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# Some log and weak majorization inequalities in Euclidean Jordan algebras

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May 11, 2020

## Abstract

Motivated by Horn's log-majorization (singular value) inequality  $s(AB) \prec_{\log} s(A) * s(B)$  and the related weak-majorization inequality  $s(AB) \prec_w s(A) * s(B)$  for square complex matrices, we consider their Hermitian analogs  $\lambda(\sqrt{AB}\sqrt{A}) \prec_{\log} \lambda(A) * \lambda(B)$  for positive semidefinite matrices and  $\lambda(|A \circ B|) \prec_w \lambda(|A|) * \lambda(|B|)$  for general (Hermitian) matrices, where  $A \circ B$  denotes the Jordan product of  $A$  and  $B$  and  $*$  denotes the componentwise product in  $\mathcal{R}^n$ . In this paper, we extended these inequalities to the setting of Euclidean Jordan algebras in the form  $\lambda(P_{\sqrt{a}}(b)) \prec_{\log} \lambda(a) * \lambda(b)$  for  $a, b \geq 0$  and  $\lambda(|a \circ b|) \prec_w \lambda(|a|) * \lambda(|b|)$  for all  $a$  and  $b$ , where  $P_u$  and  $\lambda(u)$  denote, respectively, the quadratic representation and the eigenvalue vector of an element  $u$ . We also describe inequalities of the form  $\lambda(|A \bullet b|) \prec_w \lambda(\text{diag}(A)) * \lambda(|b|)$ , where  $A$  is a real symmetric

positive semidefinite matrix and  $A \bullet b$  is the Schur product of  $A$  and  $b$ . In the form of applications, we prove the generalized Hölder type inequality  $\|a \circ b\|_p \leq \|a\|_r \|b\|_s$ , where  $\|x\|_p := \|\lambda(x)\|_p$  denotes the spectral  $p$ -norm of  $x$  and  $p, q, r \in [1, \infty]$  with  $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$ . We also give precise values of the norms of the Lyapunov transformation  $L_a$  and  $P_a$  relative to two spectral  $p$ -norms.

**Key Words:** Euclidean Jordan algebra, log and weak majorization, Schur product.

**AMS Subject Classification:** 15A42, 17C20.

## 1 Introduction

In matrix theory, the well-known Horn's log-majorization inequality ([13], Corollary 6.14) asserts that for any two  $n \times n$  complex matrices  $A$  and  $B$ ,

$$s(AB) \underset{\log}{\prec} s(A) * s(B), \quad (1)$$

where  $s(X)$  denotes the vector of singular values of  $X$  written in the decreasing order and  $*$  denotes the componentwise product of vectors in  $\mathcal{R}^n$ . A simple consequence of (1) is the weak-majorization inequality

$$s(AB) \underset{w}{\prec} s(A) * s(B). \quad (2)$$

When  $A$  and  $B$  are Hermitian, the product  $AB$  need not be Hermitian and so, to stay within the realm of Hermitian matrices, we consider the following Hermitian analogs.

- When  $A$  and  $B$  are (Hermitian and) positive semidefinite,

$$\lambda\left(\sqrt{AB}\sqrt{A}\right) \underset{\log}{\prec} \lambda(A) * \lambda(B),$$

where, for a complex Hermitian matrix  $X$ ,  $\lambda(X)$  denotes the vector of eigenvalues of  $X$  written in the decreasing order. This can be seen from (1) by using the fact that the eigenvalues of  $AB$  are the same as those of  $\sqrt{AB}\sqrt{A}$ , see the proof of Corollary III.4.6 in [2].

As an immediate consequence, we have

$$\lambda\left(\sqrt{AB}\sqrt{A}\right) \underset{w}{\prec} \lambda(A) * \lambda(B).$$

- When  $A$  and  $B$  are Hermitian,

$$\lambda(|A \circ B|) \underset{w}{\prec} \lambda(|A|) * \lambda(|B|),$$

where  $A \circ B := \frac{AB+BA}{2}$  is the Jordan product of  $A$  and  $B$ , and  $|A| := \sqrt{A^*A}$ , etc. This inequality can be seen from (1) by using the majorization inequality  $s(X+Y) \underset{w}{\prec} s(X) + s(Y)$

([13], Corollary 6.12), see Section 4 for an elaboration.

Now, these Hermitian inequalities can be viewed as describing (pointwise) eigenvalue majorization inequalities of two linear transformations  $L_A$  and  $P_A$  on the space  $\mathcal{H}^n$  of all  $n \times n$  complex Hermitian matrices, where

$$L_A(X) := A \circ X \quad \text{and} \quad P_A(X) := AXA \quad (X \in \mathcal{H}^n).$$

Noting that  $\mathcal{H}^n$  is a Euclidean Jordan algebra with the Jordan product  $X \circ Y := \frac{XY+YX}{2}$  and the inner product  $\langle X, Y \rangle := \text{tr}(XY)$ , we ask if such inequalities can be established in the setting of general Euclidean Jordan algebras. The main objective of this paper is to provide an affirmative answer.

Let  $(\mathcal{V}, \circ, \langle \cdot, \cdot \rangle)$  denote a Euclidean Jordan algebra of rank  $n$  with  $\circ$  denoting the Jordan product and  $\langle \cdot, \cdot \rangle$  denoting the (trace) inner product;  $\mathcal{S}^n$ , the space of all  $n \times n$  real symmetric matrices, and  $\mathcal{H}^n$ , the space of all  $n \times n$  complex Hermitian matrices are two primary examples. For  $x \in \mathcal{V}$ , let  $\lambda(x)$  denote the vector of eigenvalues of  $x$  written in the decreasing order. Given  $x, y \in \mathcal{V}$ , we say that  $x$  is *weakly majorized* by  $y$  and write  $x \prec_w y$  if  $\lambda(x) \prec_w \lambda(y)$  in  $\mathcal{R}^n$ , which means that for each index  $k$ ,  $1 \leq k \leq n$ ,  $\sum_{i=1}^k \lambda_i(x) \leq \sum_{i=1}^k \lambda_i(y)$ ; additionally, if the equality holds when  $k = n$ , we say that  $x$  is *majorized* by  $y$  and write  $x \prec y$ . Replacing sums by products, one defines *weak log-majorization* and *log-majorization*, respectively denoted by  $x \prec_{w \log} y$  and  $x \prec_{\log} y$ . The above concepts have been extensively studied in matrix theory and other areas, see for example, [18, 2, 13]. In the setting of Euclidean Jordan algebras, they have been recently studied in several papers [19, 20, 5, 6, 7].

On  $\mathcal{V}$  we consider two linear transformations,  $L_a$ , the *Lyapunov transformation of  $a$* , and  $P_a$ , the *quadratic representation of  $a$* , defined on  $\mathcal{V}$  as follows:

$$L_a(x) := a \circ x \quad \text{and} \quad P_a(x) := 2a \circ (a \circ x) - a^2 \circ x \quad (a, x \in \mathcal{V}). \quad (3)$$

(It turns out that in the algebras  $\mathcal{S}^n$  and  $\mathcal{H}^n$ ,  $P_A(X) = AXA$ .)

Generalizing the (Hermitian) matrix inequalities stated above, we show the following: For  $a, b \in \mathcal{V}$ ,

$$\lambda(P_{\sqrt{a}}(b)) \prec_{\log} \lambda(a) * \lambda(b) \quad (a, b \geq 0), \quad (4)$$

$$\lambda(|P_a(b)|) \prec_w \lambda(a^2) * \lambda(|b|), \quad (5)$$

and

$$\lambda(|L_a(b)|) = \lambda(|a \circ b|) \prec_w \lambda(|a|) * \lambda(|b|). \quad (6)$$

One novelty here is that our proofs do not rely on matrix theory results. Rather, they are based on

general results on Euclidean Jordan algebras and on well known majorization inequalities in  $\mathcal{R}^n$ . The proof of (4) is modeled after a proof of Gelfand-Naimark Theorem presented in [12] (see [13], Chapter 6, where it is attributed to [16]). We prove (5) by combining (4) with the inequality

$$|P(b)| \underset{w}{\prec} P(|b|) \quad (7)$$

that is valid for all positive linear transformations  $P$  on  $\mathcal{V}$  and  $b$ . The inequality (6) is established by relying heavily on the so-called Fan-Theobald-von Neumann inequality [1]

$$\langle x, y \rangle \leq \langle \lambda(x), \lambda(y) \rangle \quad (x, y \in \mathcal{V}).$$

As we shall elaborate in the subsequent sections, both  $P_a(x)$  and  $L_a(x)$  can be described by means of Schur products of the form  $A \bullet x$  (relative to the Peirce decomposition coming from a Jordan frame), where  $A = [a_i a_j]$  in the case of  $P_a$  and  $A = [\frac{a_i + a_j}{2}]$  in the case of  $L_a$ , with  $a_1, a_2, \dots, a_n$  denoting the eigenvalues of  $a$ . Then, the inequalities (5) and (6) take the form

$$\lambda(|A \bullet b|) \underset{w}{\prec} \lambda(|\text{diag}(A)|) * \lambda(|b|), \quad (8)$$

where  $\text{diag}(A)$  denotes the diagonal vector of  $A$ .

Going beyond the case of  $A = [a_i a_j]$ , we show that (8) is valid for all real symmetric positive semidefinite matrices  $A$  and raise the *problem of characterizing  $A$  for which (8) holds*.

In the form of applications, we describe some norm inequalities. Given  $p \in [1, \infty]$ , we consider the spectral  $p$ -norm on  $\mathcal{V}$ :  $\|x\|_p := \|\lambda(x)\|_p$ , where the latter norm is computed in  $\mathcal{R}^n$ . Under the assumption that (8) holds for all  $b$ , we show that

$$\|A \bullet b\|_p \leq \|\text{diag}(A)\|_r \|b\|_s \quad (b \in \mathcal{V})$$

whenever  $p, q, r \in [1, \infty]$  with  $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$ . When specialized, this yields the generalized Hölder type inequality

$$\|a \circ b\|_p \leq \|a\|_r \|b\|_s \quad (a, b \in \mathcal{V}).$$

We also compute the norms of  $L_a$  and  $P_a$  relative to two spectral  $p$ -norms.

The organization of the paper is as follows. In Section 2, we cover some preliminary material on Euclidean Jordan algebras and majorization results in  $\mathcal{R}^n$ . In Section 3, we will provide the proof of the log-majorization inequality (4), describe results on positive transformations, and prove the inequality (5). Here, we will also prove the inequality (8) for real symmetric positive semidefinite matrices. Section 4 deals with the proof of the inequality (6). Applications, describing the generalized Hölder type inequality and norms of  $L_a$  and  $P_a$  will be covered in Section 5.

## 2 Preliminaries

Throughout this paper,  $\mathcal{R}^n$  denotes the Euclidean  $n$ -space whose elements are regarded as column vectors or row vectors depending on the context. For elements  $p, q \in \mathcal{R}^n$ ,  $p * q$  denotes their componentwise product. For  $p \in \mathcal{R}^n$ ,  $p^\downarrow$  and  $|p|$  denote, respectively, the decreasing rearrangement and the vector of absolute values of entries of  $p$ . Borrowing the notation used in the Introduction, we recall the following results.

**Proposition 2.1** ([2], Problem II.5.16, Example II.3.5, Exercise II.3.2; [18], A.7.(ii), p. 173 and [3], p.136)

- (a) Let  $p, q \in \mathcal{R}^n$ ,  $r \in \mathcal{R}_+^n$ , and  $p \prec_w q$ . Then  $p^\downarrow * r^\downarrow \prec_w q^\downarrow * r^\downarrow$ .
- (b) Let  $p, q \in \mathcal{R}^n$ . Then  $p \prec q \Rightarrow |p| \prec_w |q|$ .
- (c) Let  $p, q \in \mathcal{R}^n$  and  $I$  be an interval in  $\mathcal{R}$  that contains all the entries of  $p$  and  $q$ . Then  $p \prec_w q \iff \sum_{i=1}^n \phi(p_i) \leq \sum_{i=1}^n \phi(q_i)$  for every increasing convex function  $\phi : I \rightarrow \mathcal{R}$ . Moreover, if  $p \prec_w q$  and  $\sum_{i=1}^n \phi(p_i) = \sum_{i=1}^n \phi(q_i)$  for some increasing strictly convex function  $\phi$ , then  $p^\downarrow = q^\downarrow$ .

Throughout, we let  $(\mathcal{V}, \circ, \langle \cdot, \cdot \rangle)$  denote a Euclidean Jordan algebra of rank  $n$  with unit element  $e$  [4, 9]; the Jordan product and inner product of elements  $x$  and  $y$  in  $\mathcal{V}$  are respectively denoted by  $x \circ y$  and  $\langle x, y \rangle$ . It is well known [4] that any Euclidean Jordan algebra is a direct product/sum of simple Euclidean Jordan algebras and every simple Euclidean Jordan algebra is isomorphic to one of five algebras, three of which are the algebras of  $n \times n$  real/complex/quaternion Hermitian matrices. The other two are: the algebra of  $3 \times 3$  octonion Hermitian matrices and the Jordan spin algebra. In the algebras  $\mathcal{S}^n$  (of all  $n \times n$  real symmetric matrices) and  $\mathcal{H}^n$  (of all  $n \times n$  complex Hermitian matrices), the Jordan product and the inner product are given, respectively, by

$$X \circ Y := \frac{XY + YX}{2} \quad \text{and} \quad \langle X, Y \rangle := \text{tr}(XY),$$

where the trace of a real/complex matrix is the sum of its diagonal entries.

According to the *spectral decomposition theorem* [4], any element  $x \in \mathcal{V}$  has a decomposition

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n,$$

where the real numbers  $x_1, x_2, \dots, x_n$  are (called) the eigenvalues of  $x$  and  $\{e_1, e_2, \dots, e_n\}$  is a Jordan frame in  $\mathcal{V}$ . (An element may have decompositions coming from different Jordan frames, but the eigenvalues remain the same.) Then,  $\lambda(x)$  – called the *eigenvalue vector* of  $x$  – is the vector of eigenvalues of  $x$  written in the decreasing order. We write

$$\lambda(x) = (\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x)).$$

It is known that  $\lambda : \mathcal{V} \rightarrow \mathcal{R}^n$  is continuous [1].

We recall some standard definitions/results. The *rank* of an element  $x$  is the number of nonzero eigenvalues. An element  $x$  is said to be *invertible* if all its eigenvalues are nonzero; such elements form a dense subset of  $\mathcal{V}$ . We use the notation  $x \geq 0$  ( $x > 0$ ) when all the eigenvalues of  $x$  are nonnegative (respectively, positive) and  $x \geq y$  when  $x - y \geq 0$ , etc. The set of all elements  $x \geq 0$  (called the symmetric cone of  $\mathcal{V}$ ) is a self-dual cone. Given the spectral decomposition  $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$ , we define  $|x| := |x_1| e_1 + |x_2| e_2 + \cdots + |x_n| e_n$  and  $\sqrt{x} := \sqrt{x_1} e_1 + \sqrt{x_2} e_2 + \cdots + \sqrt{x_n} e_n$  when  $x \geq 0$ . The *trace and determinant* of  $x$  are defined by  $\text{tr}(x) := x_1 + x_2 + \cdots + x_n$  and  $\det(x) := x_1 x_2 \cdots x_n$ . Also, for  $p \in [1, \infty]$ , we define the corresponding *spectral norm*  $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$  when  $p < \infty$  and  $\|x\|_\infty = \max_i |x_i|$ .

For any  $a \in \mathcal{V}$ , we define  $L_a$  and  $P_a$  as in (3). We say that two elements  $a$  and  $b$  *operator commute* if  $L_a$  and  $L_b$  commute, or equivalently, if  $a$  and  $b$  have their spectral representations with respect to the same Jordan frame.

An element  $c \in \mathcal{V}$  is an *idempotent* if  $c^2 = c$ ; it is said to be a *primitive idempotent* if it is nonzero and cannot be written as the sum of two other nonzero idempotents. We write  $\mathcal{J}(\mathcal{V})$  for the set of all primitive idempotents and  $\mathcal{J}^{(k)}(\mathcal{V})$  ( $1 \leq k \leq n$ ) for the set of all idempotents of rank  $k$ .

It is known that  $(x, y) \mapsto \text{tr}(x \circ y)$  defines another inner product on  $\mathcal{V}$  that is compatible with the Jordan product. *Throughout this paper, we assume that the inner product on  $\mathcal{V}$  is this trace inner product, that is,  $\langle x, y \rangle = \text{tr}(x \circ y)$ .* In this inner product, the norm of any primitive element is one and so any Jordan frame in  $\mathcal{V}$  is an orthonormal set. Additionally,  $\text{tr}(x) = \langle x, e \rangle$  for all  $x \in \mathcal{V}$ .

Given an idempotent  $c$ , we have the *Peirce (orthogonal) decomposition* [4]:  $\mathcal{V} = \mathcal{V}(c, 1) + \mathcal{V}(c, \frac{1}{2}) + \mathcal{V}(c, 0)$ , where  $\mathcal{V}(c, \gamma) = \{x \in \mathcal{V} : x \circ c = \gamma x\}$  with  $\gamma \in \{1, \frac{1}{2}, 0\}$ . Then, any  $b \in \mathcal{V}$  can be decomposed as  $b = u + v + w$ , where  $u \in \mathcal{V}(c, 1)$ ,  $v \in \mathcal{V}(c, \frac{1}{2})$ , and  $w \in \mathcal{V}(c, 0)$ .

Below, we record some standard results that are needed in the sequel. We emphasize that  $\mathcal{V}$  has rank  $n$  and carries the trace inner product.

**Proposition 2.2** ([19], Theorem 6.1 or [5], page 54) *If  $c$  is an idempotent and  $x = u + v + w$ , where  $u \in \mathcal{V}(c, 1)$ ,  $v \in \mathcal{V}(c, \frac{1}{2})$  and  $w \in \mathcal{V}(c, 0)$ , then  $\lambda(u + w) \prec \lambda(x)$ .*

**Proposition 2.3** *Let  $x, y, c, u \in \mathcal{V}$ . Then,*

$$(i) \ S_k(x) := \lambda_1(x) + \lambda_2(x) + \cdots + \lambda_k(x) = \max_{c \in \mathcal{J}^{(k)}(\mathcal{V})} \langle x, c \rangle.$$

$$(ii) \ \langle x, y \rangle \leq \langle \lambda(x), \lambda(y) \rangle \leq \langle \lambda(|x|), \lambda(|y|) \rangle.$$

(iii)  $x \leq y \Rightarrow \lambda(x) \leq \lambda(y)$ .

(iv)  $P_c$  is a positive transformation, that is,  $x \geq 0 \Rightarrow P_c(x) \geq 0$ .

(v)  $\det P_c(u) = (\det c)^2 \det u$ .

(vi)  $\|x \circ y\|_p \leq \|x\|_p \|y\|_\infty$ , where  $p \in [1, \infty]$ . In particular,  $\|x \circ y\|_\infty \leq \|x\|_\infty \|y\|_\infty$ , or equivalently,  $\lambda_1(|x \circ y|) \leq \lambda_1(|x|) \lambda_1(|y|)$ .

In the above proposition, Item (i) is stated in [1], Lemma 20. The first inequality in (ii), known as the Fan-Theobald-von Neumann inequality, can be found in [1], Theorem 23; the second inequality – a particular case of the first inequality – is a simple consequence of the rearrangement inequality of Hardy-Littlewood-Pólya in  $\mathcal{R}^n$ :  $\langle p, q \rangle \leq \langle p^\downarrow, q^\uparrow \rangle$ . Item (iii) is a consequence of the well-known min-max theorem of Hirzebruch [14, 10]. Items (iv) and (v) are well-known properties of  $P_c$ , see, e.g., [4]. Items in (vi) follow from Theorem 3.2 in [6].

Given a Jordan frame  $\{e_1, e_2, \dots, e_n\}$  in  $\mathcal{V}$ , we consider the corresponding Peirce decomposition of  $\mathcal{V}$  and  $x \in \mathcal{V}$  ([4], Theorem IV.2.1):  $\mathcal{V} = \sum_{i \leq j} \mathcal{V}_{ij}$  and  $x = \sum_{i \leq j} x_{ij}$ , where  $x_{ij} \in \mathcal{V}_{ij}$  for all  $1 \leq i \leq j \leq n$ . Here,  $\mathcal{V}_{ii} := \mathcal{R} e_i$  for all  $i$  and  $\mathcal{V}_{ij} := \mathcal{V}(e_i, \frac{1}{2}) \cap \mathcal{V}(e_j, \frac{1}{2})$  for  $i < j$ . Then, for any matrix  $A = [a_{ij}] \in \mathcal{S}^n$ , we define

$$A \bullet x := \sum_{i \leq j} a_{ij} x_{ij}. \quad (9)$$

We call this the *Schur product of  $A$  and  $x$*  relative to the Jordan frame  $\{e_1, e_2, \dots, e_n\}$ . (This is a generalization of the usual Schur/Hadamard product of matrices in  $\mathcal{S}^n$ .)

Two primary examples: For any  $a \in \mathcal{V}$  with spectral decomposition  $a = \sum_{i=1}^n a_i e_i$ ,

$$L_a(x) = A \bullet x, \text{ where } A = \left[ \frac{a_i + a_j}{2} \right] \text{ and } P_a(x) = A \bullet x, \text{ where } A = [a_i a_j].$$

We refer to [11] for further examples and properties. We record a recent result that connects Schur products and quadratic representations.

**Proposition 2.4** ([7], Corollary 3.4) *Consider a Jordan frame  $\{e_1, e_2, \dots, e_n\}$ . Suppose  $A = [a_{ij}] \in \mathcal{S}^n$  is positive semidefinite and let  $a = a_{11}e_1 + \dots + a_{nn}e_n$ . Then,  $A \bullet x \prec P_{\sqrt{a}}(x)$  for all  $x \in \mathcal{V}$ .*

### 3 A log-majorization inequality

In this section, we prove the inequality (4). Its proof (along with that of Lemma 3.3 below) is modeled after techniques given in [12] (see also, [13], Theorem 6.13 and [16]). First, we present several lemmas.



**Lemma 3.1** ([17], Corollary 9) For  $a, b \geq 0$  in  $\mathcal{V}$ ,  $\lambda(P_{\sqrt{a}}(b)) = \lambda(P_{\sqrt{b}}(a))$ .

**Lemma 3.2** For  $a, b \geq 0$  in  $\mathcal{V}$ ,  $\lambda(P_{\sqrt{a}}(b)) \leq \|a\|_\infty \lambda(b)$ .

**Proof.** As  $a \geq 0$ , we have  $\lambda_1(a) = \|a\|_\infty$  and so,  $a \leq \lambda_1(a)e = \|a\|_\infty e$ . Since for any  $c$ ,  $P_c$  is a positive (linear) transformation,  $P_{\sqrt{b}}(a) \leq \|a\|_\infty P_{\sqrt{b}}(e) = \|a\|_\infty b$ . From Lemma 3.1 and Item (iii) in Proposition 2.3,

$$\lambda(P_{\sqrt{a}}(b)) = \lambda(P_{\sqrt{b}}(a)) \leq \lambda(\|a\|_\infty b) = \|a\|_\infty \lambda(b).$$

□

**Lemma 3.3** Suppose  $a \in \mathcal{V}$  is invertible with its spectral decomposition

$$a = \sum_{i=1}^n a_i e_i = \sum_{i=1}^n |a_i| \varepsilon_i e_i,$$

where  $|a_1| \geq |a_2| \geq \dots \geq |a_n|$  and  $\varepsilon_i = \text{sgn}(a_i)$  for all  $i$ . For  $1 \leq k \leq n$ , let

$$x := \sum_{i=1}^k \frac{|a_i|}{|a_k|} e_i + \sum_{j=k+1}^n e_j \quad \text{and} \quad y := \sum_{i=1}^k |a_k| \varepsilon_i e_i + \sum_{j=k+1}^n a_j e_j.$$

Then, the following statements hold:

- (i)  $x \geq e$ ,
- (ii)  $x$  and  $y$  operator commute,
- (iii)  $P_{\sqrt{x}}(y) = a$ ,  $P_x(y^2) = a^2$ ,
- (iv)  $(\det x) \|y\|_\infty^k = \prod_{i=1}^k |a_i|$ .

This lemma follows from direct verification.

**Theorem 3.4** Let  $a, b \geq 0$  in  $\mathcal{V}$ . Then,

$$\lambda(P_{\sqrt{a}}(b)) \underset{\log}{\prec} \lambda(a) * \lambda(b).$$

Consequently,  $\lambda(P_{\sqrt{a}}(b)) \underset{w}{\prec} \lambda(a) * \lambda(b)$ .

**Proof.** We have to show that for all  $k \in \{1, 2, \dots, n\}$ ,

$$\prod_{i=1}^k \lambda_i(P_{\sqrt{a}}(b)) \leq \prod_{i=1}^k \lambda_i(a) \lambda_i(b) \tag{10}$$

with equality when  $k = n$ . By continuity, it is enough to prove this statement for  $a, b > 0$ . So, in the rest of the proof, we assume that  $a, b > 0$  and fix  $k$ .

Corresponding to the spectral decomposition  $a = \sum_{i=1}^n a_i e_i$  (with  $a_1 \geq a_2 \geq \dots \geq a_n > 0$ ) and  $k$ ,

we define  $x$  and  $y$  as in Lemma 3.3:

$$x := \sum_{i=1}^k \frac{a_i}{a_k} e_i + \sum_{j=k+1}^n e_j \quad \text{and} \quad y := \sum_{i=1}^k a_k e_i + \sum_{j=k+1}^n a_j e_j.$$

In addition to the properties of  $x$  and  $y$  listed in Lemma 3.3, we observe that  $x, y \geq 0$ ,  $\sqrt{y} \circ \sqrt{x} = \sqrt{a}$ , and  $P_{\sqrt{y}} P_{\sqrt{x}} = P_{\sqrt{a}}$ . Now,  $x \geq e$  implies, via Lemma 3.1 and Proposition 2.3,

$$\lambda_i(P_{\sqrt{x}}(b)) = \lambda_i(P_{\sqrt{b}}(x)) \geq \lambda_i(P_{\sqrt{b}}(e)) = \lambda_i(b) \quad \text{for all } i = 1, 2, \dots, n.$$

Hence,

$$\frac{\lambda_i(P_{\sqrt{x}}(b))}{\lambda_i(b)} \geq 1 \quad \text{for all } i = 1, 2, \dots, n.$$

It follows that

$$\prod_{i=1}^k \frac{\lambda_i(P_{\sqrt{x}}(b))}{\lambda_i(b)} \leq \prod_{i=1}^n \frac{\lambda_i(P_{\sqrt{x}}(b))}{\lambda_i(b)} = \frac{\det P_{\sqrt{x}}(b)}{\det b} = \det x.$$

This implies that

$$\prod_{i=1}^k \lambda_i(P_{\sqrt{x}}(b)) \leq (\det x) \prod_{i=1}^k \lambda_i(b).$$

On the other hand, for any index  $i \in \{1, 2, \dots, n\}$ , from Lemma 3.2,

$$\lambda_i(P_{\sqrt{a}}(b)) = \lambda_i(P_{\sqrt{y}} P_{\sqrt{x}}(b)) \leq \|y\|_\infty \lambda_i(P_{\sqrt{x}}(b)).$$

Hence,

$$\prod_{i=1}^k \lambda_i(P_{\sqrt{a}}(b)) \leq \|y\|_\infty^k \prod_{i=1}^k \lambda_i(P_{\sqrt{x}}(b)) \leq \|y\|_\infty^k (\det x) \prod_{i=1}^k \lambda_i(b).$$

As  $(\det x) \|y\|_\infty^k = \prod_{i=1}^k a_i = \prod_{i=1}^k \lambda_i(a)$  (from Lemma 3.3), we see that

$$\prod_{i=1}^k \lambda_i(P_{\sqrt{a}}(b)) \leq \prod_{i=1}^k \lambda_i(a) \lambda_i(b).$$

This proves the inequality (10). Now suppose  $k = n$ . Then,

$$\prod_{i=1}^n \lambda_i(P_{\sqrt{a}}(b)) = \det P_{\sqrt{a}}(b) = (\det a) \det b = \prod_{i=1}^n \lambda_i(a) \lambda_i(b).$$

Finally, the weak-majorization inequality is an immediate consequence of the log-majorization inequality. This completes the proof.  $\square$

As a consequence of the above log-majorization inequality, we now prove a weak-majorization inequality dealing with quadratic representations. While our primary focus here is to prove (5),

it turns out that a general result dealing with Löwner maps of sublinear functions can be obtained without much difficulty. First, some relevant definitions.

For a function  $\phi : \mathcal{R} \rightarrow \mathcal{R}$ , the corresponding *Löwner map* (also denoted by)  $\phi : \mathcal{V} \rightarrow \mathcal{V}$  is defined as follows: For any  $x \in \mathcal{V}$  with spectral decomposition  $x = \sum_{i=1}^n x_i e_i$ ,  $\phi(x) := \sum_{i=1}^n \phi(x_i) e_i$ . We make one simple observation (using [2], Corollary II.3.4): When  $\phi$  is convex,

$$x \prec y \Rightarrow \phi(x) \prec_w \phi(y).$$

A function  $\phi : \mathcal{R} \rightarrow \mathcal{R}$  is said to be *sublinear* if

1.  $\phi(\mu t) = \mu \phi(t)$  for all  $\mu \geq 0$  and  $t \in \mathcal{R}$ ;
2.  $\phi(t + s) \leq \phi(t) + \phi(s)$  for all  $t, s \in \mathcal{R}$ .

It is easy to see that sublinear functions on  $\mathcal{R}$  are of the form  $\phi(t) = \alpha t$  for  $t \geq 0$  and  $\phi(t) = \beta t$  for  $t \leq 0$ , where (constants)  $\alpha, \beta \in \mathcal{R}$  satisfy  $\beta \leq \alpha$ . Particular examples are  $\phi(t) = |t|$ ,  $\max\{t, 0\}$ , and  $\max\{-t, 0\}$ .

**Theorem 3.5** *Let  $\phi : \mathcal{R} \rightarrow \mathcal{R}$  be a nonnegative sublinear function. Then, for all  $a, b \in \mathcal{V}$ ,*

$$\lambda(\phi(P_a(b))) \prec_w \lambda(a^2) * \lambda(\phi(b)).$$

*In particular,  $\lambda(|P_a(b)|) \prec_w \lambda(a^2) * \lambda(|b|)$ .*

We prove the above result by relying on the following lemma that may be of independent interest. It is motivated by a recent result in [15] which states that  $\phi(P_u(x)) \prec_w P_u(\phi(x))$  when  $\phi$  is a convex function on  $\mathcal{R}$  with  $\phi(0) \leq 0$  and  $u^2 \leq e$ . In the result below, we replace  $P_u$  by a positive linear transformation, but restrict  $\phi$  to sublinear functions.

**Lemma 3.6** *Suppose  $P : \mathcal{V} \rightarrow \mathcal{V}$  is a positive linear transformation and  $\phi : \mathcal{R} \rightarrow \mathcal{R}$  is sublinear. Then,*

$$\phi(P(x)) \prec_w P(\phi(x)) \quad (x \in \mathcal{V}).$$

**Proof.** We first claim, for any  $x \in \mathcal{V}$  and  $a \geq 0$ , the inequality

$$\phi(\langle x, a \rangle) \leq \langle \phi(x), a \rangle.$$

To see this, we write the spectral decomposition of  $x$  and use the definition of  $\phi(x)$ :

$$x = \sum_{i=1}^n x_i e_i \quad \text{and} \quad \phi(x) = \sum_{i=1}^n \phi(x_i) e_i.$$

Since  $\langle e_i, a \rangle \geq 0$  for all  $i$  and  $\phi$  is sublinear, it follows that

$$\phi(\langle x, a \rangle) = \phi\left(\sum_{i=1}^n x_i \langle e_i, a \rangle\right) \leq \sum_{i=1}^n \phi(x_i) \langle e_i, a \rangle = \langle \phi(x), a \rangle.$$

Hence the claim.

Now, we write the spectral decomposition of  $P(x)$  and use the definition of  $\phi(P(x))$ :

$$P(x) = \sum_{i=1}^n \alpha_i f_i, \quad \phi(P(x)) = \sum_{i=1}^n \phi(\alpha_i) f_i.$$

Since the eigenvalues of  $\phi(P(x))$  are  $\phi(\alpha_1), \dots, \phi(\alpha_n)$ , there exists a permutation  $\sigma$  on the set  $\{1, 2, \dots, n\}$  such that  $\lambda_i(\phi(P(x))) = \phi(\alpha_{\sigma(i)}) = \phi(\langle P(x), f_{\sigma(i)} \rangle)$  for all  $i$ . To simplify the notation, we let  $i' := \sigma(i)$ . Then, for any index  $k$ ,  $1 \leq k \leq n$ , we have

$$S_k(\phi(P(x))) = \sum_{i=1}^k \lambda_i(\phi(P(x))) = \sum_{i=1}^k \phi(\langle P(x), f_{i'} \rangle) = \sum_{i=1}^k \phi(\langle x, P^*(f_{i'}) \rangle),$$

where  $P^*$  is the adjoint of  $P$ . As the symmetric cone of  $\mathcal{V}$  is self-dual, we see that  $P^*$  is also a positive linear transformation, hence  $P^*(f_{i'}) \geq 0$ . By applying the above claim, we see that

$$\phi(\langle x, P^*(f_{i'}) \rangle) \leq \langle \phi(x), P^*(f_{i'}) \rangle = \langle P(\phi(x)), f_{i'} \rangle.$$

Hence,

$$S_k(\phi(P(x))) \leq \sum_{i=1}^k \langle P(\phi(x)), f_{i'} \rangle \leq \max_{c \in \mathcal{J}^{(k)}(\mathcal{V})} \langle P(\phi(x)), c \rangle = S_k(P(\phi(x))),$$

where the last equality follows from Item (i) in Proposition 2.3. As this inequality holds for all  $1 \leq k \leq n$ , we have  $\phi(P(x)) \prec_w P(\phi(x))$ .  $\square$

**Example 3.7** Let  $P$  be as in the above lemma. Taking  $\phi(t) = |t|$ ,  $\phi(t) = \max\{t, 0\}$ , or  $\phi(t) = \max\{-t, 0\}$ , we get the inequalities

$$|P(x)| \prec_w P(|x|), \quad P(x)^+ \prec_w P(x^+), \quad \text{and} \quad P(x)^- \prec_w P(x^-),$$

for any  $x \in \mathcal{V}$ .

**Proof of Theorem 3.5:** For the given  $a, b \in \mathcal{V}$ , we have  $a^2, |b| \geq 0$  and  $\sqrt{a^2} = |a|$ . By Theorem 3.4,  $\lambda(P_{\sqrt{a^2}}(|b|)) \prec_{\log} \lambda(a^2) * \lambda(|b|)$ . Since the inequality  $p \prec_{\log} q$  in  $\mathcal{R}_+^n$  implies that  $p \prec_w q$  (see [2], Example II.3.5), we see that

$$\lambda(P_{|a|}(|b|)) \prec_w \lambda(a^2) * \lambda(|b|).$$

Now, it is known that  $P_a(x) \prec P_{|a|}(x)$  for all  $x$ , see e.g., [7], page 11. Using this and the above lemma with  $P_a$  in place of  $P$ , we have

$$\lambda(\phi(P_a(b))) \prec_w \lambda(P_a(\phi(b))) \prec_w \lambda(P_{|a|}(\phi(b))) \prec_w \lambda(a^2) * \lambda(\phi(b)),$$

where we have used the condition that  $\phi$  is nonnegative (so  $\phi(t) \geq 0$  for all  $t \in \mathcal{R}$ , consequently,

$\phi(b) \geq 0$ .) Finally, by putting  $\phi(t) = |t|$ , we get  $\lambda(|P_a(b)|) \prec_w \lambda(a^2) * \lambda(|b|)$ .  $\square$

**Remark.** In a recent paper [6], it was shown that  $P_a(x) \prec L_{a^2}(x)$  for all  $a, x \in \mathcal{V}$ . This, in particular, implies that

$$\lambda(P_{\sqrt{a}}(b)) \prec \lambda(a \circ b) \quad \text{for all } a \geq 0 \text{ and } b \in \mathcal{V}. \quad (11)$$

In the next section, we will prove the inequality  $\lambda(|a \circ b|) \prec_w \lambda(|a|) * \lambda(|b|)$ . Based on these two results, we can give an alternative proof of the absolute-value case in the above theorem as follows:

$$P_a(b) \prec a^2 \circ b \Rightarrow \lambda(|P_a(b)|) \prec_w \lambda(|a^2 \circ b|) \prec_w \lambda(a^2) * \lambda(|b|).$$

We now extend Theorem 3.5 to Schur products. In what follows, for a matrix  $A \in \mathcal{S}^n$ ,  $\text{diag}(A)$  denotes the diagonal vector of  $A$  and (by abuse of notation)  $\lambda(\text{diag}(A))$  is the decreasing rearrangement of  $\text{diag}(A)$ .

**Theorem 3.8** *Let  $\phi : \mathcal{R} \rightarrow \mathcal{R}$  be a nonnegative sublinear function. Suppose  $A \in \mathcal{S}^n$  is a positive semidefinite matrix and  $b \in \mathcal{V}$ . Then, relative to any Jordan frame,  $\phi(A \bullet b) \prec_w A \bullet \phi(b)$  and*

$$\lambda(\phi(A \bullet b)) \prec_w \lambda(\text{diag}(A)) * \lambda(\phi(b)).$$

*In particular,*

$$\lambda(|A \bullet b|) \prec_w \lambda(\text{diag}(A)) * \lambda(|b|).$$

**Proof.** We fix a Jordan frame  $\{e_1, e_2, \dots, e_n\}$  relative to which all Schur products are defined. Since  $A$  is positive semidefinite, the map  $P : x \mapsto A \bullet x$  is a positive linear transformation ([11], Prop.2.2). The inequality  $\phi(A \bullet b) \prec_w A \bullet \phi(b)$  comes from an application of the above lemma. Now, letting  $A = [a_{ij}]$ , we define  $a := a_{11}e_1 + \dots + a_{nn}e_n$  and note that  $a \geq 0$ . From Proposition 2.4 we have  $A \bullet b \prec P_{\sqrt{a}}(b)$  for all  $b \in \mathcal{V}$ . This implies, by the convexity of  $\phi$ ,  $\phi(A \bullet b) \prec_w \phi(P_{\sqrt{a}}(b))$ . Applying the previous theorem,

$$\lambda(\phi(A \bullet b)) \prec_w \lambda(a) * \lambda(\phi(b)) = \lambda(\text{diag}(A)) * \lambda(\phi(b)).$$

$\square$

## 4 A weak-majorization inequality

In this section, we prove the following.

**Theorem 4.1** *Let  $a, b \in \mathcal{V}$ . Then*

$$\lambda(|a \circ b|) \prec_w \lambda(|a|) * \lambda(|b|). \quad (12)$$

As in the case of Theorem 3.4, the motivation comes from matrix theory. Consider two matrices  $A, B \in \mathcal{H}^n$ . Then, in the algebra  $\mathcal{H}^n$ ,  $A \circ B = \frac{AB+BA}{2}$  and so,

$$\lambda(|A \circ B|) = s(A \circ B) = s\left(\frac{AB + BA}{2}\right),$$

where we recall that  $s(X)$  denotes the vector of singular values of a matrix  $X$  written in the decreasing order. Invoking the inequality  $s(X + Y) \prec_w s(X) + s(Y)$  ([13], Corollary 6.12), we see that

$$s\left(\frac{AB + BA}{2}\right) \prec_w \frac{s(AB) + s(BA)}{2} \prec_w \frac{s(A) * s(B) + s(B) * s(A)}{2} = \lambda(|A|) * \lambda(|B|),$$

as  $s(A) = \lambda(|A|)$ ,  $s(AB) \prec_w s(A) * s(B)$ , etc. Thus,  $\lambda(|A \circ B|) \prec_w \lambda(|A|) * \lambda(|B|)$ . Theorem 4.1 is a generalization of this to Euclidean Jordan algebras. Before considering its proof, we present several lemmas.

In what follows, we let  $\varepsilon$  (likewise,  $\varepsilon'$ ) be an element in  $\mathcal{V}$  such as  $\varepsilon^2 = e$ . Clearly, such an element is of the form  $\sum_{i=1}^n \varepsilon_i e_i$ , where  $\varepsilon_i = 1$  for  $i = 1, 2, \dots, k$  and  $\varepsilon_j = -1$  for  $j = k + 1, k + 2, \dots, n$  ( $1 \leq k \leq n$ ) for some Jordan frame  $\{e_1, e_2, \dots, e_n\}$ .

**Lemma 4.2** *Let  $\varepsilon \in \mathcal{V}$  with  $\varepsilon^2 = e$  and  $b \in \mathcal{V}$ . Then  $\lambda(|b \circ \varepsilon|) \prec_w \lambda(|b|) * \lambda(|\varepsilon|) = \lambda(|b|)$ .*

**Proof.** As the result is obvious when  $\varepsilon = e$  or  $-e$ , we assume the spectral decomposition of  $\varepsilon$  in the form  $\varepsilon = \sum_{i=1}^n \varepsilon_i e_i$ , where  $1 \leq k < n$ ,  $\varepsilon_i = 1$  for  $i = 1, 2, \dots, k$  and  $\varepsilon_j = -1$  for  $j = k + 1, k + 2, \dots, n$ . We let  $c = \sum_{i=1}^k e_i$  and consider the Peirce decomposition  $b = u + v + w$ , where  $u \in \mathcal{V}(c, 1)$ ,  $v \in \mathcal{V}(c, \frac{1}{2})$  and  $w \in \mathcal{V}(c, 0)$ . A direct calculation leads to  $b \circ \varepsilon = u - w$ . Now, writing the spectral decompositions of  $u$  and  $w$  in the (sub)algebras  $\mathcal{V}(c, 1)$  and  $\mathcal{V}(c, 0)$  in the form  $u = \sum_{i=1}^k u_i f_i$  and  $w = \sum_{i=k+1}^n w_i f_i$  for some Jordan frame  $\{f_1, f_2, \dots, f_n\}$  in  $\mathcal{V}$ , we see that

$$|u - w| = \left| \sum_{i=1}^k u_i f_i - \sum_{i=k+1}^n w_i f_i \right| = \sum_{i=1}^k |u_i| f_i + \sum_{i=k+1}^n |w_i| f_i = |u + w|.$$

Thus,  $\lambda(|u - w|) = \lambda(|u + w|)$ . By Proposition 2.2,  $u + w \prec b$  and so  $\lambda(u + w) \prec \lambda(b)$ . By Item (b) in Proposition 2.1,  $|\lambda(u + w)| \prec_w |\lambda(b)|$ . Since  $\lambda(|b|)$  is the decreasing rearrangement of the vector  $|\lambda(b)|$ , we see that  $\lambda(|b \circ \varepsilon|) = \lambda(|u - w|) = \lambda(|u + w|) \prec_w \lambda(|b|)$ . This completes the proof.  $\square$

We note a simple consequence.

**Corollary 4.3** *For elements  $\varepsilon$  and  $\varepsilon'$  with  $\varepsilon^2 = e = (\varepsilon')^2$  and any idempotent  $c$  in  $\mathcal{V}$ , we have*

$$\lambda(|c \circ \varepsilon|) \prec_w \lambda(c) \quad \text{and} \quad \lambda(|\varepsilon' \circ (c \circ \varepsilon)|) \prec_w \lambda(c).$$

**Lemma 4.4** *For elements  $\varepsilon$  and  $\varepsilon'$  with  $\varepsilon^2 = e = (\varepsilon')^2$  and idempotents  $c$  and  $c'$  in  $\mathcal{V}$ , we have*

$$\lambda(|(\varepsilon' \circ c') \circ (c \circ \varepsilon)|) \prec_w \lambda(c) * \lambda(c').$$

**Proof.** Let  $m = \min\{\text{rank}(c), \text{rank}(c')\}$ . Note that  $\lambda(c) * \lambda(c')$  is a vector in  $\mathcal{R}^n$  with ones in the first  $m$  slots and zeros elsewhere. From the previous corollary,  $\lambda_1(|c \circ \varepsilon|) \leq 1$  and  $\lambda_1(|\varepsilon' \circ c'|) \leq 1$ ; hence, by Item (vi) in Proposition 2.3,  $\lambda_1(|(\varepsilon' \circ c') \circ (c \circ \varepsilon)|) \leq \lambda_1(|\varepsilon' \circ c'|) \lambda_1(|c \circ \varepsilon|) \leq 1$ . Thus,

$$\lambda_i(|(\varepsilon' \circ c') \circ (c \circ \varepsilon)|) \leq 1, \quad i = 1, 2, \dots, n.$$

Additionally, for some  $\varepsilon''$  with  $(\varepsilon'')^2 = e$ ,

$$\begin{aligned} \text{tr}(|(\varepsilon' \circ c') \circ (c \circ \varepsilon)|) &= \langle [(\varepsilon' \circ c') \circ (c \circ \varepsilon)] \circ \varepsilon'', e \rangle = \langle c \circ \varepsilon, \varepsilon'' \circ (\varepsilon' \circ c') \rangle \\ &\leq \langle \lambda(c \circ \varepsilon), \lambda(\varepsilon'' \circ (\varepsilon' \circ c')) \rangle \\ &\leq \langle \lambda(|c \circ \varepsilon|), \lambda(|\varepsilon'' \circ (\varepsilon' \circ c')|) \rangle \\ &\leq m, \end{aligned}$$

where the first two inequalities come from item (ii) in Proposition 2.3 and the last inequality is due to the previous lemma and Item (a) in Proposition 2.1. Hence,

$$\lambda(|(\varepsilon' \circ c') \circ (c \circ \varepsilon)|) \prec_w \lambda(c) * \lambda(c').$$

□

**Lemma 4.5** For any  $\varepsilon$  with  $\varepsilon^2 = e$ , idempotent  $c$ , and  $b$  in  $\mathcal{V}$ , we have

$$\lambda(|b \circ (c \circ \varepsilon)|) \prec_w \lambda(|b|) * \lambda(c).$$

**Proof.** We write the spectral decomposition  $b \circ (c \circ \varepsilon) = \sum_{i=1}^n \lambda_i e_i$ . As the conclusion of the lemma remains the same if  $b$  is replaced by  $-b$ , we may assume that some  $\lambda_i$  is nonnegative. Without loss of generality, let  $\lambda_i \geq 0$ , for  $i = 1, 2, \dots, k$  and  $\lambda_j < 0$  for  $j = k + 1, k + 2, \dots, n$ , where  $k \in \{1, 2, \dots, n\}$ . (This includes the possibility that  $k = n$ , in which case, there is no  $\lambda_j < 0$ .) Define  $\varepsilon' := \sum_{i=1}^n \varepsilon'_i e_i$ , where  $\varepsilon'_i = 1$  for  $i = 1, 2, \dots, k$  and  $\varepsilon'_j = -1$  for  $j = k + 1, k + 2, \dots, n$ . Then, for any idempotent  $f$  of rank  $l$ , that is,  $f \in \mathcal{J}^{(l)}(\mathcal{V})$ , we have

$$\begin{aligned} \langle |b \circ (c \circ \varepsilon)|, f \rangle &= \langle (b \circ (c \circ \varepsilon)) \circ \varepsilon', f \rangle = \langle b, (c \circ \varepsilon) \circ (\varepsilon' \circ f) \rangle \\ &\leq \langle \lambda(b), \lambda((c \circ \varepsilon) \circ (\varepsilon' \circ f)) \rangle \\ &\leq \langle \lambda(|b|), \lambda(|(c \circ \varepsilon) \circ (\varepsilon' \circ f)|) \rangle \\ &\leq \langle \lambda(|b|), \lambda(c) * \lambda(f) \rangle \\ &= \langle \lambda(|b|) * \lambda(c), \lambda(f) \rangle \\ &= \sum_{i=1}^l (\lambda(|b|) * \lambda(c))_i, \end{aligned}$$

where the first two inequalities come from item (ii) in Proposition 2.3 and the last inequality is from the previous Lemma. Now, taking the maximum over  $f \in \mathcal{J}^{(l)}(\mathcal{V})$  and using Item (i) in

Proposition 2.3, we get

$$\sum_{i=1}^l \lambda_i(|b \circ (c \circ \varepsilon)|) \leq \sum_{i=1}^l (\lambda(|b|) * \lambda(c))_i = \sum_{i=1}^l \lambda_i(|b|) \lambda_i(c),$$

that is,

$$\lambda(|b \circ (c \circ \varepsilon)|) \prec_w \lambda(|b|) * \lambda(c).$$

□

**Proof of Theorem 4.1.** We write the spectral decomposition  $a \circ b = \sum_{i=1}^n \lambda_i(a \circ b) e_i$ . As the conclusion of the theorem remains the same if  $b$  is replaced by  $-b$ , we may assume that some  $\lambda_i(a \circ b)$  is nonnegative. Without loss of generality, we assume that for some  $k \in \{1, 2, \dots, n\}$ ,  $\lambda_i(a \circ b) \geq 0$ , for  $i = 1, 2, \dots, k$  and  $\lambda_j(a \circ b) < 0$  for  $j = k + 1, k + 2, \dots, n$ . ((This includes the possibility that  $k = n$ .) Define  $\varepsilon := \sum_{i=1}^n \varepsilon_i e_i$ , where  $\varepsilon_i = 1$  for  $i = 1, 2, \dots, k$  and  $\varepsilon_j = -1$  for  $j = k + 1, k + 2, \dots, n$ . We fix an index  $l$ ,  $1 \leq l \leq n$ , and  $c \in \mathcal{J}^{(l)}(\mathcal{V})$ . Then,

$$\begin{aligned} \langle |a \circ b|, c \rangle &= \langle (a \circ b) \circ \varepsilon, c \rangle \\ &= \langle a \circ b, c \circ \varepsilon \rangle \\ &= \langle a, b \circ (c \circ \varepsilon) \rangle \\ &\leq \langle \lambda(|a|), \lambda(|b \circ (c \circ \varepsilon)|) \rangle \\ &\leq \langle \lambda(|a|), \lambda(|b|) * \lambda(c) \rangle \\ &= \sum_1^l \lambda_i(|a|) \lambda_i(|b|), \end{aligned}$$

where the last inequality is due to Lemma 4.5 and Item (a) of Proposition 2.1. Taking the maximum over all such  $c$  and using Item (i) in Proposition 2.3, we see that

$$\sum_1^l \lambda_i(|a \circ b|) \leq \sum_1^l \lambda_i(|a|) \lambda_i(|b|).$$

This gives the inequality (12). □

The following example show that the inequalities  $\lambda(|a \circ b|) \prec_w \lambda(|a| \circ |b|)$  and  $\lambda(|a| \circ |b|) \prec_w \lambda(|a \circ b|)$  need not hold.

**Example 4.6** In  $\mathcal{S}^2$ , consider two matrices

$$A = \begin{bmatrix} 8 & 3 \\ 3 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 3 \\ 3 & 8 \end{bmatrix}.$$

Then, a direct (or Matlab) calculation shows that  $\lambda(|A \circ B|) = (33, 15)$  and  $\lambda(|A| \circ |B|) = (44.52, -3.48)$ . Clearly, the inequalities  $\lambda(|A \circ B|) \prec_w \lambda(|A| \circ |B|)$  and  $\lambda(|A| \circ |B|) \prec_w \lambda(|A \circ B|)$



do not hold.

**Remarks.** Theorem 3.8 shows that the inequality  $\lambda(|A \bullet b|) \prec_w \lambda(|\text{diag}(A)|) * \lambda(|b|)$  holds for any  $A \in \mathcal{S}^n$  that is positive semidefinite. Now, the inequality  $\lambda(|a \circ b|) \prec_w \lambda(|a|) * \lambda(|b|)$  can be viewed as  $\lambda(|A \bullet b|) \prec_w \lambda(|\text{diag}(A)|) * \lambda(|b|)$ , where  $a$  has the spectral decomposition  $a = a_1 e_1 + a_2 e_2 + \dots + a_n e_n$ ,  $A = [\frac{a_i + a_j}{2}]$ , and the Schur product  $A \bullet b$  is defined relative to the Jordan frame  $\{e_1, e_2, \dots, e_n\}$ . However, this  $A$  is not, in general, positive semidefinite. Motivated by this observation/result, we raise the following

**Problem:** Characterize  $A \in \mathcal{S}^n$  such that  $\lambda(|A \bullet b|) \prec_w \lambda(|\text{diag}(A)|) * \lambda(|b|)$  holds for all  $b$ .

A related weaker problem could be: Characterize  $A \in \mathcal{S}^n$  such that  $\lambda(A \bullet b) \prec_w \lambda(|\text{diag}(A)|) * \lambda(b)$  holds for all  $b \geq 0$ .

## 5 A generalized Hölder type inequality and norms of Lyapunov and quadratic representations

Recall that for  $p \in [1, \infty]$ , the spectral  $p$ -norm on  $\mathcal{V}$  is given by  $\|x\|_p := \|\lambda(x)\|_p$ , where the latter norm is computed in  $\mathcal{R}^n$ . When numbers  $r, s \in [1, \infty]$  are conjugates, that is, when  $\frac{1}{r} + \frac{1}{s} = 1$ , the following Hölder type inequality holds [6]:

$$\|x \circ y\|_1 \leq \|x\|_r \|y\|_s \quad (x, y \in \mathcal{V}).$$

It was conjectured in [8], Page 9, that a generalized version, namely,

$$\|x \circ y\|_p \leq \|x\|_r \|y\|_s \quad (x, y \in \mathcal{V})$$

holds for  $p, r, s \in [1, \infty]$  with  $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$ . In what follows, we settle this conjecture in the affirmative. For a linear transformation  $T : (\mathcal{V}, \|\cdot\|_r) \rightarrow (\mathcal{V}, \|\cdot\|_s)$ , we define the corresponding norm by

$$\|T\|_{r \rightarrow s} := \sup_{0 \neq x \in \mathcal{V}} \frac{\|T(x)\|_s}{\|x\|_r}.$$

The problem of finding these norms for  $L_a$  and  $P_a$  has been addressed in two recent papers [6, 8], where only partial results were given. As a consequence of the following result, we give a complete description of the norms of  $L_a$  and  $P_a$  relative to two spectral norms.

**Theorem 5.1** Given  $A \in \mathcal{S}^n$  and a Jordan frame in  $\mathcal{V}$ , we consider the Schur product  $A \bullet x$  for any  $x \in \mathcal{V}$  and define the linear transformation  $D_A$  on  $\mathcal{V}$  by  $D_A(x) := A \bullet x$ . Suppose the inequality

$$\lambda(|A \bullet x|) \prec_w \lambda(|\text{diag}(A)|) * \lambda(|x|) \tag{13}$$

holds for all  $x \in \mathcal{V}$ . Let  $p, r, s \in [1, \infty]$  with  $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$ . Then,

$$\|A \bullet x\|_p \leq \|\text{diag}(A)\|_r \|x\|_s \quad (x \in \mathcal{V}). \quad (14)$$

Consequently,

$$\|D_A\|_{r \rightarrow s} = \begin{cases} \|\text{diag}(A)\|_\infty & \text{if } r \leq s, \\ \|\text{diag}(A)\|_{\frac{rs}{r-s}} & \text{if } s < r. \end{cases}$$

In particular, the above conclusions hold when  $A$  is positive semidefinite.

**Proof.** Consider a fixed nonzero  $x \in \mathcal{V}$ . To simplify the notation, let  $z := \lambda(|A \bullet x|)$ ,  $u := \lambda(|\text{diag}(A)|)$ , and  $v := \lambda(|x|)$ , which are nonnegative vectors (with decreasing entries) in  $\mathcal{R}^n$ . Then, by our assumption,  $z \prec u * v$  in  $\mathcal{R}^n$ . We consider two cases:

*Case 1:*  $p = \infty$  (so  $r = s = \infty$ ).

Then, by comparing the first components in  $u, v, z$ , we have  $z_1 \leq u_1 v_1$ , that is,  $\|z\|_\infty \leq \|u\|_\infty \|v\|_\infty$ , where the norms are computed in  $\mathcal{R}^n$ . This gives,  $\|A \bullet x\|_\infty \leq \|\text{diag}(A)\|_\infty \|x\|_\infty$ .

*Case 2:*  $p < \infty$ .

Then,

$$\|z\|_p \leq \|u * v\|_p \leq \|u\|_r \|v\|_s,$$

where the first inequality comes from Item (c) in Proposition 2.1 by considering the increasing convex function  $t \mapsto t^p$  and the second inequality comes from the classical generalized Hölder's inequality in  $\mathcal{R}^n$ . From this, we get  $\|A \bullet x\|_p \leq \|\text{diag}(A)\|_r \|x\|_s$ .

Thus we have proved (14) in both cases.

Now for the computation of  $\|D_A\|_{r \rightarrow s}$ . Let  $\{e_1, e_2, \dots, e_n\}$  be the Jordan frame that defines our Schur products. Let  $A = [a_{ij}]$ . We consider two cases:

*Case (i):*  $r \leq s$ .

Then,

$$\frac{\|A \bullet x\|_s}{\|x\|_r} \leq \frac{\|A \bullet x\|_s}{\|x\|_s} \leq \|\text{diag}(A)\|_\infty,$$

where the first inequality is due to the fact that  $r \leq s \Rightarrow \|x\|_s \leq \|x\|_r$  (because, in  $\mathcal{R}^n$ , for a fixed vector  $v$ , the norm  $\|v\|_t$  decreases in  $t$ ) and the second inequality is due to the fact that  $\|z\|_s \leq \|u * v\|_s \leq \|u\|_\infty \|v\|_s$  in  $\mathcal{R}^n$ . Thus,

$$\|D_A\|_{r \rightarrow s} \leq \|\text{diag}(A)\|_\infty.$$

To see the reverse inequality, we observe that  $A \bullet e_i = a_{ii} e_i$  for all  $1 \leq i \leq n$ . As 1 is the only nonzero eigenvalue of  $e_i$ ,  $\|e_i\|_r = 1$  and so,  $|a_{ii}| = \frac{\|A \bullet e_i\|_s}{\|e_i\|_r}$  for all  $i$ . Thus,

$$\|\text{diag}(A)\|_\infty \leq \max_{1 \leq i \leq n} \frac{\|A \bullet e_i\|_s}{\|e_i\|_r} \leq \|D_A\|_{r \rightarrow s}.$$

Hence,

$$\|D_A\|_{r \rightarrow s} = \|\text{diag}(A)\|_\infty.$$

*Case (ii):*  $s < r$  (so  $s < \infty$ ).

We define  $t \in [1, \infty)$  by  $\frac{1}{s} = \frac{1}{t} + \frac{1}{r}$  so that  $t = \frac{rs}{r-s}$  (which is taken to be 1 when  $r = \infty$ ). Then, by an application of (14),  $\|A \bullet x\|_s \leq \|\text{diag}(A)\|_t \|x\|_r$ . This implies

$$\frac{\|A \bullet x\|_s}{\|x\|_r} \leq \frac{\|\text{diag}(A)\|_t \|x\|_r}{\|x\|_r} = \|\text{diag}(A)\|_t.$$

From this, we get

$$\|D_A\|_{r \rightarrow s} \leq \|\text{diag}(A)\|_t.$$

As this turns into equality when  $\text{diag}(A) = 0$ , we prove the reverse inequality by assuming  $\text{diag}(A) \neq 0$ . In this setting, let  $x = \sum_{i=1}^n |a_{ii}|^{\frac{t}{r}} (\text{sgn } a_{ii}) e_i$  (using the convention  $0^0 = 1$ , if needed, when  $r = \infty$ ), where  $\text{sgn } a_{ii}$  denotes the sign of  $a_{ii}$ . Then,  $A \bullet x = \sum_{i=1}^n |a_{ii}|^{\frac{t}{r}+1} e_i$ . So,  $\|A \bullet x\|_s = \|\text{diag}(A)\|_t^{\frac{t}{s}}$ . Also,  $\|x\|_r = \|\text{diag}(A)\|_t^{\frac{t}{r}}$ . Thus, for this  $x$ ,

$$\frac{\|A \bullet x\|_s}{\|x\|_r} = \frac{\|\text{diag}(A)\|_t^{\frac{t}{s}}}{\|\text{diag}(A)\|_t^{\frac{t}{r}}} = \|\text{diag}(A)\|_t.$$

It follows that  $\|D_A\|_{r \rightarrow s} \geq \|\text{diag}(A)\|_t$ . Hence,

$$\|D_A\|_{r \rightarrow s} = \|\text{diag}(A)\|_t.$$

Finally, when  $A$  is positive semidefinite, condition (13) holds thanks to Theorem 3.8. Hence (14) and norm statements hold in this special case as well.  $\square$

We now describe the norms of  $L_a$  and  $P_a$ .

**Corollary 5.2** *Consider  $p, r, s \in [1, \infty]$  with  $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$ . Then the following statements hold for all  $a, b \in \mathcal{V}$ :*

(i)

$$\|a \circ b\|_p \leq \|a\|_r \|b\|_s \quad \text{and} \quad \|L_a\|_{r \rightarrow s} = \begin{cases} \|a\|_\infty & \text{if } r \leq s, \\ \|a\|_{\frac{rs}{r-s}} & \text{if } s < r. \end{cases}$$

(ii)

$$\|P_a(b)\|_p \leq \|a^2\|_r \|b\|_s \quad \text{and} \quad \|P_a\|_{r \rightarrow s} = \begin{cases} \|a^2\|_\infty & \text{if } r \leq s, \\ \|a^2\|_{\frac{rs}{r-s}} & \text{if } s < r. \end{cases}$$

**Proof.** We prove Item (i). For a fixed  $a \in \mathcal{V}$ , we consider its spectral decomposition  $a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$  and define all Schur products relative to  $\{e_1, e_2, \dots, e_n\}$ . Then, with  $A = [\frac{a_i + a_j}{2}]$ ,  $L_a(x) = A \bullet x$  for all  $x \in \mathcal{V}$ . Since condition (13) holds (thanks to Theorem 4.1), we can apply

Theorem 5.1. As the diagonal entries of  $A$  are  $a_1, a_2, \dots, a_n$ , which are the eigenvalues of  $a$ , we get both the items in (i). Based on the matrix  $A = [a_i a_j]$ , a similar argument gives (ii).  $\square$

## Acknowledgment

The second author was financially supported by the National Research Foundation of Korea NRF-2016R1A5A1008055.

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