High-Fidelity Quantum Logic Operations Using Linear Optical Elements

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Knill, Laflamme, and Milburn [Nature (London) **409**, 46 (2001)] have shown that quantum logic operations can be performed using linear optical elements and additional ancilla photons. Their approach is probabilistic in the sense that the logic devices fail to produce an output with a failure rate that scales as $1/n$, where n is the number of ancilla. Here we present an alternative approach in which the logic devices always produce an output with an intrinsic error rate that scales as $1/n^2$, which may have several advantages in quantum computing applications.

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In an earlier paper [1], Knill, Laflamme, and Milburn (KLM) proposed a method for implementing probabilistic quantum logic gates using linear optical elements, additional ancilla photons, and postselection based on the results of measurements made on the ancilla. The output of these devices is known to be correct (in principle) when certain measurement results are obtained, but no output is produced for the remaining events, which occur with a failure rate that scales as $1/n$ where n is the number of ancilla. Here we propose an alternative approach in which the logic devices always produce an output with an intrinsic error rate that scales as $1/n^2$, which is expected to have several advantages in quantum computing applications.

Logic operations are inherently nonlinear, whereas nonlinear interactions at the single-photon level are difficult to achieve because of the low electric field associated with a single photon. The KLM approach makes use of the fact that the measurement process itself is nonlinear. By accepting only those events in which measurements made on the ancilla yield a specified result, the system can be left in a postselected final state that corresponds to the desired quantum logic operation. Depending on the results of the measurements, it may be necessary to apply a classical correction to the output states in order to obtain the desired logical output. We have experimentally demonstrated several quantum logic devices of this kind [2,3] that have a probability of success of 1/2.

Quantum error correction requires relatively small error rates, however, and KLM showed that the failure probability could decrease as $1/n$ in the limit of large n. Our approach is motivated by the fact that quantum error correction automatically identifies the presence of an error via the appropriate syndrome [4], in which case there may be little practical advantage in independently knowing that a logic operation has succeeded based on measurements made on the ancilla. In addition, even when the KLM approach indicates that a logic operation has succeeded, there may still be errors from technical sources, such as photon absorption or decoherence. As a result, our strategy is to accept the output of all of the logic operations while choosing the initial state of the ancilla photons to minimize the overall error rate, which is equivalent to maximizing the fidelity of the output. The ancilla will still be prepared using postselection, but the calculations themselves will be performed with the maximum fidelity. We will refer to our approach as the high-fidelity or "Hi-Fi" approach in order to distinguish it from the original KLM method; the two approaches differ only in their overall strategy and the choice of the entangled ancilla state.

Gottesman and Chuang [5] showed that quantum logic operations could be performed using quantum teleportation, where the desired logic operation is applied to the entangled pair of ancilla rather than to the input qubits. Probabilistic techniques [1–3,6–8] can be applied repeatedly until the necessary ancilla state has been generated, after which the logic operation can be successfully performed by teleporting the qubits of interest. As a result, we begin by describing a quantum teleportation protocol with an error rate that scales as $2/n^2$.

The input qubit q will be represented by a single optical mode, such as a single-mode optical fiber. The absence of a photon will represent a logical value of 0, while the presence of a single photon will represent a logical value of 1. The input qubit to be teleported is assumed to be in an arbitrary initial state $|q\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$. Unlike the original quantum teleportation protocol [9], our approach makes use of n ancilla photons distributed among two sets of modes, which we will refer to as the x and y registers, as illustrated in the upper part of Fig. 1. Each of these registers contains n modes and their states will be represented by $|x\rangle$ and $|y\rangle$. The modes in register x will be labeled by index l with values from 1 to n while the modes in register y will be labeled by index l with values from $n+1$ to $2n$. The mode containing the initial qubit will be labeled $l = 0$.

Our use of the x and y ancilla registers is the same as in the KLM approach [1], except that we assume a more general form for the initial entangled ancilla state $|\psi_{A0}\rangle$ given by

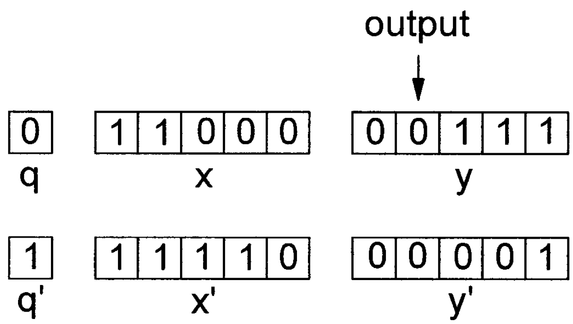

FIG. 1. Ancilla registers x, y, x', and y' used in the quantum teleportation of two input qubits q and q' [1]. The example shown here corresponds to $n = 5$ and the values of the qubits correspond to one term in the entangled state where $q = 0$, $j = 2$, $q' = 1$, and $j' = 4$. If a total of $k = 2$ photons are found in q and x after the Fourier transform, for example, then the output of the teleported state for q would be taken from mode $l = n + k$, as indicated by the arrow. It can be seen that this corresponds to the correct value of $q = 0$. If a total of two photons were found but the input value of q was equal to 1, that would correspond to $j = 1$ and the output qubit at that location would have had the value 1, as required in that case. The projective nature of the measurement process gives the appropriate superposition of these two output values.

$$|\psi_{A0}\rangle = \sum_{j=0}^{n} f(j)|1\rangle^j|0\rangle^{n-j}\ |0\rangle^j|1\rangle^{n-j}. \tag{1}$$

Here the $f(j)$ are arbitrary coefficients whose values will be chosen to maximize the fidelity. The contents of modes 1 through $2n$ are listed from left to right, where $|1\rangle^j$ indicates that there is one photon in the first j modes, $|0\rangle^{n-j}$ indicates that there are no photons in the next $n - j$ modes, etc. We have left a space in Eq. (1) to separate the states of the x and y registers. It can be seen that $|\psi_{A0}\rangle$ contains exactly n ancilla photons and that the y register is the same as the x register except that all of its qubits have been flipped, as illustrated in Fig. 1 for the case of $n = 5$ and $j = 2$.

The first step in the teleportation process is to apply a Fourier transform $\hat{F}$ to the combination of q and the x register, which we will refer to as register c, i.e., $|c\rangle = |q\rangle|x\rangle$. The Fourier transform corresponds to the operator transformation

$$\hat{a}_l^\dagger \rightarrow \frac{1}{\sqrt{n+1}} \sum_{p=0}^{n} e^{2\pi i p l/(n+1)} \hat{a}_p^\dagger, \tag{2}$$

where $i = \sqrt{-1}$. $\hat{F}$ can be implemented efficiently using a set of beam splitters and phase shifters [10,11]. After the Fourier transform, the numbers r_l of photons in modes 0 through n are measured and the total number of photons will be denoted by $k = \sum r_l$.

Consider the case in which the measurements yield a particular value of k. The measurement process is then a projection onto the subspace $|S_k\rangle$ of the original state vector that contains a total of k photons:

$$|S_k\rangle = \alpha_0|c_0\rangle|y_0\rangle + \alpha_1|c_1\rangle|y_1\rangle, \tag{3}$$

where

$$\begin{aligned} |c_0\rangle &= f(k)\hat{F}|0\rangle|1\rangle^k|0\rangle^{n-k} \qquad |y_0\rangle = |0\rangle^k|1\rangle^{n-k} \\ |c_1\rangle &= f(k-1)\hat{F}|1\rangle^1|1\rangle^{k-1}|0\rangle^{n-k+1} \\ |y_1\rangle &= |0\rangle^{k-1}|1\rangle^{n-k+1}. \end{aligned} \tag{4}$$

Here $|c_0\rangle$ corresponds to the case in which no photons were initially present in q and k photons were present in x, while $|c_1\rangle$ corresponds to the case in which there was one photon initially present in q and $k - 1$ photons present in x. We use the convention that $f(j) = 0$ unless $0 \leq j \leq n$, which eliminates one of the terms in Eqs. (3) and (4) if $k = 0$ or $k = n + 1$.

As noted by KLM, applying a phase shift of $\exp[2\pi i l r_l/(n+1)]$ to each mode l in register c after applying $\hat{F}$ is equivalent to shifting modes 0 through n right by one location before applying $\hat{F}$. As a result, the relevant terms in $|c_0\rangle$ and $|c_1\rangle$ differ only by a phase factor that can be compensated by applying a classical correction using Pockels cells, for example. After the phase correction, the projective measurement leaves the system in the state $|\psi_k\rangle$ given by

$$|\psi_k\rangle = c_n[\alpha_0 f(k)|y_0\rangle + \alpha_1 f(k-1)|y_1\rangle]|M\rangle. \tag{5}$$

Here c_n is a normalization constant and $|M\rangle$ is the final state of the measurement device, which is the same for the α_0 and α_1 terms.

The contents of mode $n + k$ in the $|y\rangle$ register is now selected as the output. It can be seen from the arrow in Fig. 1 that this qubit will have the correct value of 0 or 1 and that the remaining modes of register y will be in the same state $|y_R\rangle$ for both terms in Eq. (5). The final output state $|\psi_{\text{out}}\rangle$ from the teleportation is thus given by

$$|\psi_{\text{out}}\rangle = c_n[\alpha_0 f(k)|0\rangle + \alpha_1 f(k-1)|1\rangle]|M\rangle|y_R\rangle. \tag{6}$$

The states $|M\rangle$ and $|y_R\rangle$ are common to both terms and have no effect on any subsequent computations.

It can be seen that Eq. (6) would correspond to the correctly teleported state if $f(k) = f(k-1)$. In the KLM approach, all of these coefficients are taken to be equal. In that case the teleportation succeeds with certainty unless $k = 0$ or $k = n + 1$, in which case the input state has been determined and the output of the logic operation is rejected. This occurs with probability $1/n$ in the limit of large n and gives a correspondingly large failure rate. In our approach, the output is accepted in all cases and we choose the values of the $f(j)$ to minimize the average error rate. The probability that the output will be in the correct state is $P_S = |\langle q|\psi_{\text{out}}\rangle|^2$, which is the square of the fidelity [4]. The probability that an error would be detected by an error correction algorithm, for example, is $P_E = 1 - P_S$.

The intuitive idea behind our approach is that it would be better to evenly distribute the probability amplitude for an error over all of the terms in the entangled ancilla state rather than concentrating it on just a few ($k = 0$ or $k = n + 1$). Since the error probability depends on the squares of the amplitudes, one might expect to reduce the overall error by a factor of $1/n$ by distributing the errors evenly. The expected error reduction is roughly analogous to the quantum Zeno effect [12].

With that in mind, we assume that $f(j)$ is a slowly varying function in the limit of large n, in which case the approximation

$$f(k-1) = f(k) - \frac{df}{dk} \tag{7}$$

holds, where the derivative has been multiplied by $\delta k = -1$. We also define a parameter ε by writing

$$\frac{df}{dk} = f'(k) = \varepsilon(k) f(k) \tag{8}$$

and we note that ε can be made small in the limit of large n, since k will take on large values. Evaluating the normalization constant c_n and taking the projection gives an expression for P_S:

$$P_S = \left[\frac{|\alpha_0|^2 f(k) + |\alpha_1|^2 f(k-1)}{[|\alpha_0|^2 f(k)^2 + |\alpha_1|^2 f(k-1)^2]^{1/2}}\right]^2. \tag{9}$$

Making use of Eqs. (7) and (8) allows this to be rewritten as

$$P_S = \frac{[|\alpha_0|^2 + |\alpha_1|^2(1-\varepsilon)]^2}{|\alpha_0|^2 + |\alpha_1|^2(1-\varepsilon)^2}. \tag{10}$$

Expanding to second order in ε gives

$$P_S = 1 - |\alpha_0|^2|\alpha_1|^2\varepsilon^2 = 1 - P_0 P_1\left(\frac{f'(k)}{f(k)}\right)^2, \tag{11}$$

where we have defined $P_0 = |\alpha_0|^2$ and $P_1 = |\alpha_1|^2$, and used the definition of ε.

If we assume that P_0 and P_1 are uniformly distributed between 0 and 1, then the average value of $P_0 P_1$ is equal to 1/6. In the limit of large n, the probability $P(k)$ of obtaining k photons is $f(k)^2$, in which case the average value of ε^2 is given by

$$\overline{\varepsilon^2} = \int_0^n P(k)\varepsilon^2 dk = \int_0^n f'(k)^2 dk. \tag{12}$$

Combining Eqs. (11) and (12) gives an average error rate of

$$P_E = \frac{1}{6}\int_0^n f'(k)^2 dk \qquad (n \gg 1). \tag{13}$$

In order to evaluate the average error rate, we will consider the simple case of a triangular-shaped function for which $f(j)$ is zero for $j = 0$ or $j = n$ and increases linearly to a maximum value at $j = n/2$ as determined by the normalization requirement that $\sum f(j)^2 = 1$. The probability of a large error near $k = 0$ or $k = n + 1$ becomes negligibly small in the limit of large n and the probability amplitude for an error (f') is evenly distributed across all of the terms in the entangled ancilla state. Evaluating the integral in Eq. (13) for this choice of $f(k)$ gives $P_E = 2/n^2$ ($n \gg 1$). The optimal choice of $f(k)$ can be found numerically and gives an error rate that is 18% smaller.

Having described a high-fidelity approach for quantum teleportation, we can now consider the implementation of two-qubit quantum logic gates as originally suggested by Gottesman and Chuang [5]. We will take q to be the control qubit and introduce a second qubit q' as the target, where q' is initially in the arbitrary state $|q'\rangle = \alpha_0'|0'\rangle + \alpha_1'|1'\rangle$. As illustrated in Fig. 1, q' will have associated ancilla registers x' and y' that will be used in its teleportation along with the teleportation of q using registers x and y as described above. The goal will be to apply the desired logic operation to the ancilla in registers y and y' before the teleportation in order to implement the desired logic operations on q and q'.

Here we consider the implementation of a controlled sign flip [1] in which the sign of the $\alpha_1'|1'\rangle$ term is to be flipped if $q = 1$. In order to implement this operation, we take the two sets of ancilla to be in an entangled state $|\psi_{AA'}\rangle$ given by

$$|\psi_{AA'}\rangle = \sum_{j=0}^{n} f(j)|1\rangle^j|0\rangle^{n-j}|0\rangle^j|1\rangle^{n-j}\sum_{j'=0}^{n}(-1)^{jj'}f(j')|1\rangle^{j'}|0\rangle^{n-j'}|0\rangle^{j'}|1\rangle^{n-j'}, \tag{14}$$

which is similar to the entangled state used by KLM except for the factors of $f(j)$ and $f(j')$. This state would correspond to the tensor product of two sets of ancilla in the initial state of Eq. (1) except that the factor of $(-1)^{jj'}$ entangles the two. Here we perform separate Fourier transforms $\hat{F}$ and $\hat{F}'$ on the two combined registers c and c' and we measure the numbers r_l and $r_{l'}$ of photons in each mode, with totals denoted k and k'. After the same phase corrections described above, the measurement projects the system into the state $|\psi_C\rangle$ given by

$$\begin{aligned}|\psi_C\rangle = {}& c_n(-1)^{kk'}[\alpha_0\alpha_0' f(k)f(k')|0\rangle|0'\rangle + (-1)^{-k}\alpha_0\alpha_1' f(k)f(k'-1)|0\rangle|1'\rangle \\ &+ (-1)^{-k'}\alpha_1\alpha_0' f(k-1)f(k')|1\rangle|0'\rangle + (-1)^{-k-k'+1}\alpha_1\alpha_1' f(k-1)f(k'-1)|1\rangle|1'\rangle]|y_R\rangle|y_R'\rangle|M\rangle|M'\rangle.\end{aligned} \tag{15}$$

This state differs from the desired output state by various sign factors, which can be corrected using a Pockels cell and the observed values of k and k'. With these corrections, the system is left in a final state given by

$$\begin{aligned}|\psi_{\text{out}}\rangle = c_n[\alpha_0\alpha_0' f(k)f(k')|0\rangle|0'\rangle + \alpha_0\alpha_1' f(k)f(k'-1)|0\rangle|1'\rangle \\ + \alpha_1\alpha_0' f(k-1)f(k')|1\rangle|0'\rangle - \alpha_1\alpha_1' f(k-1)f(k'-1)|1\rangle|1'\rangle]|y_R\rangle|y_R'\rangle|M\rangle|M'\rangle,\end{aligned} \tag{16}$$

which corresponds to the desired result aside from the factors of f. The total error rate can be shown to be the sum of the error rates from the two teleportations in the limit of large n, giving $P_E = 4/n^2$ for the choice of coefficients described above. The optimal choice of $f(j)$ gives an error rate that is 32% less. A controlled-NOT gate can be implemented by applying simple single-qubit transformations before and after the controlled sign flip gate.

It is interesting to note that our approach does not rely on postselection in the sense that all of the logic operations are accepted after deterministic corrections have been applied. The nonlinearity still originates in the reduction of the state vector during the measurement process, but the process might be better referred to as postcorrection rather than postselection.

Although the occurrence of a failure in the original KLM approach can be identified, that does not mean that the use of quantum error correction techniques can be avoided even in the ideal case. Known failures of that kind are equivalent to z-measurement errors, which can be corrected using a simple two-bit concatenated code with a relatively high error threshold [13]. The same two-bit code can also correct for phase-shift errors and possibly for photon losses, but not for more general errors, such as those that would be introduced by imperfect generation of the entangled ancilla states, for example. If more general errors cannot be neglected, which seems likely [14], then the advantage of identifiable failures is lost, since more general codes [4,15,16] would have to be used and those codes can correct for all errors whether they have been identified in some other way or not. In that case, the total rate of errors and/or failures must be kept below the error threshold and a reasonable figure of merit is the fidelity (before postselection if any is applied). Our approach allows the error threshold to be met for a smaller value of n than does the original KLM approach. For example, if our approach requires $n = 20$, then the original KLM approach would require $n = 400$ assuming that general error correction is needed. The resources required to generate the required ancilla states for a given value of n can be shown [17] to be the same in our approach as in the original KLM approach, so that minimizing n will minimize the resources required to generate the ancilla. But more importantly, reducing the value of n also reduces the probability of an error in the logic operations or in the generation of the ancilla, which in turn makes it much easier to meet the error threshold, which we feel is the major advantage of our approach.

In summary, we have described a high-fidelity approach to linear optics quantum computing that does not rely on postselection and may have practical advantages in the implementation of a quantum computer.

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