} given by

$$\bar{u} = \mathcal{H}^{-1}(\mathcal{S}^* y_d). \quad (1.18)$$

The challenge of this approach is to find efficient solution methods for solving the linear systems representing discrete versions of (1.17). To solve the discrete optimization problem we will introduce special discrete versions of \mathcal{S} , \mathcal{S}^* , and \mathcal{H} , and

we will define and analyze two-grid preconditioners for the discrete version of \mathcal{H} .

1.2.2 The discrete control-to-state map and the discrete reduced problem

The discussion in this section follows closely [32]. For $h \in (0, h_0](h_0 > 0)$, let \mathcal{T} denote a quasi-uniform triangulation of Ω with mesh size h , i.e., $\mathcal{T} = \{\tau\}$ is a partition of Ω into cells (triangles) τ of diameter h_τ such that for $h = \max_\tau h_\tau$,

$$\text{diam}(\tau) \leq h \leq C|\tau|^{\frac{1}{d}}, \quad \forall \tau \in \mathcal{T},$$

hold. Let \tilde{V}_h be the space of continuous piecewise linear functions with respect to \mathcal{T} , and $V_h = \tilde{V}_h \cap H_0^1(\Omega)$. We introduce the following notation: $I = (0, T)$, $I_m = (t_{m-1}, t_m]$, $k_m = t_m - t_{m-1}$, $0 = t_0 < t_1 < \dots < t_{M-1} < t_M = T$. To obtain the fully discrete approximation we consider the space-time finite element space

$$Y_{k,h}^{0,1} = \{y_{kh} : y_{kh}|_{I_m} \in V_h, m = 1, 2, \dots, M\}, \quad (1.19)$$

$$U_{k,h}^{0,1} = \{u_{kh} : u_{kh}|_{I_m} \in \tilde{V}_h, m = 1, 2, \dots, M\}. \quad (1.20)$$

More precisely, $Y_{k,h}^{0,1}$ and $U_{k,h}^{0,1}$ consist of functions that are piecewise constant in time with respect to the partition I_m and continuous piecewise linear in Ω with respect

to \mathcal{T} . We also define the following:

$$[y_{kh}]_m = y_{kh,m}^+ - y_{kh,m}^-,$$

$$y_{kh,m}^+ = \lim_{\epsilon \rightarrow 0^+} y_{kh}(t_m + \epsilon),$$

$$y_{kh,m}^- = \lim_{\epsilon \rightarrow 0^+} y_{kh}(t_m - \epsilon).$$

We define the following bilinear form

$$B(y_{kh}, \phi_{kh}) = (\nabla y_{kh}, \nabla \phi_{kh}) + \sum_{m=2}^M ([y_{kh}]_{m-1}, \phi_{kh,m-1}^+)_{L^2(\Omega)}. \quad (1.21)$$

By rearranging the terms in (1.21), we obtain an equivalent (dual) expression of B :

$$\begin{aligned} B(y_{kh}, \phi_{kh}) &= (\nabla y_{kh}, \nabla \phi_{kh}) + \sum_{m=1}^M \left((y_{kh,m}^-, \phi_{kh,m}^-)_{L^2(\Omega)} - (y_{kh,m-1}^+, \phi_{kh,m-1}^+)_{L^2(\Omega)} \right) \\ &+ \sum_{m=2}^M \left((y_{kh,m-1}^+, \phi_{kh,m-1}^+)_{L^2(\Omega)} - (y_{kh,m-1}^-, \phi_{kh,m-1}^-)_{L^2(\Omega)} \right) + (y_{kh,0}^+, \phi_{kh,0}^+). \end{aligned}$$

By adding like terms we obtain:

$$B(y_{kh}, \phi_{kh}) = (\nabla y_{kh}, \nabla \phi_{kh}) - \sum_{m=1}^{M-1} (y_{kh,m}^-, [\phi_{kh}]_m)_{L^2(\Omega)} + (y_{kh,M}^-, \phi_{kh,M}^-)_{L^2(\Omega)}. \quad (1.22)$$

Following [32], the dG(0)cG(1) discretization of the state equation (1.2) for given

$u \in U_{k,h}^{0,1}$ has the form: Find a state $y_{kh} = \mathcal{S}_{kh}(u) \in Y_{k,h}^{0,1}$ such that

$$B(y_{kh}, \phi_{kh}) = (u, \phi_{kh}) \quad \forall \phi_{kh} \in Y_{k,h}^{0,1}. \quad (1.23)$$

We note that the test space is the same as the solution space and the method reduces to a modified backward Euler method [40]. It is noted that there exist a variety of space-time finite element formulations for the discrete parabolic equation. For example, in [27] a certain formulation based on continuous finite elements in space and time gives rise to the standard Crank-Nicolson Galerkin method. To construct the matrix equation corresponding to (1.23), we introduce the nodal basis $\{\phi_j\}$ of V_h associated to the N_h interior nodes of \mathcal{T} numbered in some fashion, hence

$$y|_{I_m} = \sum_{j=1}^{N_h} y_j^m \phi_j,$$

where the coefficient y_j^m are the nodal values of $y|_{I_m}$. Let ϕ_j with $j = N_h + 1, \dots, \tilde{N}_h$, be basis functions associated with the boundary nodes. Hence

$$u|_{I_m} = \sum_{j=1}^{\tilde{N}_h} u_j^m \phi_j,$$

where the coefficients u_j^m are the nodal values of $u|_{I_m}$. We denote $y_m = (y_j^m)$ to be the vector of coefficients. We define $N_h \times N_h$ mass matrix M , stiffness matrix K , and the $N_h \times 1$ vector b by

$$(M)_{ij} = (\phi_j, \phi_i)_{L^2(\Omega)}, (K)_{ij} = (\nabla \phi_j, \nabla \phi_i)_{L^2(\Omega)}, (b)_i = (u, \phi_i),$$

for $1 \leq i, j \leq N_h$. We also define $\tilde{N}_h \times \tilde{N}_h$ mass matrix \tilde{M} with coefficients $(\tilde{M})_{ij} = (\phi_j, \phi_i)_{L^2(\Omega)}$, for $1 \leq i, j \leq \tilde{N}_h$. Since $y_m = y_{m-1}^+$ and $y_{m-1} = y_{m-1}^-$, the discrete

equation for the dG(0)cG(1) approximation on I_m is:

$$(M + k_m K)y_m = My_{m-1} + \tilde{M}u_m, \quad (1.24)$$

where $u_m = (u_j^m)_{j=1, \dots, \tilde{N}_h}$. Define the following matrix

$$A = \begin{bmatrix} M + k_1 K & 0 & 0 & \dots & 0 \\ -M & M + k_2 K & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & -M & M + k_M K \end{bmatrix}.$$

The above equation can be written as

$$Ay = Cu, \quad (1.25)$$

where C is a matrix that contains \tilde{M} as diagonal blocks.

The following theorem appears in [41].

Theorem 1.2.4 *Let $u \in U_{k,h}^{0,1}$, the discrete problem*

$$B(y_{kh}, \phi_{kh}) = (u_{kh}, \phi_{kh}) \quad \forall \phi_{kh} \in Y_{k,h}^{0,1}, \quad (1.26)$$

has a unique solution $y_{kh} \in Y_{k,h}^{0,1}$.

The existence result for the discrete problem in Theorem 1.2.4 ensures the existence of a control-to-state mapping $u \rightarrow y = \mathcal{S}_{kh}(u)$ defined in (1.23). By means

of this mapping we introduce the discrete reduced cost functional $\hat{J}_{kh} : U_{k,h}^{0,1} \rightarrow \mathbb{R}$

$$\hat{J}_{kh}(u) := J(u, y_{kh}(u)).$$

Hence the optimal control problem can be reformulated as

$$\min_{u_{kh} \in U_{k,h}^{0,1}} \hat{J}_{kh}(u_{kh}). \quad (1.27)$$

The following result appears as Remark 3.5 in [35], ensures the solvability of the discrete optimal control problem (1.27).

Theorem 1.2.5 *The discrete optimal control problem (1.27) admits for $\beta > 0$ a unique solution $u_{kh} \in U_{k,h}^{0,1}$.*

The uniquely determined optimal solution of (1.27) is denoted by $\bar{u}_{kh} \in U_{k,h}^{0,1}$.

The optimal control $\bar{u}_{kh} \in U_{k,h}^{0,1}$ fulfills the first order optimality condition

$$\hat{J}'_{kh}(\bar{u}_{kh})(\delta u) = (\nabla \hat{J}(\bar{u}_{kh}), \delta u) = 0 \quad \forall \delta u \in U. \quad (1.28)$$

Due to the linear-quadratic structure of the optimal control problem this condition is also sufficient for optimality. Since the reduced discrete problem is

$$\hat{J}_{kh}(u) := \frac{1}{2} \|\mathcal{S}_{kh}u - y_d\|^2 + \frac{\beta}{2} \|u\|^2, \quad (1.29)$$

the first derivative of it can be expressed as

$$\hat{J}'_{kh}(u)(\delta u) = (\beta u + \mathcal{S}_{kh}^*(\mathcal{S}_{kh}u - y_d), \delta u), \quad (1.30)$$

where \mathcal{S}_{kh}^* is the adjoint of the discrete version of \mathcal{S} that is

$$(\mathcal{S}_{kh}^*u, v) = (u, \mathcal{S}_{kh}v), \quad \forall u, v \in U_{k,h}^{0,1}.$$

Therefore

$$\nabla \hat{J}_{kh}(u) = \beta u + \mathcal{S}_{kh}^*(\mathcal{S}_{kh}u - y_d).$$

Theorem 1.2.6 *The adjoint of \mathcal{S}_{kh} is given by $z_{kh} = \mathcal{S}_{kh}^*v$ if and only if*

$$B(\phi_{kh}, z_{kh}) = (\phi_{kh}, v) \quad \forall \phi_{kh} \in Y_{k,h}^{0,1}. \quad (1.31)$$

Proof. Given $v \in U_{k,h}^{0,1}$, let ϕ_{kh} in (1.31) be y_{kh} from (1.23), then

$$B(y_{kh}, z_{kh}) = (y_{kh}, v).$$

From (1.23), $B(y_{kh}, z_{kh}) = (u, z_{kh})$ and $y_{kh} = \mathcal{S}_{kh}u$, hence

$$(u, z_{kh}) = (\mathcal{S}_{kh}u, v) = (u, \mathcal{S}_{kh}^*v).$$

■

Remark that proof is so simple due to the fact that the test space for the discrete

variational inequality (1.23) is same as the solution space. As in the continuous case, the second variation of the reduced discrete cost functional defines the Hessian operator

$$(\mathcal{H}_{kh}(u)v_1, v_2) = \hat{J}''(u)(v_1, v_2), \quad \forall v_1, v_2 \in U_{k,h}^{0,1}.$$

Since \hat{J} is quadratic,

$$\hat{J}''(u)(v_1, v_2) = ((\mathcal{S}_{kh}^* \mathcal{S}_{kh} + \beta I)v_1, v_2), \quad \forall v_1, v_2 \in U_{k,h}^{0,1}. \quad (1.32)$$

It follows that

$$\mathcal{H}_{kh} = \mathcal{S}_{kh}^* \mathcal{S}_{kh} + \beta I. \quad (1.33)$$

We note that the discrete Hessian operator is independent of u and is symmetric positive definite because

$$(\mathcal{H}_{kh}u, u) \geq \beta(u, u), \quad \forall u \in U_{k,h}^{0,1}$$

1.3 Estimates

We define a projection operator $\hat{\pi}_k : C(I, L^2(\Omega)) \rightarrow P_0(L^2(\Omega))$, where $P_0(L^2(\Omega))$ is the space of functions that are piecewise constant in time and square integrable in space by

$$\hat{\pi}_k y(x, t) = y(x, t_m^-), \quad \forall x \in I_m, x \in \Omega. \quad (1.34)$$

We also define $P_h : L^2(\Omega) \rightarrow V_h$ to be the orthogonal L^2 -projection, and R_h to be the Ritz projection, namely $R_h : H_0^1(\Omega) \rightarrow V_h$, and $(\cdot, \cdot)_{I_m \times \Omega}$ is the L^2 space-time inner-products on the subinterval I_m and Ω . We introduce the discrete Laplace operator $\Delta_h : V_h \rightarrow V_h$ by

$$(-\Delta_h v_h, \mathcal{X})_{L^2(\Omega)} = (\nabla v_h, \nabla \mathcal{X})_{L^2(\Omega)}, \quad \forall \mathcal{X} \in V_h.$$

The following result is extracted from Theorem 12 in [42].

Theorem 1.3.1 *Let y be the solution to (1.2) with $y \in C(I; L^2(\Omega))$ and y_{kh} be the $dG(0)cG(1)$ solution. Then there exists a constant C independent of k and h such that*

$$\|y - y_{kh}\| \leq C \ln \frac{T}{k} (\|y - \hat{\pi}_k y\| + \|P_h y - y\| + \|R_h y - y\|).$$

The following estimate appears as Corollary 4 in [42].

Corollary 1.3.2 *If the solution y to (1.2) satisfies $y \in H^1(I; L^2(\Omega)) \cap L^2(I; H^2(\Omega))$, then there exists a constant C independent of k and h such that*

$$\|y - y_{kh}\| \leq C \ln \frac{T}{k} (k \|y\|_{H^1(I; L^2(\Omega))} + h^2 \|y\|_{L^2(I; H^2(\Omega))}). \quad (1.35)$$

Proof. The following estimates hold for Ritz projection, projection π_k , and

projection P_h ,

$$\|R_h y - y\| \leq c_1 h^2 \|y\|_{L^2(I; H^2(\Omega))} \quad (1.36)$$

$$\|\hat{\pi}_k y - y\| \leq c_2 k \|y\|_{H^1(I; L^2(\Omega))} \quad (1.37)$$

$$\|P_h y - y\| \leq c_3 h^2 \|y\|_{L^2(I; H^2(\Omega))}, \quad (1.38)$$

where c_1, c_2 , and c_3 are independent of k and h . The estimates for the Ritz projection and the L^2 projection P_h are the standard estimates and the estimate for $\hat{\pi}_k$ is taken from [42]. From Theorem 1.3.1 and using (1.36), (1.37), and (1.38), the estimate (1.35) can be obtained. \blacksquare

By applying the Maximal parabolic regularity from Theorem 1.2.1 to Corollary 1.3.2 we obtain the following.

Corollary 1.3.3 *If $u \in U$, then the solution $y = \mathcal{S}u$ to (1.2) satisfies $y \in H^1(I; L^2(\Omega)) \cap L^2(I; H^2(\Omega))$, and there exists a constant C independent of k and h such that*

$$\|y - y_{kh}\| \leq C \ln \frac{T}{k} (k + h^2) \|u\|. \quad (1.39)$$

We define $\pi_{kh} : U \rightarrow U_{k,h}^{0,1}$ to be the L^2 -projection. The discrete solution operator \mathcal{S}_{kh} is naturally from $U_{k,h}^{0,1}$ to $Y_{k,h}^{0,1}$. We extract its definition from U to U by $\widetilde{\mathcal{S}}_{kh} = \mathcal{S}_{kh} \circ \pi_{kh} : U \rightarrow U$. By Corollary 1.3.3 the following estimate holds for $\widetilde{\mathcal{S}}_{kh}$

$$\|(\mathcal{S} - \widetilde{\mathcal{S}}_{kh})u\| \leq C(h^2 + k)\|u\|. \quad (1.40)$$

A standard duality argument (see, e.g., [43]) implies,

$$\|(\mathcal{S} - \widetilde{\mathcal{S}}_{kh})^* u\| = \|(\mathcal{S} - \widetilde{\mathcal{S}}_{kh})u\|, \quad (1.41)$$

and by (1.40), we have

$$\|(\mathcal{S} - \widetilde{\mathcal{S}}_{kh})^* u\| = \|(\mathcal{S}^* - \mathcal{S}_{kh}^* \circ \pi_{kh})u\| \leq C(k + h^2)\|u\|. \quad (1.42)$$

Therefore, the estimate that holds for the solution operator \mathcal{S} (Corollary 1.3.3) and its discrete counterpart \mathcal{S}_{kh}^* also holds for the adjoint operators \mathcal{S}^* and \mathcal{S}_{kh}^* .

1.3.1 The two-grid preconditioner

In this section we introduce and analyze the preconditioner for the operator denoted

$$\mathcal{H}_{kh} = \mathcal{S}_{kh}^* \mathcal{S}_{kh} + \beta I : U_{k,h}^{0,1} \rightarrow Y_{k,h}^{0,1},$$

where \mathcal{S}_{kh}^* , is the adjoint of the discrete version of \mathcal{S} introduced in the previous section. In fact, the operator \mathcal{H}_{kh} introduced here is the Hessian of the reduced cost functional in (1.29).

Note that $\pi_{2k,2h}$ is the orthogonal projection onto $U_{2k,2h}^{0,1}$, which is a coarser (both in space and in time) version of $U_{k,h}^{0,1}$. When the L^2 - projector applies to a function in U , we regard this operator as the extraction of its smooth component. On the other hand $(I - \pi_{2k,2h})$ extracts the oscillatory component of a function.

Since $\mathcal{S}_{kh}^* \mathcal{S}_{kh}$ approximates the smoothing operator, we have

$$(\beta I + \mathcal{S}_{kh}^* \mathcal{S}_{kh})(I - \pi_{2k2h}) \approx \beta(I - \pi_{2k2h}).$$

Therefore we have

$$\begin{aligned} \mathcal{H}_{kh} &= (\pi_{2k2h} + (I - \pi_{2k2h}))(\mathcal{S}_{kh}^* \mathcal{S}_{kh} + \beta I)(\pi_{2k2h} + (I - \pi_{2k2h})) \\ &\approx \pi_{2k2h}(\mathcal{S}_{kh}^* \mathcal{S}_{kh} + \beta I) + \beta(I - \pi_{2k2h}). \end{aligned}$$

Hence, we anticipate that the discrete Hessian \mathcal{H}_{kh} is well approximated by

$$\mathcal{M}_{kh} = \mathcal{H}_{2k2h} \pi_{2k2h} + \beta(I - \pi_{2k2h}).$$

Note that the inverse of \mathcal{M}_{kh} is computed as

$$\mathcal{M}_{kh}^{-1} = \mathcal{H}_{2k2h}^{-1} \pi_{2k2h} + \beta^{-1}(I - \pi_{2k2h}).$$

To evaluate the quality of the preconditioner we search for an estimate of the type

$$1 - \frac{C(k^\delta + h^\alpha)}{\beta} \leq \frac{(\mathcal{H}_{kh} u, u)}{(\mathcal{M}_{kh} u, u)} \leq 1 + \frac{C(k^\delta + h^\alpha)}{\beta}, \quad \forall u \neq 0 \quad (1.43)$$

where h is a mesh size, C is independent of h , k , and may depend on $\ln k$, k is the time-step, δ is related to the time-discretization order, and an exponent $\alpha \geq 0$ (is related to the spatial discretization) is as large as possible. In general, α and δ are

not expected to exceed the order of convergence of the discretization. Numerical results suggests that $\alpha = 2$ and $\delta = 1$, so the preconditioner is of optimal-order since we applied dG(0)cG(1) discretization. The goal is to show that the generalized eigenvalues of \mathcal{H}_{kh} and \mathcal{M}_{kh} are in an interval of center 1 and radius $C(k + h^2)/\beta$. Given that

$$(\mathcal{H}_{kh}u, u) \geq \beta\|u\|^2,$$

it is sufficient to find a norm $\|\cdot\|$ on $U_{k,h}^{0,1}$ so that the symmetric (same norm on both sides) estimate holds

$$\|(\mathcal{H}_{kh} - \mathcal{M}_{kh})u\| \leq C(k + h^2)\|u\|, \quad \forall u \in U_{k,h}^{0,1}.$$

The rest of this section is devoted to showing that the above estimate holds with $\|\cdot\| = \|\cdot\|$.

The first result shows a symmetric approximation between of the continuous reduced Hessian and the discrete operator \mathcal{H}_{kh} in the $\|\cdot\|$ -norm.

Theorem 1.3.4 *The following estimate holds:*

$$\|(\pi_{kh}\mathcal{H} - \mathcal{H}_{kh})u\| \leq C(k + h^2)\|u\|, \quad \forall u \in U_{k,h}^{0,1},$$

where $\pi_{kh} : U \rightarrow U_{k,h}^{0,1}$ is the L^2 projection.

Proof. We have

$$\pi_{kh}\mathcal{H} - \mathcal{H}_{kh} = \pi_{kh}\mathcal{S}^*\mathcal{S} - \mathcal{S}_{kh}^*\mathcal{S}_{kh} = \underbrace{\pi_{kh}(\mathcal{S}^* - \mathcal{S}_{kh}^*)\mathcal{S}}_A + \underbrace{\pi_{kh}\mathcal{S}_{kh}^*(\mathcal{S} - \mathcal{S}_{kh})}_B.$$

For the term A we have

$$\|\pi_{kh}(\mathcal{S}^* - \mathcal{S}_{kh}^*)\mathcal{S}u\| \leq C\|(\mathcal{S}^* - \mathcal{S}_{kh}^*)\mathcal{S}u\| \stackrel{(1.42)}{\leq} C(k+h^2)\|\mathcal{S}u\| \stackrel{(1.4)}{\leq} C(k+h^2)\|u\|.$$

To estimate the term B we use the stability of \mathcal{S}_{kh}^* that is obtained by the approximation:

$$\begin{aligned} \|\mathcal{S}_{kh}^*v\| &\leq \|(\mathcal{S}_{kh}^* - \mathcal{S}^*)v\| + \|\mathcal{S}^*v\| \\ &\stackrel{(1.42)}{\leq} C_1(k+h^2)\|v\| + C_2\|v\| \leq C\|v\|. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\pi_{kh}\mathcal{S}_{kh}^*(\mathcal{S} - \mathcal{S}_{kh})u\| &\leq C\|\mathcal{S}_{kh}^*(\mathcal{S} - \mathcal{S}_{kh})u\| \\ &\leq \|(\mathcal{S} - \mathcal{S}_{kh})u\| \\ &\stackrel{\text{Corollary 1.3.3}}{\leq} C(k+h^2)\|u\|. \end{aligned}$$

■

Theorem 1.3.5 *The following estimate holds:*

$$\|\mathcal{S}^*\mathcal{S}(I - \pi_{kh})u\| \leq C(h^2 + k)\|u\|, \quad \forall u \in U_{k,h}^{0,1}. \quad (1.44)$$

Proof. Define \mathcal{J} to be an interpolant operator such that $\hat{y}(x, t) = P_h(y(\cdot, t))$, $\mathcal{J}(y) = \hat{\pi}_k \hat{y}$ and for $y = \mathcal{S}u$ or $y = \mathcal{S}^*u$ satisfies the following estimate

$$\|y - \mathcal{J}y\| \leq C(k + h^2)\|u\|. \quad (1.45)$$

The above estimate is the result of (1.15) and Theorem 1.2.1. Therefore

$$\begin{aligned} (\mathcal{S}^* \mathcal{S}(I - \pi_{kh})u, u) &= ((I - \pi_{kh})u, \mathcal{S}^* \mathcal{S}u) \\ &= ((I - \pi_{kh})u, \mathcal{S}^* \mathcal{S}u - \mathcal{J}(\mathcal{S}^* \mathcal{S}u)) \leq C(k + h^2)\|(I - \pi_{kh})u\|\|\mathcal{S}u\| \\ &\leq C(k + h^2)\|u\|. \end{aligned}$$

By dividing both sides by $\|u\|$ and taking the supremum, we arrive at the estimate.

■

Theorem 1.3.6 *In the conditions of Theorems 1.3.4 and 1.3.5 the following holds:*

$$\|\mathcal{H}_{kh} - \mathcal{M}_{kh}\| \leq C(k + h^2)\|u\|. \quad (1.46)$$

Proof. The difference between the two-grid preconditioner and the Hessian is:

$$\begin{aligned} \mathcal{H}_{kh} - \mathcal{M}_{kh} &= \mathcal{H}_{kh} + \beta I - (\mathcal{H}_{2k2h} + \beta I)\pi_{2k2h} - \beta(I - \pi_{2k2h}) \\ &= \mathcal{H}_{kh} - \mathcal{H}_{2k2h}\pi_{2k2h} \\ &= \mathcal{H}_{kh} - \mathcal{H} + \mathcal{H} - \mathcal{H}\pi_{2k2h} + (\mathcal{H} - \mathcal{H}_{2k2h})\pi_{2k2h} \\ &= (\mathcal{H}_{kh} - \mathcal{H}) + \mathcal{H}(I - \pi_{2k2h}) + (\mathcal{H} - \mathcal{H}_{2k2h})\pi_{2k2h}. \end{aligned}$$

We showed that

$$\|(\mathcal{H} - \mathcal{H}_{kh})u\| \leq C(k + h^2)\|u\|, \quad \forall u \in U_{k,h}^{0,1}.$$

Similarly, we obtain

$$\|(\mathcal{H} - \mathcal{H}_{2k2h})\pi_{2k2h}u\| \leq C_1(2k + (2h)^2)\|\pi_{2k2h}u\| \leq C_2(k + h^2)\|u\|, \quad \forall u \in U_{k,h}^{0,1}.$$

The second term bounded as in (1.44), therefore

$$\|(\mathcal{H}_{kh} - \mathcal{M}_{kh})u\| \leq C(k + h^2)\|u\|.$$

■

1.4 Numerical results

In this section, we numerically show how well the preconditioner approximates the Hessian. We consider two different projections. First, we define π to be the space-time projection π_{2k2h} , as in Section (1.3). Second, we experiment with a space-only projection π .

1.4.1 Space-time projection

We use the orthogonal projection and the two-grid preconditioner that we defined in Section 1.3.1 to verify $O(k + h^2)$ numerically. We conduct numerical

experiments in one space dimension, $\Omega = [0, 1]$ with variable T . We build the matrices representing \mathcal{H}_{kh} and \mathcal{M}_{kh} and compute the joint spectrum. Finally, we compute

$$d_{kh} = \max\{|\ln \lambda| : \lambda \in \sigma(\mathcal{H}_{kh}, \mathcal{M}_{kh})\}.$$

We hope to verify numerically the validity of estimate (1.46). In order to do so we consider

$$h_0 = 2^{-3}, h_n = 2^{-n}h_0, k_0 = 2^{-4}, k_n = 4^{-n}k_0.$$

The theoretical estimate predict

$$d_{k_n h_n} / d_{k_{n+1} h_{n+1}} \approx 4.$$

Conclusion: numerics are consistent with the theoretical results.

$\beta = 1, T = 1$			
n	0	1	2
$d_{k_n h_n}$	1.726×10^{-3}	5.629×10^{-4}	1.560×10^{-4}
$d_{k_n h_n} / d_{k_{n+1} h_{n+1}}$	3.0672	3.6083	–
$\beta = 1, T = 2$			
$d_{k_n h_n}$	2.887×10^{-3}	1.017×10^{-3}	3.009×10^{-4}
$d_{k_n h_n} / d_{k_{n+1} h_{n+1}}$	2.8380	3.3814	–
$\beta = 10^{-2}, T = 1$			
$d_{k_n h_n}$	1.377×10^{-1}	4.322×10^{-2}	1.182×10^{-2}
$d_{k_n h_n} / d_{k_{n+1} h_{n+1}}$	3.1866	3.6558	–
$\beta = 10^{-2}, T = 2$			
$d_{k_n h_n}$	2.388×10^{-1}	7.964×10^{-2}	2.304×10^{-2}
$d_{k_n h_n} / d_{k_{n+1} h_{n+1}}$	2.9979	3.4573	–

1.4.2 Space projection

In this section, We define $\pi_{k2h} : U \rightarrow U_{k,2h}^{0,1}$ to project on the coarse finite element space only. Hence the preconditioner is,

$$\mathcal{M}_{kh} = \mathcal{H}_{k2h} \pi_{k2h} + \beta(I - \pi_{k2h}).$$

The way to do this is to cut h in $\frac{1}{2}$ and fix $k = 2^{-4}$ and $k = 2^{-5}$ and hope to see a decrease by a factor of 2. We begin with $h_0 = 2^{-5}$ and $h_n = 2^{-n}h_0$. From the tables it is observed that the spectral distance between constructed preconditioner and the Hessian is $O(h^2)$. We fix $T = 1$ and consider two different values for the regularization parameter β and obtain the following results.

$\beta = 10^{-2}, k = 2^{-4}$			
n	0	1	2
d_{kh_n}	2.463×10^{-3}	1.624×10^{-4}	1.541×10^{-4}
$d_{kh_n}/d_{kh_{n+1}}$	3.9968	3.9992	–
$\beta = 1, k = 2^{-4}$			
d_{kh_n}	4.379×10^{-1}	1.097×10^{-1}	2.745×10^{-2}
$d_{kh_n}/d_{kh_{n+1}}$	3.9908	3.9977	–
$\beta = 1, k = 2^{-5}$			
d_{kh_n}	0.112×10^{-4}	0.028×10^{-4}	0.007×10^{-4}
$d_{kh_n}/d_{kh_{n+1}}$	3.9977	3.9994	–
$\beta = 10^{-2}, k = 2^{-5}$			
d_{kh_n}	0.624×10^{-3}	0.156×10^{-3}	0.039×10^{-3}
$d_{kh_n}/d_{kh_{n+1}}$	3.9992	3.9998	–

1.5 Future work

In this work, we presented the numerical analysis for a specific discretization (dG(0)cG(1)) of the state equation, with no control or state constraints. Our analysis can be extended in a number of directions. We would like to observe the behavior of the two-grid preconditioner when we apply dG(1)cG(1) as a discretization of the state equation. In the present work, we did not consider additional constraints on the state y or the control u . We intend to consider control-constrained optimal control problems constrained by parabolic PDEs. We also intend to study the more challenging case of state constrained problems using Lavrentiev- and Moreau- Yosida regularization.

Chapter 2: Multigrid methods for Dirichlet boundary control of elliptic equations

2.1 Introduction

In this work we design and analyze multigrid preconditioners for the elliptic boundary control problem

$$\begin{cases} \text{minimize } J(y, u) = \frac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \frac{\beta}{2}\|u\|_{L^2(\partial\Omega)}^2, \\ \text{subject to : } \mathcal{A}y = 0, \quad y|_{\partial\Omega} = u, \quad (y, u) \in H^1(\Omega) \times L^2(\partial\Omega) \end{cases} \quad (2.1)$$

where $\Omega \subset \mathbb{R}^d$, $d = 2$ is a bounded domain, and \mathcal{A} is a second-order linear uniformly elliptic operator in divergence form

$$\mathcal{A}y(x) = - \sum_{i=1}^d \partial_i \left(\sum_{j=1}^d a_{ij}(x) \partial_j y(x) + b_i(x) y(x) \right) + c(x) y(x), \quad (2.2)$$

where a_{ij} , b_i , and c are assumed to be sufficiently smooth functions. Since the constraint in (2.1) is a well-posed elliptic PDE, we define the solution $u \rightarrow y = \mathcal{S}u$ which allows us to replace y by $\mathcal{S}u$ in the cost functional, and obtain the reduced

form of the problem

$$\text{minimize } \hat{J}(u) := \frac{1}{2} \|\mathcal{S}u - y_d\|_{L^2(\Omega)}^2 + \frac{\beta}{2} \|u\|_{L^2(\partial\Omega)}^2. \quad (2.3)$$

Since (2.3) is quadratic, the Hessian of the reduced cost functional (called the reduced Hessian) is independent of u , and is given by

$$\mathcal{H} = \mathcal{S}^* \mathcal{S} + \beta I,$$

where \mathcal{S}^* is the adjoint of the solution operator \mathcal{S} (see next section for derivation).

Given that the solution of the elliptic boundary control problem is

$$u_{\min} = \mathcal{H}^{-1}(\mathcal{S}^* y_d),$$

the challenge is to find efficient solution methods for solving the linear systems representing discrete version of

$$\mathcal{H}u = \mathcal{S}^* y_d. \quad (2.4)$$

Solving a discrete version of (2.4) is very expensive due to the potentially very large cost of applying the operators \mathcal{S} and \mathcal{S}^* , each of them involving a PDE solve. The discrete representations of \mathcal{S} and \mathcal{S}^* are dense, therefore solving (2.4) has to rely on matrix-free Krylov space solvers associated with efficient matrix-free preconditioners. We developed efficient two-grid preconditioners for the reduced Hessian, and we

observed numerically for the above optimization problem with $(a_{ij} = 1, b_i = 0, c = 0)$ a degrading by at least one order of the expected optimality of the preconditioners. Our goal is to analyze the behavior of the two-grid preconditioners in theory.

2.2 Problem description

We denote by $\mathcal{Y} = H^{\frac{1}{2}}(\Omega)$, $\mathcal{Y}_0 = H_0^1(\Omega)$, $\mathcal{U} = L^2(\partial\Omega)$. Furthermore (\cdot, \cdot) denotes the $L^2(\Omega)$ -inner product, while $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\partial\Omega)$.

2.2.1 The Dirichlet problem and its discretization

We begin with constructing a solution operator $\mathcal{S} : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\Omega)$. The PDE-constraint in (2.1) is given by the Dirichlet problem

$$\text{Find } y \in H^1(\Omega) \quad \text{s.t.} \quad \mathcal{A}y = 0 \quad \text{in } \Omega, \quad y|_{\partial\Omega} = u, \quad (2.5)$$

with $u \in H^{\frac{1}{2}}(\partial\Omega)$ given. If we define $a : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$ to be the standard bilinear form associated with \mathcal{A}

$$a(y, \varphi) = \sum_{i,j} \int_{\Omega} a_{ij} \partial_j y \partial_i \varphi, \quad (2.6)$$

then the weak solution y of (2.5) is found as

$$y = y_0 + \mathcal{E}u,$$

where $\mathcal{E} : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\Omega)$ is a bounded linear extension operator, and $\mathcal{S}_0 u = y_0 \in \mathcal{Y}_0$ satisfies the variational inequality

$$a(y_0, \varphi) = -a(\mathcal{E}u, \varphi), \quad \forall \varphi \in \mathcal{Y}_0. \quad (2.7)$$

We denote by $\mathcal{S} : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\Omega)$ the solution operator of (2.5); hence

$$\mathcal{S} = \mathcal{E} + \mathcal{S}_0. \quad (2.8)$$

Note that the definition of \mathcal{S} is independent of the extension operator \mathcal{E} . To extend \mathcal{S} from \mathcal{U} to \mathcal{Y} we consider the very weak formulation of elliptic equation (2.2),

$$-(y, \Delta\phi) + \langle u, \partial_n\phi \rangle = 0, \quad \forall \phi \in \mathcal{Y}_0 \cap H^2(\Omega). \quad (2.9)$$

For this formulation of the state equation, we recall the following result on existence that is taken from [2]:

Theorem 2.2.1 *There exists an operator $\mathcal{S} : \mathcal{U} \rightarrow \mathcal{Y}$ such that $y = \mathcal{S}u$ solves (2.1).*

We use the following a priori bound for the solution operator \mathcal{S} that is extracted from [44] Lemma 2.2.

Lemma 2.2.2 *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain with boundary $\partial\Omega$. For $0 \leq s \leq 1$ the solution operator \mathcal{S} is continuously defined from*

$H^s(\partial\Omega) \rightarrow H^{s+1/2}(\Omega)$, and satisfies

$$\|\mathcal{S}u\|_{H^{s+\frac{1}{2}}(\Omega)} \leq c\|u\|_{H^s(\partial\Omega)}. \quad (2.10)$$

The following theorem that is taken from [44] Lemma 2.4, proves the solvability of the optimal control problem (2.1).

Lemma 2.2.3 *The optimization problem (2.1) together with the very weak formulation (2.9) of the state equation possesses a uniquely determined solution $\{\bar{y}, \bar{u}\} \in \mathcal{Y} \times \mathcal{U}$.*

The existence result for the state equation in Theorem (2.2.1) ensures the existence of a control-to-state mapping $y = \mathcal{S}u$ defined through (2.8). By means of this mapping we introduce the reduced cost functional $J : \mathcal{U} \rightarrow \mathbb{R}$:

$$\hat{J}(u) := J(u, \mathcal{S}u).$$

The optimal control problem can then be equivalently reformulated as

$$\text{minimize}_{u \in \mathcal{U}} \hat{J}(u). \quad (2.11)$$

The first order necessary optimality condition for (2.11) reads as

$$\hat{J}'(\bar{u})(\delta u) = \langle \nabla \hat{J}(\bar{u}), \delta u \rangle = 0, \quad \forall \delta u \in \mathcal{U}. \quad (2.12)$$

Due to the quadratic structure of the optimal control problem this condition is also sufficient for optimality. Since the reduced cost functional is

$$\text{minimize } \hat{J}(u) := \frac{1}{2} \|\mathcal{S}u - y_d\|_{L^2(\Omega)}^2 + \frac{\beta}{2} \|u\|_{L^2(\partial\Omega)}^2,$$

the first derivative of it can be expressed as

$$\hat{J}'(u)(\delta u) = \langle \beta u + \mathcal{S}^*(\mathcal{S}u - y_d), \delta u \rangle,$$

where \mathcal{S}^* is the adjoint of \mathcal{S} , that is

$$\langle \mathcal{S}^*u, v \rangle = \langle u, \mathcal{S}v \rangle \quad \forall u, v \in \mathcal{U}.$$

Therefore

$$\nabla \hat{J}(u) = \beta u + \mathcal{S}^*(\mathcal{S}u - y_d). \tag{2.13}$$

The second variation of the reduced cost functional defines the Hessian operator

$$\langle \mathcal{H}(u)v_1, v_2 \rangle = \hat{J}''(u) \langle v_1, v_2 \rangle, \quad \forall v_1, v_2 \in \mathcal{U},$$

since

$$\hat{J}''(u) \langle v_1, v_2 \rangle = \langle (\mathcal{S}^*\mathcal{S} + \beta I)v_1, v_2 \rangle, \quad \forall v_1, v_2 \in \mathcal{U}, \tag{2.14}$$

it follows that

$$\mathcal{H} = \mathcal{S}^* \mathcal{S} + \beta I. \quad (2.15)$$

We note that the Hessian operator is symmetric positive definite because

$$\langle \mathcal{H}u, u \rangle \geq \beta \langle u, u \rangle$$

and is independent of u . The optimality condition (2.12) is equivalent, due to (2.13)

to

$$\mathcal{H}u = (\mathcal{S}^* \mathcal{S} + \beta I)u = \mathcal{S}^* y_d. \quad (2.16)$$

Hence the reduced problem has a unique solution given by

$$\bar{u} = \mathcal{H}^{-1}(\mathcal{S}^* y_d), \quad \bar{y} = \mathcal{S}\bar{u}. \quad (2.17)$$

The challenge of this approach is to find efficient solution methods for solving the linear systems representing discrete version of (2.16).

To discretize (2.16) we consider a quasi-uniform sequence of triangulations $(\mathcal{T}_h)_{h \in J}$ of Ω with $J = \{h_0, h_1, \dots\}$, where h is the mesh size of \mathcal{T}_h , and \mathcal{T}_{h_j} is a refinement of $\mathcal{T}_{h_{j-1}}$. Let \mathcal{Y}_h be the space of continuous piecewise linear functions on \mathcal{T}_h , and

$$\mathcal{Y}_{0,h} = \{u \in \mathcal{Y}_h : u|_{\partial\Omega} = 0\}.$$

For the discrete control problem we consider the $(d - 1)$ -dimensional triangulations $(\mathcal{T}_h^\partial)_{h \in J}$ defined on $\partial\Omega$ by \mathcal{T}_h^∂ , and the spaces \mathcal{U}_h of continuous piecewise linear functions (with respect to \mathcal{T}_h^∂) on $\partial\Omega$. Note that $\mathcal{U}_h \subset \mathcal{U}$. Let $\mathcal{E}_h : \mathcal{U}_h \rightarrow \mathcal{Y}_h$ be the natural extension, that is, for each $\varphi \in \mathcal{U}_h$ we set

$$(\mathcal{E}_h \varphi)(p) = \begin{cases} \varphi(p) & , \text{ for } p \text{ vertex on } \partial\Omega \\ 0 & , \text{ for } p \text{ vertex in } \text{Int}(\Omega) . \end{cases}$$

For $u \in \mathcal{U}_h$, the finite element solution of (3.5) is given by $y_h = y_{0,h} + \mathcal{E}_h u$, where $\mathcal{S}_{0,h} u = y_{0,h} \in \mathcal{Y}_{0,h}$ satisfies the variational inequality

$$a(y_{0,h}, \varphi) = -a(\mathcal{E}_h u, \varphi), \quad \forall \varphi \in \mathcal{Y}_{0,h} . \quad (2.18)$$

If we denote by $\mathcal{S}_h : \mathcal{U}_h \rightarrow \mathcal{Y}_h$ the discrete solution operator then

$$\mathcal{S}_h = \mathcal{E}_h + \mathcal{S}_{0,h} . \quad (2.19)$$

The existence of a control-to-state mapping $u \rightarrow y = \mathcal{S}_h u$ defined (2.19), allows us to introduce the discrete reduced cost functional $\hat{J}_h : \mathcal{U}_h \rightarrow \mathbb{R}$

$$\hat{J}_h(u) := J(u, \mathcal{S}_h u).$$

The discrete optimization problem can be reformulated as

$$\text{minimize}_{u_h \in \mathcal{U}_h} \hat{J}_h(u_h). \quad (2.20)$$

The following result which is taken from [44], ensures the solvability of the discrete optimal control problem.

Theorem 2.2.4 *The discrete optimal control problem (2.20) admits for $\beta > 0$ a unique solution $u_h \in \mathcal{U}_h$.*

The optimal control $\bar{u}_h \in \mathcal{U}_h$ fulfills the first order optimality condition

$$\hat{J}'_h(\bar{u}_h)(\delta u) = \langle \nabla \hat{J}_h(\bar{u}_h), \delta u \rangle = 0 \quad \forall \delta u \in \mathcal{U}_h. \quad (2.21)$$

Due to the quadratic structure of the optimal control problem this condition is also sufficient for optimality. Since the reduced discrete cost functional is

$$\hat{J}_h(u) := \frac{1}{2} \|\mathcal{S}_h u - y_d\|_{L^2(\Omega)}^2 + \frac{\beta}{2} \|u\|_{L^2(\partial\Omega)}^2,$$

the first derivative of it can be expressed as

$$\hat{J}'_h(u)(\delta u) = \langle \beta u + \mathcal{S}_h^*(\mathcal{S}_h u - y_d), \delta u \rangle, \quad (2.22)$$

where \mathcal{S}_h^* is the adjoint of \mathcal{S}_h . Therefore

$$\nabla \hat{J}_h(u) = \beta u + \mathcal{S}_h^*(\mathcal{S}_h u - y_d). \quad (2.23)$$

The second variation of the reduced discrete cost functional defines the Hessian operator

$$\langle \mathcal{H}_h(u)v_1, v_2 \rangle = \hat{J}''(u) \langle v_1, v_2 \rangle \quad \forall v_1, v_2 \in \mathcal{U}_h, \quad (2.24)$$

since

$$\hat{J}''_h(u) \langle v_1, v_2 \rangle = \langle (\mathcal{S}_h^* \mathcal{S}_h + \beta I)v_1, v_2 \rangle, \quad \forall v_1, v_2 \in \mathcal{U}_h. \quad (2.25)$$

It follows that

$$\mathcal{H}_h = \mathcal{S}_h^* \mathcal{S}_h + \beta I. \quad (2.26)$$

We note that the discrete Hessian \mathcal{H}_h is symmetric positive definite also with respect to $\langle \cdot, \cdot \rangle$ and is independent of u .

2.2.2 Computation of adjoints

Given $\tilde{y} \in L^2(\Omega)$, we define $z \in \mathcal{Y}_0$ to be the weak solution of the variational inequality

$$a(\varphi, z) = (\varphi, \tilde{y}), \quad \forall \varphi \in \mathcal{Y}_0. \quad (2.27)$$

We denote the solution operator of (2.27) by $\widehat{\mathcal{S}}$. Due to elliptic regularity $z \in H^2(\Omega)$, therefore $\widehat{\mathcal{S}} \in \mathfrak{L}(L^2(\Omega), H^2(\Omega) \cap \mathcal{Y}_0)$. Hence

$$-\sum_{i,j=1}^d \partial_j \left(\sum_{j=1}^d a_{ij} \partial_i z \right) = \tilde{y} \quad \text{a.e.} \quad (2.28)$$

After multiplying in (2.28) with a test function $\varphi \in H^1(\Omega)$ and applying Green's theorem to the first term we obtain

$$-\langle \varphi, \mathcal{N}z \rangle + a(\varphi, z) = (\varphi, \tilde{y}), \quad (2.29)$$

where $\mathcal{N} : H^2(\Omega) \rightarrow \mathcal{U}$ is defined by

$$\mathcal{N}z = \sum_{i,j=1}^d (a_{ij} \partial_i z)|_{\partial\Omega} n_j, \quad (2.30)$$

with $\vec{n} = (n_j)_{j=1,\dots,d}$ being the unit outer normal vector on $\partial\Omega$ which is defined at all but finitely many points of $\partial\Omega$.

We are now computing the adjoint of the operator \mathcal{S} , regarded as $\mathcal{S} : \mathcal{U} \rightarrow \mathcal{U}$.

We need $u \in H^{\frac{1}{2}}(\partial\Omega)$, $\tilde{y} \in L^2(\Omega)$, and $y \in H^1(\Omega)$ for this formulation

$$\begin{aligned} (\mathcal{S}u, \tilde{y}) &= (y, \tilde{y}) = (y_0 + \mathcal{E}u, \tilde{y}) \stackrel{(2.27)}{=} a(y_0, z) + (\mathcal{E}u, \tilde{y}) \\ &\stackrel{(2.7)}{=} -a(\mathcal{E}u, z) + (\mathcal{E}u, \tilde{y}) \stackrel{(2.29)}{=} \langle u, -\mathcal{N}z \rangle = \left\langle u, -(\mathcal{N} \circ \widehat{\mathcal{S}})\tilde{y} \right\rangle. \end{aligned}$$

Since $H^{\frac{1}{2}}(\partial\Omega)$ is dense in $L^2(\partial\Omega)$, The above holds for $u \in L^2(\partial\Omega)$. Therefore

$$\mathcal{S}^* = -\mathcal{N} \circ \widehat{\mathcal{S}}. \quad (2.31)$$

The following estimates hold for \mathcal{S}^* .

Lemma 2.2.5 *The adjoint operator \mathcal{S}^* is defined from $L^2(\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$, and satisfies*

$$\|\mathcal{S}^*v\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq c\|v\|_{L^2(\Omega)} \quad \forall v \in L^2(\Omega), \quad (2.32)$$

$$\|\mathcal{S}^*v\|_{L^2(\Omega)} \leq c\|v\|_{H^{-\frac{1}{2}}(\partial\Omega)} \quad \forall v \in L^2(\Omega). \quad (2.33)$$

Proof. Since the adjoint operator \mathcal{S}^* is naturally defined from $L^2(\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$, we have the first estimate (2.32). For the second estimate (2.33), we have

$$\begin{aligned} \|\mathcal{S}^*v\|_{L^2(\partial\Omega)} &= \sup_{\phi \in \mathcal{U}} \frac{(\mathcal{S}^*v, \phi)}{\|\phi\|_{L^2(\partial\Omega)}} \\ &= \sup_{\phi \in \mathcal{U}} \frac{\langle v, \mathcal{S}\phi \rangle}{\|\phi\|_{L^2(\partial\Omega)}} \\ &\leq \sup_{\phi \in \mathcal{U}} \frac{c\|v\|_{H^{-\frac{1}{2}}(\Omega)} \|\mathcal{S}\phi\|_{H^{\frac{1}{2}}(\Omega)}}{\|\phi\|_{L^2(\partial\Omega)}} \\ &\stackrel{(2.10)(s=0)}{\leq} c\|v\|_{H^{-\frac{1}{2}}(\Omega)}. \end{aligned}$$

■

We extract from [45], the following estimate.

Lemma 2.2.6 *The following estimate holds for the adjoint operator \mathcal{S}^* .*

$$\|\mathcal{S}^*y\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq c\|y\|_{H^{-1}(\Omega)} \quad \forall y \in L^2(\Omega). \quad (2.34)$$

Proof. We first consider the general case that the following inequality

$$\left\| \frac{\partial y}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq c\|y\|_{H^1(\Omega)} \leq c\|u\|_{H^{-1}(\Omega)}$$

holds for the following problem

$$\int_{\Omega} \nabla y \cdot \nabla v = \langle u, v \rangle \quad \forall v \in \mathcal{Y}_0.$$

We have the extension $\mathcal{E} : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\Omega)$ so that $\mathcal{E}(v) = v$ on the boundaries and

$$\|\mathcal{E}(v)\|_{H^1(\Omega)} \leq c\|v\|_{H^{\frac{1}{2}}(\partial\Omega)}.$$

We have

$$\int_{\Omega} \nabla y \cdot \nabla \mathcal{E}(v) dx = \int_{\Omega} u \mathcal{E}(v) dx + \int_{\partial\Omega} \frac{\partial y}{\partial \nu} v,$$

hence

$$\begin{aligned} \left| \int_{\partial\Omega} \frac{\partial y}{\partial \nu} v \right| &\leq \|y\|_{H^1(\Omega)} \|\mathcal{E}(v)\|_{H^1(\Omega)} + \|u\|_{H^{-1}(\Omega)} \|\mathcal{E}(v)\|_{H^1(\Omega)} \\ &\leq (\|y\|_{H^1(\Omega)} + \|u\|_{H^{-1}(\Omega)}) \|v\|_{H^{\frac{1}{2}}(\partial\Omega)}. \end{aligned}$$

By dividing both sides by $\|v\|_{H^{\frac{1}{2}}(\partial\Omega)}$ and taking the supremum, we get

$$\left\| \frac{\partial y}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq \|y\|_{H^1(\Omega)} + \|u\|_{H^{-1}(\Omega)}. \quad (2.35)$$

For the adjoint solution operator, we have $\mathcal{S}^*y = v$, where $v = -\frac{\partial z}{\partial \nu}|_{\partial\Omega}$, and z is the solution of (2.27). By (2.35), we have

$$\|\mathcal{S}^*y\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq c (\|z\|_{H^1(\Omega)} + \|y\|_{H^{-1}(\Omega)}) \leq c \|y\|_{H^{-1}(\Omega)}.$$

■

The adjoint of \mathcal{S}_h is computed somewhat similarly, the difference arising from the fact that the discrete solution is not in $H^2(\Omega)$. Given $\tilde{y} \in L^2(\Omega)$, let $z_h \in \mathcal{Y}_{0,h}$ satisfy

$$a(\varphi, z_h) = (\varphi, \tilde{y}), \quad \forall \varphi \in \mathcal{Y}_{0,h}. \quad (2.36)$$

For $u \in \mathcal{U}_h$ let $y_h = \mathcal{S}_h u$. Then

$$\begin{aligned} (\mathcal{S}_h u, \tilde{y}) &= (y_h, \tilde{y}) = (y_{0,h} + \mathcal{E}_h u, \tilde{y}) \stackrel{(2.36)}{=} a(y_{0,h}, z_h) + (\mathcal{E}_h u, \tilde{y}) \\ &\stackrel{(2.7)}{=} -a(\mathcal{E}_h u, z_h) + (\mathcal{E}_h u, \tilde{y}) \stackrel{\text{def}}{=} \langle u, g_h \rangle, \end{aligned}$$

where we used the Riesz representation theorem to define g_h as the only element in \mathcal{U}_h for which

$$-a(\mathcal{E}_h u, z_h) + (\mathcal{E}_h u, \tilde{y}) = \langle u, g_h \rangle, \quad u \in \mathcal{U}_h.$$

Since g_h depends linearly on \tilde{y} , we have $\mathcal{S}_h^* \tilde{y} = g_h$.

2.3 The two-grid preconditioner

In this section we discuss the analysis of the preconditioner for the Hessian operator \mathcal{H}_h defined in (2.26).

We now introduce the orthogonal projector $\pi_{2h} : \mathcal{U} \rightarrow \mathcal{U}_{2h}$. Cf. [20]. When the L^2 -projection applies to a discrete function it extracts the smooth part of it and $I - \pi_{2h}$ extracts the oscillatory part of the discrete function. The $\mathcal{S}^* \mathcal{S}$ is a smoothing operator so $(\beta I + \mathcal{S}_h^* \mathcal{S}_h)(I - \pi_{2h}) \approx \beta(I - \pi_{2h})$. Therefore we have

$$\mathcal{H}_h = (\pi_{2h} + (I - \pi_{2h}))(\mathcal{S}_h^* \mathcal{S}_h + \beta I)(\pi_{2h} + (I - \pi_{2h})) \approx \pi_{2h}(\mathcal{S}_h^* \mathcal{S}_h + \beta I) + \beta(I - \pi_{2h}).$$

Hence the discrete Hessian is well approximated by

$$\mathcal{M}_h = \mathcal{H}_{2h} \pi_{2h} + \beta(I - \pi_{2h}). \tag{2.37}$$

Note that

$$\mathcal{M}_h^{-1} = \mathcal{H}_{2h}^{-1}\pi_{2h} + \beta^{-1}(I - \pi_{2h}). \quad (2.38)$$

To evaluate the quality of the preconditioner we use the spectral distance introduced in [20] defined as

$$1 - \frac{Ch^\alpha}{\beta} \leq \frac{\langle \mathcal{H}_h u, u \rangle}{\langle \mathcal{M}_h u, u \rangle} \leq 1 + \frac{Ch^\alpha}{\beta}, \quad \forall u \neq 0 \quad (2.39)$$

where h is a mesh size, C is independent of h , and an exponent $\alpha \geq 0$ is as large as possible. In general, α is not expected to exceed the order of convergence of the discretization. Since in discretization of the PDE we used continuous piecewise linear polynomials for control and state, we expect to see a maximal rate of $\alpha = 2$ which would be the optimal order. Numerical results performed on a number of examples suggest $\alpha \in [1/2, 1]$, so the preconditioner is of suboptimal-order. The goal is to show that the generalized eigenvalues of \mathcal{H}_h and \mathcal{M}_h are in an interval of center 1 and radius Ch^α/β . Given that

$$\langle \mathcal{H}_h u, u \rangle \geq \beta \|u\|^2,$$

it is sufficient to find a norm $\|\cdot\|$ on \mathcal{U}_h so that the symmetric (same norm on both sides) estimate holds

$$\|(\mathcal{H}_h - \mathcal{M}_h)u\| \leq Ch\|u\|, \quad \forall u \in \mathcal{U}_h.$$

However, the estimate resulting from our analysis appears to be less than the one we observed numerically, namely

$$\|(\mathcal{H}_h - \mathcal{M}_h)u\| \leq Ch^{\frac{1}{2}}\|u\|, \quad \forall u \in \mathcal{U}_h.$$

The rest of this chapter is devoted to showing that the above estimate holds with

$$\|\cdot\| = \|\cdot\|_{L^2(\partial\Omega)}.$$

2.3.1 Error estimates

We extract from [44] the following estimate(s) where they appear as Lemma 3.3 parts (i) and (iv).

Lemma 2.3.1 *Let $\mathcal{A} = -\Delta$ and $\Omega \subset \mathbb{R}^2$ be a convex domain, there exists a constant C independent of h so that*

$$\|\mathcal{S}_h u\|_{H^1(\Omega)} \leq C\|u\|_{H^{\frac{1}{2}}(\partial\Omega)}, \quad \forall u \in \mathcal{U}_h, \quad (2.40)$$

$$\|(\mathcal{S} - \mathcal{S}_h)u\|_{H^1(\Omega)} \leq C\|u\|_{H^{\frac{1}{2}}(\partial\Omega)}, \quad \forall u \in \mathcal{U}_h, \quad (2.41)$$

$$\|(\mathcal{S} - \mathcal{S}_h)u\|_{L^2(\Omega)} \leq Ch\|u\|_{H^{\frac{1}{2}}(\partial\Omega)}, \quad \forall u \in \mathcal{U}_h, \quad (2.42)$$

$$\|(\mathcal{S} - \mathcal{S}_h)u\|_{L^2(\Omega)} \leq Ch^{1/2}\|u\|_{L^2(\partial\Omega)}, \quad \forall u \in \mathcal{U}_h. \quad (2.43)$$

The following estimate satisfies for the solution operator \mathcal{S} .

Lemma 2.3.2 *The solution operator \mathcal{S} satisfies*

$$\|\mathcal{S}u\|_{L^2(\Omega)} \leq c\|u\|_{H^{-\frac{1}{2}}(\partial\Omega)} \quad \forall u \in \mathcal{U}. \quad (2.44)$$

Proof. Let $u \in \mathcal{U}$,

$$\begin{aligned} \|\mathcal{S}u\|^2 = (\mathcal{S}u, \mathcal{S}u) &= \langle u, \mathcal{S}^* \mathcal{S}u \rangle \\ &\leq c\|u\|_{H^{-\frac{1}{2}}(\partial\Omega)} \|\mathcal{S}^* \mathcal{S}u\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\stackrel{(2.32)}{\leq} c\|u\|_{H^{-\frac{1}{2}}(\partial\Omega)} \|\mathcal{S}u\|_{L^2(\Omega)}. \end{aligned}$$

By dividing both sides by $\|\mathcal{S}u\|_{L^2(\Omega)}$, we obtain

$$\|\mathcal{S}u\|_{L^2(\Omega)} \leq c\|u\|_{H^{-\frac{1}{2}}(\partial\Omega)}$$

■

The following estimate holds.

Lemma 2.3.3 *There exists a constant C independent of h so that*

$$\|(\mathcal{S} - \mathcal{S}_h)u\|_{H^{-\frac{1}{2}}(\Omega)} \leq Ch^{\frac{1}{2}}\|u\|_{L^2(\partial\Omega)}, \quad \forall u \in \mathcal{U}_h. \quad (2.45)$$

Proof. We have

$$y = \mathcal{E}_h u + y_0, \quad y_h = \mathcal{E}_h u + y_{0,h},$$

where y_0 and $y_{0,h}$ satisfy

$$a(y_0, v) = -(\mathcal{E}_h u, v), \quad \forall v \in \mathcal{Y}_0, \tag{2.46}$$

$$a(y_{0,h}, v) = -(\mathcal{E}_h u, v), \quad \forall v \in \mathcal{Y}_{0,h}. \tag{2.47}$$

We obtain that

$$y - y_h = y_0 - y_{0,h} = e_h$$

and

$$a(e_h, v) = 0, \quad \forall v \in \mathcal{Y}_{0,h}.$$

For a given $\phi \in \mathcal{U}$, let $\hat{z} \in \mathcal{Y}_0$

$$a(v, \hat{z}) = (v, \phi), \quad \forall v \in \mathcal{Y}_0.$$

We now estimate $\|e_h\|_{H^{-\frac{1}{2}}(\Omega)}$

$$\begin{aligned}
\|e_h\|_{H^{-\frac{1}{2}}(\Omega)} &= \sup_{\phi \in H^{\frac{1}{2}}(\Omega)} \frac{(e_h, \phi)}{\|\phi\|_{H^{\frac{1}{2}}(\Omega)}} \\
&= \sup_{\phi \in H^{\frac{1}{2}}(\Omega)} \frac{a(e_h, \hat{z})}{\|\phi\|_{H^{\frac{1}{2}}(\Omega)}} = \sup_{\phi \in H^{\frac{1}{2}}(\Omega)} \frac{a(e_h, \hat{z} - \hat{z}_h)}{\|\phi\|_{H^{\frac{1}{2}}(\Omega)}} \\
&\leq \frac{C\|e_h\|_{H^1(\Omega)}\|\hat{z} - \hat{z}_h\|_{H^1(\Omega)}}{\|\phi\|_{H^{\frac{1}{2}}(\Omega)}} \\
&\stackrel{(2.41)}{\leq} Ch\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \frac{\|\phi\|_{L^2(\Omega)}}{\|\phi\|_{H^{\frac{1}{2}}(\Omega)}} \\
&\leq Ch^{\frac{1}{2}}\|u\|_{L^2(\partial\Omega)},
\end{aligned}$$

where for the last inequality, we used the inverse estimate [46]. ■

Lemma 2.3.4 *There exists a constant c independent of h such that*

$$\|(\mathcal{S}^* - \mathcal{S}_h^*)y\|_{L^2(\partial\Omega)} \leq ch^{\frac{1}{2}}\|y\|_{L^2(\Omega)} \quad \forall y \in \mathcal{Y} \quad (2.48)$$

Proof. Let $y \in \mathcal{Y}$,

$$\begin{aligned}
\langle (\mathcal{S}^* - \mathcal{S}_h^*)y, u \rangle_{L^2(\partial\Omega)} &= (y, (\mathcal{S} - \mathcal{S}_h)u)_{L^2(\Omega)} \\
&\leq \|y\|_{L^2(\Omega)} \|(\mathcal{S} - \mathcal{S}_h)u\|_{L^2(\Omega)} \\
&\stackrel{(2.43)}{\leq} \|y\|_{L^2(\Omega)} Ch^{\frac{1}{2}}\|u\|_{L^2(\partial\Omega)},
\end{aligned} \quad (2.49)$$

by dividing both sides by $\|u\|_{L^2(\partial\Omega)}$ and taking the supremum over $u \in \mathcal{U}$, we get the estimate.

■

2.3.2 Projection estimates and stability

We define the L^2 projection π_h that denotes either $\pi_h^u : \mathcal{U} \rightarrow \mathcal{U}_h$ or $\pi_h^y : L^2(\Omega) \rightarrow \mathcal{Y}_h$.

Lemma 2.3.5 *There exists a constant C independent of h so that*

$$\|\pi_h z\|_k \leq C \|z\|_k, \quad k = -1, 0, 1 \quad (2.50)$$

$$\|(I - \pi_h)z\|_{-1} \leq Ch \|z\|_0, \quad (2.51)$$

$$\|(I - \pi_h)z\|_0 \leq Ch^{\frac{1}{2}} \|z\|_{\frac{1}{2}}, \quad (2.52)$$

where $\|\cdot\|_1$ is either $\|\cdot\|_{H^1(\Omega)}$ or $\|\cdot\|_{H^1(\partial\Omega)}$, $\|\cdot\|_{-1}$ is either $\|\cdot\|_{H^{-1}(\Omega)}$ or $\|\cdot\|_{H^{-1}(\partial\Omega)}$, and $\|\cdot\|_0$ is either $\|\cdot\|_{L^2(\Omega)}$ or $\|\cdot\|_{L^2(\partial\Omega)}$.

Proof. For $k = 0$, it is trivial. We prove for $k = 1, -1$. Let \mathcal{J}_h be either $\mathcal{J}_h^y : H^1(\Omega) \rightarrow \mathcal{Y}_h$ or $\mathcal{J}_h^u : \mathcal{U} \rightarrow \mathcal{U}_h$ and \mathcal{I}_h be either $\mathcal{I}_h^y : \mathcal{Y}_h \rightarrow L^2(\Omega)$ or $\mathcal{I}_h^u : \mathcal{U}_h \rightarrow \mathcal{U}$. For $k = 1$, we have,

$$\begin{aligned} \|\pi_h z\|_1 &\leq \|(\pi_h - \mathcal{J}_h)z\|_1 + \|\mathcal{J}_h z\|_1 \\ &\leq \|\mathcal{J}_h(\pi_h - I)z\|_1 + \|\mathcal{J}_h z\|_1 \\ &\leq C_1 h^{-1} \|(\pi_h - I)z\|_0 + C_2 \|z\|_1 \\ &\leq C_3 h^{-1} h \|z\|_1 + C_2 \|z\|_1 = C \|z\|_1. \end{aligned}$$

For $k = -1$, we have,

$$\begin{aligned}
\|\pi_h z\|_{-1} &= \sup_{v \in H_0^1} \frac{(\pi_h z, v)}{\|v\|_1} \\
&= \sup_{v \in H_0^1} \frac{(z, \pi_h v)}{\|v\|_1} \\
&\leq C \frac{\|z\|_{-1} \|\pi_h v\|_1}{\|v\|_1} \\
&\stackrel{(2.50)(k=1)}{\leq} C \frac{\|z\|_{-1} \|v\|_1}{\|v\|_1} = \|z\|_{-1},
\end{aligned}$$

where H_0^1 denotes either \mathcal{Y}_0 or $H^1(\partial\Omega)$. For (2.51), we have,

$$\begin{aligned}
\|(I - \pi_h)z\|_{-1} &= \sup_{v \in H_0^1} \frac{((I - \pi_h)z, v)}{\|v\|_1} \\
&= \sup_{v \in H_0^1} \frac{(z, (I - \pi_h)v)}{\|v\|_1} \\
&\leq C \frac{\|z\|_0 \|(I - \pi_h)v\|_0}{\|v\|_1} \\
&\leq C \frac{\|z\|_0 h \|v\|_1}{\|v\|_1} = Ch \|z\|_0. \tag{2.53}
\end{aligned}$$

The last estimate (2.52) can be obtained by interpolating between the following two estimates

$$\|(I - \pi_h)z\|_0 \leq Ch^1 \|z\|_1,$$

$$\|(I - \pi_h)z\|_0 \leq C \|z\|_0.$$

■

Consider the embeddings

$$\begin{aligned}i_h^u &: \mathcal{U}_h \rightarrow \mathcal{U}, \\i_h^y &: \mathcal{Y}_h \rightarrow L^2(\Omega),\end{aligned}\tag{2.54}$$

and the orthogonal projections

$$\begin{aligned}\pi_h^u &: \mathcal{U} \rightarrow \mathcal{U}_h, \\ \pi_h^y &: L^2(\Omega) \rightarrow \mathcal{Y}_h.\end{aligned}$$

Then

$$(\pi_h^u)^* = i_h^u$$

and

$$(\pi_h^y)^* = i_h^y.$$

We introduce new discrete operators;

$$\widehat{\mathcal{S}}_h = \pi_h^y \mathcal{S} i_h^u,$$

where i_h is the embedding $\mathcal{U}_h \rightarrow \mathcal{U}$. Hence

$$\begin{aligned}
(\widehat{\mathcal{S}}_h)^* &= (i_h^u)^* \mathcal{S}^* (\pi_h^y)^* \\
&= \pi_h^u \mathcal{S}^* i_h^y \\
&= \pi_h^y \mathcal{S}^* i_h^y.
\end{aligned}$$

Lemma 2.3.6 *There exists a constant C independent of h such that*

$$\|(\pi_h^u \mathcal{S}^* - \mathcal{S}_h^*)y\|_{L^2(\partial\Omega)} \leq Ch^{\frac{1}{2}} \|y\|_{L^2(\Omega)} \quad \forall y \in \mathcal{Y}. \quad (2.55)$$

Proof. Let $y \in \mathcal{Y}$,

$$\begin{aligned}
\|(\pi_h^u \mathcal{S}^* - \mathcal{S}_h^*)y\|_{L^2(\partial\Omega)} &= \|\pi_h^u (\mathcal{S}^* - \mathcal{S}_h^*)y\|_{L^2(\partial\Omega)} \\
&\stackrel{(2.50)}{\leq} \|(\mathcal{S}^* - \mathcal{S}_h^*)y\|_{L^2(\partial\Omega)} \stackrel{\text{Lemma 2.3.4}}{\leq} Ch^{\frac{1}{2}} \|y\|_{L^2(\Omega)}. \quad (2.56)
\end{aligned}$$

■

Theorem 2.3.7 *There exists a constant c independent of h so that*

$$\|\mathcal{S}_h^* \mathcal{S}_h u - \pi_h^u \mathcal{S}^* \mathcal{S} u\|_{L^2(\partial\Omega)} \leq ch^{\frac{1}{2}} \|u\|_{L^2(\partial\Omega)} \quad \forall u \in \mathcal{U}_h. \quad (2.57)$$

Proof. Let $u \in \mathcal{U}_h$,

$$\begin{aligned}
\|\mathcal{S}_h^* \mathcal{S}_h u - \pi_h^u \mathcal{S}^* \mathcal{S} u\|_{L^2(\partial\Omega)} &\leq \|(\pi_h^u \mathcal{S}^* - \mathcal{S}_h^*) \mathcal{S}_h u\|_{L^2(\partial\Omega)} + \|\pi_h^u \mathcal{S}^* (\mathcal{S} - \mathcal{S}_h) u\|_{L^2(\partial\Omega)} \\
&\stackrel{(2.55), (2.50)}{\leq} ch^{\frac{1}{2}} \|\mathcal{S}_h u\|_{L^2(\Omega)} + c \|\mathcal{S}^* (\mathcal{S} - \mathcal{S}_h) u\|_{L^2(\Omega)} \\
&\leq ch^{\frac{1}{2}} \|u\|_{L^2(\partial\Omega)} + c \|(\mathcal{S} - \mathcal{S}_h) u\|_{L^2(\partial\Omega)} \\
&\stackrel{(2.43)}{\leq} ch^{\frac{1}{2}} \|u\|_{L^2(\partial\Omega)} + ch^{\frac{1}{2}} \|u\|_{L^2(\partial\Omega)} \\
&\leq Ch^{\frac{1}{2}} \|u\|_{L^2(\partial\Omega)}.
\end{aligned}$$

■

2.3.3 The two-grid preconditioner

In this section we estimate the quality of the two-grid preconditioner

$$\mathcal{M}_h \stackrel{\text{def}}{=} \beta(I - \pi_{2h}^u) + \mathcal{H}_{2h} \pi_{2h}^u. \quad (2.58)$$

This type of preconditioner was introduced in [20], and it is expected to be efficient if the operator $\mathcal{S}_h^* \mathcal{S}_h$ is of integral type.

Theorem 2.3.8 *There exists a constant C independent of h so that*

$$\|(\mathcal{H}_h - \mathcal{M}_h)u\|_{L^2(\partial\Omega)} \leq Ch^{\frac{1}{2}} \|u\|_{L^2(\partial\Omega)}, \quad \forall u \in \mathcal{U}_h. \quad (2.59)$$

Proof. A simple verification shows that

$$\mathcal{H}_h - \mathcal{M}_h = \mathcal{S}_h^* \mathcal{S}_h - \mathcal{S}_{2h}^* \mathcal{S}_{2h} \pi_{2h}^u .$$

Hence, for $u \in \mathcal{U}_h$ we have

$$\begin{aligned} \|(\mathcal{H}_h - \mathcal{M}_h)u\|_{L^2(\partial\Omega)} &\leq \|(\mathcal{S}_h^* \mathcal{S}_h - \pi_h^u \mathcal{S}^* \mathcal{S})u\|_{L^2(\partial\Omega)} \\ &\quad + \|(\pi_h^u \mathcal{S}^* \mathcal{S} - \pi_{2h}^u \mathcal{S}^* \mathcal{S} \pi_{2h}^u)u\|_{L^2(\partial\Omega)} \\ &\quad + \|(\pi_{2h}^u \mathcal{S}^* \mathcal{S} - \mathcal{S}_{2h}^* \mathcal{S}_{2h})\pi_{2h}^u u\|_{L^2(\partial\Omega)} . \end{aligned} \quad (2.60)$$

From Theorem 2.3.7, the first and last terms on the right-hand side of (2.60) are bound from above by $Ch^{\frac{1}{2}}\|u\|_{L^2(\partial\Omega)}$ for some appropriate constant. It remains to evaluate the middle term.

$$\begin{aligned} \|(\pi_h^u \mathcal{S}^* \mathcal{S} - \pi_{2h}^u \mathcal{S}^* \mathcal{S} \pi_{2h}^u)u\|_{L^2(\partial\Omega)} &\leq \|(\pi_h^u (I - \pi_{2h}^u) \mathcal{S}^* \mathcal{S} u)\|_{L^2(\partial\Omega)} + \|(\pi_{2h}^u \mathcal{S}^* \mathcal{S} (I - \pi_{2h}^u) u)\|_{L^2(\partial\Omega)} \\ &\stackrel{(2.50)}{\leq} C\|(I - \pi_{2h}^u) \mathcal{S}^* \mathcal{S} u\|_{L^2(\partial\Omega)} + C\|\mathcal{S}^* \mathcal{S} (I - \pi_{2h}^u) u\|_{L^2(\partial\Omega)} \\ &\stackrel{(2.52)}{\leq} Ch^{\frac{1}{2}}\|\mathcal{S}^* \mathcal{S} u\|_{H^{\frac{1}{2}}(\partial\Omega)} + C\|\mathcal{S} (I - \pi_{2h}^u) u\|_{L^2(\Omega)} \\ &\stackrel{(2.32), (2.44)}{\leq} Ch^{\frac{1}{2}}\|\mathcal{S} u\|_{L^2(\Omega)} + C\|(I - \pi_{2h}^u) u\|_{H^{-\frac{1}{2}}(\partial\Omega)} \\ &\leq Ch^{\frac{1}{2}}\|u\|_{L^2(\partial\Omega)} . \end{aligned}$$

Hence

$$\|(\mathcal{H}_h - \mathcal{M}_h)u\|_{L^2(\partial\Omega)} \leq ch^{\frac{1}{2}}\|u\|_{L^2(\partial\Omega)} .$$

■

2.4 Numerical results

We show numerically, how well the preconditioner approximate the Hessian. The following tables show the joint spectral analysis of the preconditioner and the Hessian. The spectral distance is a measure of spectral equivalence between two operators.

We build the matrices representing \mathcal{H}_h and \mathcal{M}_h and compute the joint spectrum. Finally, we compute

$$d_h = \max\{|\ln \lambda| : \lambda \in \sigma(\mathcal{H}_h, \mathcal{M}_h)\}.$$

In order to do so we consider

$$h_0 = 2^{-3}, h_n = 2^{-n} h_0.$$

We consider two different values for the regularization parameter β and obtain the following results.

$\beta = 1$						
n	0	1	2	3	4	5
d_{h_n}	0.0466	0.0238	0.012	0.00603	0.00302	0.00151
$\log_2(d_{h_n}/d_{h_{n+1}})$	0.9694	0.97104	0.98669	0.99375	0.99866	–
$\beta = 10^{-2}$						
d_{h_n}	1.71	1.22	0.79	0.47	0.264	0.141
$\log_2(d_{h_n}/d_{h_{n+1}})$	0.49	0.62	0.74	0.84	0.91	–

From the tables it is observed that the spectral distance between constructed preconditioner and the Hessian is $\mathcal{O}(h)$, which is one less than optimal.

Chapter 3: Multigrid methods for Neumann boundary control of elliptic equations

3.1 Introduction

In this work we design and analyze multigrid preconditioners for the elliptic boundary control problem

$$\begin{cases} \text{minimize } J(y, u) = \frac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \frac{\beta}{2}\|u\|_{L^2(\partial\Omega)}^2, \\ \text{subject to : } \mathcal{A}y = 0, \quad \frac{\partial y}{\partial \nu} = u, \quad (y, u) \in H^1(\Omega) \times L^2(\partial\Omega), \end{cases} \quad (3.1)$$

where $\Omega \subset \mathbb{R}^d$, $d = 2$ is a bounded convex domain, y_d denotes the desired state, y is the state which is associated with the control u by the state equation $\mathcal{A}y = 0$, $\frac{\partial y}{\partial \nu} = u$ and \mathcal{A} is a second-order linear uniformly elliptic operator in divergence form

$$\mathcal{A}y(x) = - \sum_{i=1}^d \partial_i \left(\sum_{j=1}^d a_{ij}(x) \partial_j y(x) + b_i(x) y(x) \right) + c(x) y(x), \quad (3.2)$$

where a_{ij} , b_i , and c are assumed to be sufficiently smooth functions. Since the constraint in (3.1) is a well-posed elliptic partial differential equation, we can introduce the control-to-state operator $\mathcal{S} : u \rightarrow y$. Thus, we can obtain the reduced formula-

tion of the problem which is given by

$$\underset{L^2(\partial\Omega)}{\text{minimize}} \hat{J}(u) = \frac{1}{2} \|\mathcal{S}u - y_d\|_{L^2(\Omega)}^2 + \frac{\beta}{2} \|u\|_{L^2(\partial\Omega)}^2. \quad (3.3)$$

Since (3.3) is quadratic, the Hessian of the reduced cost functional (called the reduced Hessian) is independent of u , and is given

$$\mathcal{H} = \mathcal{S}^* \mathcal{S} + \beta I,$$

where \mathcal{S}^* is the adjoint of the solution operator \mathcal{S} . Given that the solution of the elliptic boundary control problem is

$$u_{\min} = \mathcal{H}^{-1}(\mathcal{S}^* y_d),$$

the challenge is to find efficient solution methods for solving the linear systems representing discrete version of

$$\mathcal{H}u = \mathcal{S}^* y_d. \quad (3.4)$$

As mentioned in previous sections, solving a discrete version of (3.4) is very expensive due to the potentially very large cost of applying the operator \mathcal{S} and \mathcal{S}^* , each of them involving a PDE solve. Solving (3.4) has to rely on matrix-free Krylov space solvers associated with efficient matrix-free preconditioners since the discrete representations of \mathcal{S} and \mathcal{S}^* are dense. We developed efficient two-grid preconditioners

for the reduced Hessian, and we observed numerically for the above optimization problem with $(a_{ij} = \delta_{ij}, b_i = 0, c = 1)$ an optimal order behavior of the preconditioners. Our goal is to confirm the behavior of the two-grid preconditioners in theory.

3.2 Problem description

We denote by $\mathcal{Y} = H^1(\Omega)$, $\mathcal{U} = L^2(\partial\Omega)$. Furthermore (\cdot, \cdot) denotes the $L^2(\Omega)$ -inner product, while $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\partial\Omega)$.

3.2.1 The Neumann problem and its discretization

The PDE-constraint in (3.1) is given by the Neumann problem

$$\text{Find } y \in \mathcal{Y} \quad \text{s.t.} \quad \mathcal{A}y = 0 \quad \text{in } \Omega, \quad \frac{\partial y}{\partial \nu} = u, \quad (3.5)$$

with $u \in \mathcal{U}$ given and $[a_{i,j} = \delta_{i,j}, b_i = 0, c = 1.]$. Well-posedness for the Neumann problem requires that $c > 0$ a.e. in Ω . If we define $a : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}$ to be the standard bilinear form associated with \mathcal{A}

$$a(y, \varphi) = \sum_{i,j} \int_{\Omega} a_{ij} \partial_j y \partial_i \varphi + \sum_{i=1}^d \int_{\Omega} b_i y \partial_i \varphi + \int_{\Omega} c y \varphi, \quad (3.6)$$

then the weak solution $y \in \mathcal{Y}$ of (3.5) is

$$a(y, \varphi) = \langle u, \mathcal{R}\varphi \rangle, \quad \forall \varphi \in \mathcal{Y}. \quad (3.7)$$

where $\mathcal{R} : H^1(\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$ representing the restriction operator. We denote by $\mathcal{S} : \mathcal{U} \rightarrow \mathcal{Y}$ the solution operator of (3.7). The following theorem that is taken from [47] confirms the existence of the solution operator \mathcal{S} .

Theorem 3.2.1 *The continuous state equation has a unique weak solution $y \in H^1(\Omega)$ for $u \in \mathcal{U}$. Furthermore, there exists a positive constant c independent of u such that*

$$\|y\|_{H^1(\Omega)} \leq c\|u\|_{L^2(\partial\Omega)}.$$

The next theorem that is taken from [47] shows the solvability of the optimal control problem.

Theorem 3.2.2 *For a given $y_d \in \mathcal{U}$, and $\beta > 0$ the optimal control problem admits a unique solution $(\bar{u}, \bar{y}) \in (\mathcal{U}, \mathcal{Y})$.*

The existence result for the state equation in theorem (3.2.1) ensures the existence of a control-to-state mapping $u \rightarrow y = \mathcal{S}u$. By means of this mapping we introduce the reduced cost functional $\hat{J} : \mathcal{U} \rightarrow \mathbb{R}$:

$$\hat{J}(u) := J(u, \mathcal{S}u).$$

The optimal control problem can then be equivalently reformulated as

$$\text{minimize}_{u \in \mathcal{U}} \hat{J}(u). \tag{3.8}$$

The first order necessary optimality condition for (3.8) reads as

$$\hat{J}(\bar{u})(\delta u) = \langle \nabla \hat{J}(\bar{u}), \delta u \rangle = 0, \quad \forall \delta u \in \mathcal{U}. \quad (3.9)$$

Due to the quadratic structure of the optimal control problem this condition is also sufficient for optimality. Since the reduced cost functional is

$$\text{minimize}_{u \in \mathcal{U}} \hat{J}(u) := \frac{1}{2} \|\mathcal{S}u - y_d\|_{L^2(\Omega)}^2 + \frac{\beta}{2} \|u\|_{L^2(\partial\Omega)}^2,$$

the first derivative of it can be expressed as

$$\hat{J}(u)(\delta u) = \langle \beta u + \mathcal{S}^*(\mathcal{S}u - y_d), \delta u \rangle, \quad (3.10)$$

where \mathcal{S}^* is the adjoint of \mathcal{S} , that is

$$\langle \mathcal{S}^*u, v \rangle = (u, \mathcal{S}v) \quad \forall u, v \in \mathcal{U}.$$

Therefore

$$\nabla \hat{J}(u) = \beta u + \mathcal{S}^*(\mathcal{S}u - y_d). \quad (3.11)$$

The second variation of the reduced cost functional defines the Hessian operator

$$\langle \mathcal{H}(u)v_1, v_2 \rangle = \hat{J}''(u) \langle v_1, v_2 \rangle, \quad \forall v_1, v_2 \in \mathcal{U},$$

since

$$\hat{J}''(u) \langle v_1, v_2 \rangle = \langle (\mathcal{S}^* \mathcal{S} + \beta I) v_1, v_2 \rangle, \quad \forall v_1, v_2 \in \mathcal{U},$$

it follows that

$$\mathcal{H} = \mathcal{S}^* \mathcal{S} + \beta I. \quad (3.12)$$

We remark that the Hessian operator is symmetric positive definite because

$$\langle \mathcal{H}u, u \rangle \geq \beta \langle u, u \rangle$$

and is independent of u . The optimality condition (3.9) is equivalent, due to (3.11) to

$$\mathcal{H}u = (\mathcal{S}^* \mathcal{S} + \beta I)u = \mathcal{S}^* y_d. \quad (3.13)$$

Therefore the reduced problem has a unique solution given by

$$\bar{u} = \mathcal{H}^{-1}(\mathcal{S}^* y_d), \quad \bar{y} = \mathcal{S}\bar{u}. \quad (3.14)$$

The challenge of our approach is to find efficient solution methods for solving the linear systems representing discrete version of (3.13).

To discretize (3.7) we consider a quasi-uniform sequence of triangulations $(\mathcal{T}_h)_{h \in J}$ of Ω with $J = \{h_0, h_1, \dots\}$, where h is the mesh size of \mathcal{T}_h , and \mathcal{T}_{h_j} is a refinement of $\mathcal{T}_{h_{j-1}}$. Let \mathcal{Y}_h be the space of continuous piecewise linear functions on \mathcal{T}_h . For the discrete control problem we consider the $(d-1)$ -dimensional triangulations $(\mathcal{T}_h^\partial)_{h \in J}$ defined on $\partial\Omega$ by \mathcal{T}_h^∂ , and the spaces \mathcal{U}_h of discontinuous piecewise linear functions (with respect to \mathcal{T}_h^∂) on $\partial\Omega$. Note that $\mathcal{U}_h \subset \mathcal{U}$. For $u \in \mathcal{U}_h$, the finite element solution of (3.5) is given by $u \rightarrow \mathcal{Y}_h = \mathcal{S}_h u$, with $y_h \in \mathcal{Y}_h$ satisfies the variational inequality

$$a(y_h, \varphi) = \langle u, \mathcal{R}\varphi \rangle, \quad \forall \varphi \in \mathcal{Y}_h. \quad (3.15)$$

We denote by $\mathcal{S}_h : \mathcal{U}_h \rightarrow \mathcal{Y}_h$ the discrete solution operator. The following theorem is taken from [47].

Theorem 3.2.3 *The discrete state equation has a unique solution $y_h \in \mathcal{Y}_h$ for $u \in \mathcal{U}_h$. Furthermore, the estimate*

$$\|y\|_{H^1(\Omega)} \leq c \|u\|_{L^2(\partial\Omega)}$$

holds with a positive constant c independent of u .

The discrete state equation possesses a unique solution in \mathcal{Y}_h for every $u \in \mathcal{U}_h$. Therefore, we can introduce the linear and continuous discrete control-to-state operator $\mathcal{S}_h : \mathcal{U}_h \rightarrow \mathcal{Y}_h$ which maps a control $u \in \mathcal{U}_h$ to $y_h \in \mathcal{Y}_h$. The discrete reduced

cost functional $J_h : \mathcal{U}_h \rightarrow \mathbb{R}$ is now given by

$$\hat{J}_h(u) := \frac{1}{2} \|\mathcal{S}_h u - y_d\|_{L^2(\Omega)}^2 + \frac{\beta}{2} \|u\|_{L^2(\partial\Omega)}.$$

The following two theorems appear in [47].

Theorem 3.2.4 *The functional $J_h : \mathcal{U}_h \rightarrow \mathbb{R}$ is Frechet differentiable. Its derivative at $u \in \mathcal{U}_h$ in the direction $\delta u \in \mathcal{U}_h$ is given by*

$$\hat{J}'_h(u)\delta u = \langle (\mathcal{S}_h^*(\mathcal{S}_h u - y_d) + \beta u), \delta u \rangle,$$

where \mathcal{S}_h^* denotes the adjoint operator of \mathcal{S}_h .

Theorem 3.2.5 *The discrete optimal control problem*

$$\text{minimize}_{u \in \mathcal{U}_h} \hat{J}_h(u)$$

has a unique solution $\bar{u} \in \mathcal{U}_h$. Let $\bar{y} = \mathcal{S}_h \bar{u}$ be the discrete state, associated with \bar{u} .

Then the variational inequality

$$\langle \mathcal{S}_h^*(\mathcal{S}_h \bar{u} - y_d) + \beta \bar{u}, \delta u \rangle = 0 \quad \forall \delta u \in \mathcal{U}_h \tag{3.16}$$

is satisfied.

In general, the second variation of the reduced cost functional defines the Hessian operator

$$\langle \mathcal{H}_h(u)v_1, v_2 \rangle = \hat{J}_h''(u) \langle v_1, v_2 \rangle, \quad \forall v_1, v_2 \in \mathcal{U}_h, \quad (3.17)$$

since

$$\hat{J}_h''(u) \langle v_1, v_2 \rangle = \langle (\mathcal{S}_h^* \mathcal{S}_h + \beta I)v_1, v_2 \rangle \quad \forall v_1, v_2 \in \mathcal{U}_h, \quad (3.18)$$

it follows that

$$\mathcal{H}_h = \mathcal{S}_h^* \mathcal{S}_h + \beta I. \quad (3.19)$$

3.2.2 Computation of adjoints

Given $\tilde{y} \in L^2(\Omega)$, we define $z \in \mathcal{Y}$ to be the weak solution of the variational inequality

$$a(\varphi, z) = (\varphi, \tilde{y}), \quad \forall \varphi \in \mathcal{Y}. \quad (3.20)$$

which represents the weak formulation of the elliptic Neumann-boundary value problem

$$-\sum_{i,j=1}^d \partial_j \left(\sum_{j=1}^d a_{ij} \partial_i z \right) - \sum_{j=1}^d b_j \partial_i z + c z = \tilde{y} \quad \frac{\partial z}{\partial \nu} = 0. \quad (3.21)$$

If we denote by $z = \mathcal{S}_2 \tilde{y}$ the solution operator of (3.20), then

$$(\mathcal{S}u, \tilde{y}) = (y, \tilde{y}) \stackrel{(3.20)}{=} a(y, z) \stackrel{(3.7)}{=} \langle u, \mathcal{R}z \rangle = \langle u, (\mathcal{R} \cdot \mathcal{S}_2) \tilde{y} \rangle .$$

Therefore the adjoint \mathcal{S}^* of \mathcal{S} , regarded as operator in $\mathcal{L}(\mathcal{U})$, is

$$\mathcal{S}^* = \mathcal{R} \cdot \mathcal{S}_2 . \quad (3.22)$$

Given $\tilde{y} \in L^2(\Omega)$, let $z_h \in \mathcal{Y}_h$ satisfy

$$a(\varphi, z_h) = (\varphi, \tilde{y}), \quad \forall \varphi \in \mathcal{Y}_h , \quad (3.23)$$

and denote by $z_h = \mathcal{S}_{2,h} \tilde{y}$ the solution operator of (3.23). Then

$$(\mathcal{S}_h u, \tilde{y}) = (y_h, \tilde{y}) \stackrel{(3.23)}{=} a(y_h, z_h) \stackrel{(3.15)}{=} \langle u, \mathcal{R}z_h \rangle_{L^2(\partial\Omega)} = \langle u, (\mathcal{R} \cdot \mathcal{S}_{2,h}) \tilde{y} \rangle_{L^2(\partial\Omega)} .$$

Hence

$$\mathcal{S}_h^* \tilde{y} = \mathcal{R} \cdot \mathcal{S}_{2,h} .$$

The following estimate that is taken from [48], holds for \mathcal{S}^* .

Lemma 3.2.6 *The adjoint operator \mathcal{S}^* is defined from $L^2(\Omega) \rightarrow H^{\frac{3}{2}}(\partial\Omega)$ and satisfies*

$$\|\mathcal{S}^* v\|_{H^{\frac{3}{2}}(\partial\Omega)} \leq c \|v\|_{L^2(\Omega)} \forall v \in L^2(\Omega). \quad (3.24)$$

Proof. The operator \mathcal{S}_2 takes L^2 -functions to H^2 -functions and the restriction operator takes H^2 -functions to $H^{\frac{3}{2}}$ -functions. ■

3.2.3 Two-grid preconditioner for the discrete Hessian

In this section, we use the techniques developed in [20] to construct a two-grid preconditioner for the discrete Hessian \mathcal{H}_h defined in (3.19). The two-grid preconditioner can be extended to multigrid. Let $\pi_{2h} : \mathcal{U} \rightarrow \mathcal{U}_{2h}$ be the L^2 -projection. When the L^2 - projection applies to a discrete function it extracts the smooth part of it and $(I - \pi_{2h})$ extracts the oscillatory part of the discrete function. The $\mathcal{S}_h^* \mathcal{S}_h$ is a smoothing operator so $(\beta I + \mathcal{S}_h^* \mathcal{S}_h)(I - \pi_{2h}) = \beta(I - \pi_{2h})$. Therefore we have

$$\mathcal{H}_h = (\pi_{2h} + (I - \pi_{2h}))(\mathcal{S}_h^* \mathcal{S}_h + \beta I)(\pi_{2h} + (I - \pi_{2h})) \approx \pi_{2h}(\mathcal{S}_h^* \mathcal{S}_h + \beta I) + \beta(I - \pi_{2h}).$$

Hence the discrete Hessian \mathcal{H}_h is well approximated by

$$\mathcal{M}_h = \mathcal{H}_h \pi_{2h} + \beta(I - \pi_{2h}),$$

and the inverse of it is computed as

$$\mathcal{M}_h^{-1} = \mathcal{H}_h^{-1} \pi_{2h} + \beta^{-1}(I - \pi_{2h}).$$

To evaluate the quality of the preconditioner we use the spectral distance introduced in [20] defined as

$$1 - Ch^\alpha \leq \frac{\langle \mathcal{H}_h u, u \rangle}{\langle \mathcal{M}_h u, u \rangle} \leq 1 + Ch^\alpha, \quad \forall u \neq 0 \quad (3.25)$$

where h is a mesh size, C is independent of h , and an exponent $\alpha \geq 0$ is as large as possible. In general, α is not expected to exceed the order of convergence of the discretization. Since in discretization of the PDE we used discontinuous piecewise linear polynomials for control, we expect to see $\alpha = 2$ which is the optimal order. Numerical results suggests that $\alpha = 2$, so the preconditioner appears to be of optimal-order. To confirm the numerical results we need to find a norm $\|\cdot\|$ on \mathcal{U}_h such that the symmetric (same norm on both sides) estimate holds

$$\|(\mathcal{M}_h - \mathcal{H}_h)u\| \leq Ch^2 \|u\|, \quad \forall u \in \mathcal{U}_h. \quad (3.26)$$

However, the estimate resulting from our analysis appears to be less than half of the one we observed numerically, namely

$$\|(\mathcal{M}_h - \mathcal{H}_h)u\| \leq Ch^{\frac{3}{2}} \|u\|, \quad \forall u \in \mathcal{U}_h. \quad (3.27)$$

The rest of this chapter is devoted to showing that the above estimate holds with $\|\cdot\| = \|\cdot\|_{L^2(\partial\Omega)}$.

3.2.4 Error estimates

We extract from [47] the following estimate(s).

Theorem 3.2.7 *There exists a constant C independent of h so that*

$$\|(\mathcal{S} - \mathcal{S}_h)u\|_{L^2(\Omega)} \leq Ch^{\frac{3}{2}} \|u\|_{L^2(\partial\Omega)}, \quad \forall u \in \mathcal{U}_h, \quad (3.28)$$

$$\|(\mathcal{S} - \mathcal{S}_h)u\|_{H^1(\Omega)} \leq Ch^{\frac{1}{2}} \|u\|_{L^2(\partial\Omega)}, \quad \forall u \in \mathcal{U}_h. \quad (3.29)$$

We extract from [49] the following theorem.

Theorem 3.2.8 *There exists a constant C independent of h so that*

$$\|\mathcal{S}_h^* z\|_{L^\infty(\partial\Omega)} \leq C \|z\|_{L^2(\Omega)}, \quad (3.30)$$

$$\|\mathcal{R}y\|_{L^\infty(\partial\Omega)} \leq C \|\mathcal{S}_{2,h}y\|_{L^\infty(\Omega)}. \quad (3.31)$$

The following estimate holds.

Lemma 3.2.9 *The following estimate holds for \mathcal{S}*

$$\|\mathcal{S}u\|_{L^2(\Omega)} \leq c \|u\|_{H^{-\frac{3}{2}}(\partial\Omega)}. \quad (3.32)$$

Proof. Let $u \in \mathcal{U}$

$$\begin{aligned}
\|\mathcal{S}u\|_{L^2(\Omega)}^2 &= (\mathcal{S}u, \mathcal{S}u) \\
&= (u, \mathcal{S}^* \mathcal{S}u) \\
&\leq \|u\|_{H^{-\frac{3}{2}}(\partial\Omega)} \|\mathcal{S}^* \mathcal{S}u\|_{H^{\frac{3}{2}}(\partial\Omega)} \\
&\stackrel{(3.24)}{\leq} \|u\|_{H^{-\frac{3}{2}}(\partial\Omega)} \|\mathcal{S}u\|_{L^2(\Omega)}.
\end{aligned}$$

By dividing both sides by $\|\mathcal{S}u\|_{L^2(\partial\Omega)}$ and taking the supremum, we obtain (3.32).

■

Theorem 3.2.10 *There exists a constant C independent of h such that*

$$\|(\mathcal{S}^* - \mathcal{S}_h^*)y\|_{L^2(\partial\Omega)} \leq Ch^{\frac{3}{2}} \|y\|_{L^2(\Omega)} \quad \forall y \in \mathcal{Y}. \quad (3.33)$$

Proof. Let $y \in \mathcal{Y}$,

$$\begin{aligned}
\langle (\mathcal{S}^* - \mathcal{S}_h^*)y, \phi \rangle &= (y, (\mathcal{S} - \mathcal{S}_h)\phi) \\
&\leq \|y\|_{L^2(\Omega)} \|(\mathcal{S} - \mathcal{S}_h)\phi\|_{L^2(\Omega)} \\
&\stackrel{(3.28)}{\leq} Ch^{\frac{3}{2}} \|y\|_{L^2(\Omega)} \|\phi\|_{L^2(\partial\Omega)}.
\end{aligned}$$

By dividing both sides by $\|\phi\|_{L^2(\partial\Omega)}$ and taking the supremum we get,

$$\|(\mathcal{S}^* - \mathcal{S}_h^*)y\|_{L^2(\partial\Omega)} \leq Ch^{\frac{3}{2}} \|y\|_{L^2(\Omega)}.$$

■

Theorem 3.2.11 *There exists a constant C independent of h such that*

$$\|\mathcal{S}_h^*(\mathcal{S} - \mathcal{S}_h)u\|_{L^2(\partial\Omega)} \leq Ch^{\frac{3}{2}}\|u\|_{L^2(\partial\Omega)} \quad \forall u \in \mathcal{U}_h. \quad (3.34)$$

Proof. Let $u \in \mathcal{U}_h$,

$$\begin{aligned} \|\mathcal{S}_h^*(\mathcal{S} - \mathcal{S}_h)u\|_{L^2(\partial\Omega)} &\leq C\|(\mathcal{S} - \mathcal{S}_h)u\|_{L^2(\Omega)} \\ &\stackrel{(3.28)}{\leq} Ch^{\frac{3}{2}}\|u\|_{L^2(\partial\Omega)}. \end{aligned}$$

■

Theorem 3.2.12 *There exists c independent of h so that*

$$\|\mathcal{S}_h^*\mathcal{S}_hu - \mathcal{S}^*\mathcal{S}u\|_{L^2(\partial\Omega)} \leq ch^{\frac{3}{2}}\|u\|_{L^2(\partial\Omega)} \quad \forall u \in \mathcal{U}_h. \quad (3.35)$$

Proof. Let $u \in \mathcal{U}_h$,

$$\begin{aligned} \|\mathcal{S}_h^*\mathcal{S}_hu - \mathcal{S}^*\mathcal{S}u\|_{L^2(\partial\Omega)} &\leq \|(\mathcal{S}^* - \mathcal{S}_h^*)\mathcal{S}u\|_{L^2(\partial\Omega)} + \|\mathcal{S}_h^*(\mathcal{S} - \mathcal{S}_h)u\|_{L^2(\partial\Omega)} \\ &\stackrel{(3.33), (3.34)}{\leq} ch^{\frac{3}{2}}\|u\|_{L^2(\partial\Omega)} + ch^{\frac{3}{2}}\|u\|_{L^2(\partial\Omega)} \\ &\leq ch^{\frac{3}{2}}\|u\|_{L^2(\partial\Omega)}. \end{aligned}$$

■

3.3 The two-grid preconditioner

The Hessian

$$\mathcal{H}_h \stackrel{\text{def}}{=} \mathcal{S}_h^* \mathcal{S}_h + \beta I \quad (3.36)$$

would have been introduced in an earlier section as well as the L^2 -projector $\pi_h : \mathcal{U} \rightarrow \mathcal{U}_h$. In this section we estimate the quality of the two-grid preconditioner

$$\mathcal{M}_h \stackrel{\text{def}}{=} \beta(I - \pi_{2h}) + \mathcal{H}_{2h} \pi_{2h} . \quad (3.37)$$

This type of preconditioner was introduced in [20], and it is expected to be efficient if the operator $\mathcal{S}_h^* \mathcal{S}_h$ is of integral type.

Theorem 3.3.1 *There exists a constant C independent of h so that*

$$\|(\mathcal{H}_h - \mathcal{M}_h)u\|_{L^2(\partial\Omega)} \leq Ch^{\frac{3}{2}} \|u\|_{L^2(\partial\Omega)}, \quad \forall u \in \mathcal{U}_h . \quad (3.38)$$

Proof. A simple verification shows that

$$\begin{aligned} \mathcal{H}_h - \mathcal{M}_h &= \mathcal{S}_h^* \mathcal{S}_h - \mathcal{S}_{2h}^* \mathcal{S}_{2h} \pi_{2h} \\ &= (\mathcal{S}_h^* \mathcal{S}_h - \mathcal{S}^* \mathcal{S}) + \mathcal{S}^* \mathcal{S} (I - \pi_{2h}) + (\mathcal{S}^* \mathcal{S} - \mathcal{S}_{2h}^* \mathcal{S}_{2h}) \pi_{2h}. \end{aligned}$$

We showed that

$$\|(\mathcal{S}_h^* \mathcal{S}_h - \mathcal{S}^* \mathcal{S})u\|_{L^2(\partial\Omega)} \leq Ch^{\frac{3}{2}} \|u\|_{L^2(\partial\Omega)}.$$

Similarly, we obtain

$$\|(\mathcal{S}^* \mathcal{S} - \mathcal{S}_{2h}^* \mathcal{S}_{2h})\pi_{2h}u\|_{L^2(\partial\Omega)} \leq Ch^{\frac{3}{2}} \|\pi_{2h}u\|_{L^2(\partial\Omega)} \leq Ch^{\frac{3}{2}} \|u\|_{L^2(\partial\Omega)}.$$

Hence, for $u \in \mathcal{U}_h$ we have

$$\begin{aligned} \|(\mathcal{H}_h - \mathcal{M}_h)u\|_{L^2(\partial\Omega)} &\leq \|(\mathcal{S}_h^* \mathcal{S}_h - \mathcal{S}^* \mathcal{S})u\|_{L^2(\partial\Omega)} \\ &\quad + \|\mathcal{S}^* \mathcal{S}(I - \pi_{2h})u\|_{L^2(\partial\Omega)} \\ &\quad + \|(\mathcal{S}^* \mathcal{S} - \mathcal{S}_{2h}^* \mathcal{S}_{2h})\pi_{2h}u\|_{L^2(\partial\Omega)}. \end{aligned} \quad (3.39)$$

The first and last terms on the right-hand side of (3.39) are bound from above by $Ch^{\frac{3}{2}} \|u\|_{L^2(\partial\Omega)}$ for some appropriate constant. It remains to evaluate the middle term.

$$\begin{aligned} \|\mathcal{S}^* \mathcal{S}(I - \pi_{2h})u\|_{L^2(\partial\Omega)} &\leq C \|\mathcal{S}(I - \pi_{2h})u\|_{L^2(\Omega)} \\ &\stackrel{(3.32)}{\leq} C \|(I - \pi_{2h})u\|_{H^{-\frac{3}{2}}(\partial\Omega)} \\ &\leq Ch^{\frac{3}{2}} \|u\|_{L^2(\partial\Omega)}. \end{aligned}$$

Hence we obtain

$$\|\mathcal{H}_h - \mathcal{M}_h\|_{L^2(\partial\Omega)} \leq ch^{\frac{3}{2}} \|u\|_{L^2(\partial\Omega)}.$$

■

3.4 Numerical results

For large-scale problems, since the Hessian is dense, we do not form it. We can compute matrix vector products at a cost equivalent to approximately two PDE solves. We show numerically, how well the preconditioner approximate the Hessian. The following tables show the joint spectral analysis of the preconditioner and the Hessian. The spectral distance is a measure of spectral equivalence between two operators. We build the matrices representing \mathcal{H}_h and \mathcal{M}_h and compute the joint spectrum. Finally, we compute

$$d_h = \max\{|\ln \lambda| : \lambda \in \sigma(\mathcal{H}_h, \mathcal{M}_h)\}.$$

In order to do so we consider

$$h_0 = 2^{-3}, h_n = 2^{-n}h_0.$$

From the tables it is observed that the spectral distance between constructed preconditioner and the Hessian is $\mathcal{O}(h^2)$, which is optimal. As the resolution increases d_h decreases. We consider two different values for the regularization parameter β and obtain the following results.

$\beta = 1$						
n	0	1	2	3	4	5
d_{h_n}	1.59e-03	4.03-e04	1.01e-04	2.53e-05	6.32e-06	1.58e-06
$\log_2(d_{h_n}/d_{h_{n+1}})$	1.9766	1.9819	1.9960	1.9992	1.9999	–
$\beta = 10^{-2}$						
d_{h_n}	5.38e-02	1.33e-02	3.30e-03	8.25e-04	2.06e-04	5.15e-05
$\log_2(d_{h_n}/d_{h_{n+1}})$	2.0206	2.0065	2.0025	2.0010	2.0005	–

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