Commutation principles for optimization problems on spectral sets in Euclidean Jordan algebras

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Abstract

The commutation principle of Ramírez, Seeger, and Sossa [13] proved in the setting of Euclidean Jordan algebras says that when the sum of a real valued function h and a spectral function Φ is minimized/maximized over a spectral set E, any local optimizer a at which h is Fréchet differentiable operator commutes with the derivative $h'(a)$. In this paper, assuming the existence of a subgradient in place the derivative (of h), we establish 'strong operator commutativity' relations: If a solves the problem $\max_E (h+\Phi)$, then a strongly operator commutes with every element in the subdifferential of h at a; If E and h are convex and a solves the problem $\min_E h$, then a strongly operator commutes with the negative of some element in the subdifferential of h at a. These results improve known (operator) commutativity relations for linear h and for solutions of variational inequality problems. We establish these results via a geometric commutation principle that is valid not only in Euclidean Jordan algebras, but also in the broader setting of FTvN-systems.

Key Words: Euclidean Jordan algebra, spectral sets/functions, commutation principle, variational inequality problem, normal cone, subdifferential.

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1 Introduction

Let $\mathcal{V}$ be a Euclidean Jordan algebra of rank n carrying the trace inner product [4] and $\lambda : \mathcal{V} \to \mathcal{R}^n$ denote the eigenvalue map (which takes x to $\lambda(x)$, the vector of eigenvalues of x with entries written in the decreasing order). For any $a \in \mathcal{V}$, we define its λ-orbit by $[a] := \{x \in \mathcal{V} : \lambda(x) = \lambda(a)\}$. A set E in $\mathcal{V}$ is said to be a *spectral set* if it is of the form $E = \lambda^{-1}(Q)$ for some (permutation invariant) set Q in $\mathcal{R}^n$ or, equivalently, a union of λ-orbits. A function $\Phi : \mathcal{V} \to \mathcal{R}$ is said to be a *spectral function* if it is of the form $\Phi = \phi \circ \lambda$ for some (permutation invariant) function $\phi : \mathcal{R}^n \to \mathcal{R}$.

In [13], Ramírez, Seeger, and Sossa prove the following commutation principle in $\mathcal{V}$:

Theorem 1. *Suppose a is a local optimizer of the problem $\min/\max_{E}\,(h + \Phi)$, where E is a spectral set, Φ is a spectral function, and $h : \mathcal{V} \to \mathcal{R}$. If h is Fréchet differentiable at a, then a and $h'(a)$ operator commute.*

In [6], Gowda and Jeong extended the above result by assuming that E and Φ are invariant under the automorphisms of $\mathcal{V}$ and stated an analogous result in the setting of normal decomposition systems. Subsequently, certain modifications (such as replacing the sum by other combinations) and applications were given by Niezgoda [12].

The main objective of this note is to describe some analogs of the above commutation principle by assuming the existence of a subgradient in place of the derivative (of h). In each analog, this change results in a stronger commutativity relation. We derive these analogs via a geometric commutation principle. To elaborate, we first recall some definitions.

- We say that elements a and b *operator commute* in $\mathcal{V}$ if there exists a Jordan frame $\{e_1, e_2, \ldots, e_n\}$ in $\mathcal{V}$ such that the spectral decompositions of a and b are given by

  $$a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \quad \text{and} \quad b = b_1 e_1 + b_2 e_2 + \cdots + b_n e_n,$$

  where $a_1, a_2, \ldots, a_n$ are the eigenvalues of a and $b_1, b_2, \ldots, b_n$ are the eigenvalues of b. If, additionally, above decompositions hold with $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$, we say that a and b *strongly operator commute* (also said to 'simultaneously diagonalizable' [11] or said to have 'similar joint decomposition' [1]). Some equivalent formulations are described in the next section.

- Given a (nonempty) set S in $\mathcal{V}$ and $a \in S$, we define the *normal cone* of S at a by

  $$N_S(a) := \{d \in \mathcal{V} : \langle d, x - a\rangle \leq 0 \text{ for all } x \in S\}.$$

- Let $h : \mathcal{V} \to \mathcal{R} \cup \{\infty\}$, $S \subseteq \mathcal{V}$, and $a \in S \cap \operatorname{dom} h$. We define the *subdifferential of h at a*

relative to S by

$$\partial_S h(a) := \{d \in \mathcal{V} : h(x) - h(a) \geq \langle d, x - a \rangle \text{ for all } x \in S\};$$

any element of $\partial_S h(a)$ will be called a *S-subgradient of h at a*. Finally, when $S = \mathcal{V}$, we define the *subdifferential of h at a* by

$$\partial h(a) := \{d \in \mathcal{V} : h(x) - h(a) \geq \langle d, x - a \rangle \text{ for all } x \in \mathcal{V}\}.$$

We note that subdifferentials may be empty and $\partial h(a) \subseteq \partial_S h(a)$. We also note [14] that when h (defined on all of $\mathcal{V}$) is convex, the subdifferential is nonempty, compact, and convex; if h is also Fréchet differentiable at a, then $\partial h(a) = \{h'(a)\}$.

Our primary examples of Euclidean Jordan algebras are $\mathcal{R}^n$, $\mathcal{S}^n$, and $\mathcal{H}^n$. In the algebra $\mathcal{R}^n$ (with componentwise multiplication as Jordan product and usual inner product), spectral sets/functions (also called symmetric sets/functions) are precisely those that are invariant under the action of permutation matrices. In this algebra, any two elements operator commute; strong operator commutativity requires simultaneous (permutation) rearrangement with decreasing components. For example, in $\mathcal{R}^2$, the elements $(1, 0)$ and $(0, 1)$ operator commute, but not strongly. In the algebras $\mathcal{S}^n$ (of all $n \times n$ real symmetric matrices) and $\mathcal{H}^n$ (of all $n \times n$ complex Hermitian matrices), the Jordan product and the inner product are given, respectively, by

$$X \circ Y := \frac{XY + YX}{2} \quad \text{and} \quad \langle X, Y \rangle := \text{tr}(XY).$$

In $\mathcal{S}^n$ (in $\mathcal{H}^n$) spectral sets are those that are invariant under linear transformations of the form $X \mapsto UXU^*$, where U is an orthogonal (respectively, unitary) matrix. Also, two matrices X and Y in $\mathcal{S}^n$ (in $\mathcal{H}^n$) operator commute if and only if $XY = YX$, or equivalently, there exists an orthogonal (respectively, unitary) matrix U such that $X = UD_1U^*$ and $Y = UD_2U^*$, where D_1 and D_2 are diagonal matrices consisting, respectively, of eigenvalues of X and Y. If the diagonal vectors of D_1 and D_2 have decreasing components, then X and Y strongly operator commute.

We now state our *geometric commutation principle*:

Theorem 2. *Suppose E is a spectral set in $\mathcal{V}$ and $a \in E$. Then, a strongly operator commutes with every element in the normal cone $N_E(a)$. In particular, a strongly operator commutes with every element in the normal cone $N_{[a]}(a)$, where $[a]$ is the λ-orbit of a.*

We prove this result as a simple consequence of (what we call) Fan-Theobald-von Neumann inequality (1) together with its equality case, see Theorem 4 below. When E is also convex, one may prove this as a consequence of a result on subgradients of convex spectral functions such as Theorem 5.5 in [2].

Based on Theorem 2, we derive our commutation principles for optimization problems:

Theorem 3. *Suppose E is a spectral set in $\mathcal{V}$, $\Phi : \mathcal{V} \to \mathcal{R}$ is a spectral function, and $h : \mathcal{V} \to \mathcal{R}$.*

(i) If a is an optimizer of the problem $\max_{E}\,(h+\Phi)$ and h has a E-subgradient at a, then a strongly operator commutes with every element in $\partial_E\, h(a)$.

(ii) If E and h are convex and a is an optimizer of the problem $\min_{E} h$, then a strongly operator commutes with the negative of some element in $\partial\, h(a)$.

Specializing h in the above result to a linear function leads to an interesting consequence for variational inequality problems. To elaborate, consider a function $G : \mathcal{V} \to \mathcal{R}$ and a set $E \subseteq \mathcal{V}$. Then, the *variational inequality problem* [3], $\mathrm{VI}(G, E)$, is to find an element $a \in E$ such that

$$\langle G(a), x - a\rangle \geq 0 \text{ for all } x \in E.$$

When E is a closed convex cone, this becomes a *cone complementarity problem.* We now state a simple consequence of Theorem 3.

Corollary 1. *Suppose E is a spectral set in $\mathcal{V}$, Φ is a spectral function, and $h : \mathcal{V} \to R$ is convex and Fréchet differentiable. Let $c \in \mathcal{V}$ and $G : \mathcal{V} \to \mathcal{R}$. Then, the following statements hold:*

(i) If a is an optimizer of $\max_{E}\,(h + \Phi)$, then a strongly operator commutes with $h'(a)$.

(ii) If $h(x) := \langle c, x\rangle$ on $\mathcal{V}$ and a is an optimizer of $\max_{E}\,(h+\Phi)$, then a strongly operator commutes with c. Moreover, the maximum value is $\langle \lambda(c), \lambda(a)\rangle + \Phi(a)$.

(iii) If $h(x) := \langle c, x\rangle$ on $\mathcal{V}$ and a is an optimizer of $\min_{E}\,(h+\Phi)$, then a strongly operator commutes with $-c$. Moreover, the minimum value is $\langle\, \widetilde{\lambda}(c), \lambda(a)\rangle + \Phi(a)$, where $\widetilde{\lambda}(c) := -\lambda(-c)$.

(iv) If a solves VI(G, E), then a strongly operator commutes with $-G(a)$.

In our proofs, we employ standard ideas/results from convex analysis [14] and the following key result from Euclidean Jordan algebras [11, 1, 8]:

Theorem 4. *For all $x, y \in \mathcal{V}$,*

$$\langle x, y\rangle \leq \langle \lambda(x), \lambda(y)\rangle. \tag{1}$$

Equality holds in (1) if and only if x and y strongly operator commute.

While our results are stated in the setting of Euclidean Jordan algebras for simplicity and ease of proofs, it is possible to describe these in a general setting/system. This general system is formulated by turning (1) into an axiom and defining the concept of commutativity via the equality in (1). The precise formulation is as follows. A *Fan-Theobald-von Neumann system* (FTvN system, for short) [5], is a triple $(\mathcal{V}, \mathcal{W}, \lambda)$, where $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces and $\lambda : \mathcal{V} \to \mathcal{W}$ is a norm preserving map satisfying the property

$$\max\Big\{\langle c,x\rangle : x\in[u]\Big\} = \langle\lambda(c),\lambda(u)\rangle \quad (\forall\, c,u\in\mathcal{V}), \tag{2}$$

with $[u] := \{x \in \mathcal{V} : \lambda(x) = \lambda(u)\}$. This property is a combination of an inequality and a condition for equality. The inequality $\langle x,y\rangle \le \langle\lambda(x),\lambda(y)\rangle$ (that comes from (2)) is referred to as the *Fan-Theobald-von Neumann inequality* and the equality

$$\langle x,y\rangle = \langle\lambda(x),\lambda(y)\rangle$$

defines *commutativity* of x and y in this system. Spectral sets in this system are defined as sets of the form $E = \lambda^{-1}(Q)$ for some $Q \subseteq W$; spectral functions are of the form $\Phi = \phi\circ\lambda$ for some $\phi : W \to \mathcal{R}$. Examples of such systems include [5]:

- The triple $(\mathcal{V}, \mathcal{R}^n, \lambda)$, where $\mathcal{V}$ is a Euclidean Jordan algebra of rank n carrying the trace inner product with $\lambda : \mathcal{V} \to \mathcal{R}^n$ denoting the eigenvalue map. Commutativity in this FTvN system reduces to strong operator commutativity in the algebra $\mathcal{V}$.
- The triple $(\mathcal{V}, \mathcal{R}^n, \lambda)$, where $\mathcal{V}$ is a finite dimensional real vector space and p is a real homogeneous polynomial of degree n that is hyperbolic with respect to a vector $e \in \mathcal{V}$, complete and isometric, with $\lambda(x)$ denoting the vector of roots of the univariate polynomial $t \to p(te - x)$ written in the decreasing order [2].
- The triple $(\mathcal{V}, \mathcal{W}, \gamma)$, where $(\mathcal{V}, \mathcal{G}, \gamma)$ is a normal decomposition system (in particular, an Eaton triple) and $\mathcal{W} := \mathrm{span}(\gamma(\mathcal{V}))$ [10].

Based on the property (2), one can show – see Remark 1 below – that an analog of Theorem 2 holds in any FTvN system. Consequently, all of our stated results can be extended to FTvN systems.

2 Preliminaries

Throughout, we let $(\mathcal{V}, \circ, \langle\cdot,\cdot\rangle)$ denote a Euclidean Jordan algebra of rank n with unit element e [4, 7]. Additionally, we assume that the inner product is the trace inner product, that is, $\langle x,y\rangle = \mathrm{tr}(x\circ y)$, where 'tr' denotes the trace of an element (which is the sum of its eigenvalues). In this setting, every Jordan frame in $\mathcal{V}$ is orthonormal and the eigenvalue map $\lambda : \mathcal{V} \to \mathcal{R}^n$ is an isometry. It is well known that any Euclidean Jordan algebra is a direct product/sum of simple Euclidean Jordan algebras and every simple Euclidean Jordan algebra is isomorphic to one of five algebras, three of which are the algebras of $n\times n$ real/complex/quaternion Hermitian matrices. The other two are: the algebra of 3×3 octonion Hermitian matrices and the Jordan spin algebra.
It is known [9] that *when $\mathcal{V}$ is simple, spectral sets are precisely those that are invariant under automorphisms of $\mathcal{V}$ (which are invertible linear transformations from $\mathcal{V}$ to $\mathcal{V}$ that preserve Jordan products).* For an element $a \in \mathcal{V}$, we abbreviate the spectral decomposition $a = a_1e_1 + a_2e_2 + \cdots + a_ne_n$

as $a = q * \mathcal{E}$, where $q = (a_1, a_2, \ldots, a_n) \in \mathcal{R}^n$ and $\mathcal{E} := (e_1, e_2, \ldots, e_n)$ is an ordered Jordan frame. Note that (by rearranging the entries of q and $\mathcal{E}$, if necessary), we can always write $a = \lambda(a) * \mathcal{E}$ for some $\mathcal{E}$.

We recall the following equivalent formulations of commutativity.

Proposition 1. ([4], Lemma X.2.2; [5], Prop. 2.6) *Let $a, b \in \mathcal{V}$.*

(i) a and b operator commute if and only if the linear operators L_a and L_b commute, where $L_a(x) := a \circ x$, etc.

(ii) The following are equivalent:

- *a and b strongly operator commute.*
- $\langle a, b\rangle = \langle \lambda(a), \lambda(b)\rangle$.
- $\lambda(a+b) = \lambda(a) + \lambda(b)$.

Using the fact that in a simple algebra, every Jordan frame can be mapped on to any another Jordan frame by an automorphism ([4], Theorem IV.2.5), it is easily seen that: *If a and b operator commute in a simple algebra, then for some automorphism A of $\mathcal{V}$, $A(a)$ and b strongly operator commute.* A similar conclusion holds in $\mathcal{R}^n$ as well.

3 Proofs

Proof of Theorem 2: Let E be a (nonempty) spectral set in $\mathcal{V}$ and $a \in E$. Then, $E = \lambda^{-1}(Q)$ for some $Q \subseteq \mathcal{R}^n$. As $a \in E$,

$$[a] = \{x \in \mathcal{V} : \lambda(x) = \lambda(a)\} \subseteq E.$$

Since $N_E(a) \subseteq N_{[a]}(a)$, it is enough to show that a strongly operator commutes with every element in $N_{[a]}(a)$. (This will also prove the second part of the theorem.) Let $d \in N_{[a]}(a)$ so that $\langle d, x-a\rangle \leq 0$ for all $x \in [a]$. Rewriting this, we see

$$\langle x, d\rangle \leq \langle a, d\rangle \text{ for all } x \in [a]. \tag{3}$$

Now, writing the spectral decomposition of d as $d = \lambda(d) * \mathcal{E}$ for some ordered Jordan frame $\mathcal{E}$, we define $x := \lambda(a) * \mathcal{E}$. Then, $\lambda(x) = \lambda(a)$, $x \in [a]$, and (because every Jordan frame is orthonormal) $\langle x, d\rangle = \langle \lambda(x), \lambda(d)\rangle = \langle \lambda(a), \lambda(d)\rangle$. Hence, from (3),

$$\langle \lambda(a), \lambda(d)\rangle \leq \langle a, d\rangle.$$

From Theorem 4, we see the equality $\langle a, d\rangle = \langle \lambda(a), \lambda(d)\rangle$ and the strong operator commutativity of a and d. $\square$

Remark 1. In the above proof, the part beyond (3) essentially says that $\max\{\langle d, x\rangle : x \in [a]\} = \langle \lambda(d), \lambda(a)\rangle$, which is our defining property of a FTvN system. This shows that an analog of Theorem 2 holds in any FTvN system.

Proof of Theorem 3: (i) Suppose a solves the problem $\max\limits_{E}\,(h+\Phi)$, where $E = \lambda^{-1}(Q)$ for some $Q \subseteq \mathcal{R}^n$ and $\Phi = \phi \circ \lambda$ for some function $\phi : \mathcal{R}^n \to \mathcal{R}$. Then, $a \in E$ and

$$h(a) + \Phi(a) \geq h(x) + \Phi(x) \text{ for all } x \in E.$$

Now, $[a] = \{x \in \mathcal{V} : \lambda(x) = \lambda(a)\} \subseteq E$ and so

$$h(a) + \Phi(a) \geq h(x) + \Phi(x) \text{ for all } x \in [a]. \tag{4}$$

Since $\Phi(a) = \phi(\lambda(a)) = \phi(\lambda(x)) = \Phi(x)$ for all $x \in [a]$, the above expression simplifies to

$$h(a) \geq h(x) \text{ for all } x \in [a]. \tag{5}$$

Now, take any $d \in \partial_E\, h(a)$. Then,

$$h(x) - h(a) \geq \langle d, x - a\rangle \text{ for all } x \in E.$$

Since $[a] \subseteq E$, (5) leads to $\langle d, x - a\rangle \leq 0$ for all $x \in [a]$, that is, $d \in N_{[a]}(a)$. By Theorem 2, a strongly operator commutes with d.
(ii) Suppose E is convex (in addition to being spectral) and a solves the problem $\min\limits_{E} h$, where h is also convex. Let χ denote the indicator function of E (i.e., it takes the value zero on E and infinity outside of E). Then, a is a optimizer of the (global) convex problem $\min\limits_{\mathcal{V}}\,(h+\chi)$ and so

$$0 \in \partial\,(h+\chi)(a) = \partial\, h(a) + \partial\, \chi(a),$$

where the equality comes from the subdifferential sum formula ([14], Theorem 23.8). Hence, there is a $c \in \partial\, h(a)$ such that $-c \in \partial\, \chi(a)$. This c will have the property that

$$\langle -c, x - a\rangle \leq 0 \text{ for all } x \in E,$$

that is, $-c \in N_E(a)$. By Theorem 2, a strongly operator commutes with $-c$. This completes the proof. $\square$

Remark 2. In the proof of Item (i) above, we went from (4) to (5) by canceling the common term $\Phi(a)$. This type of cancellation can be carried out in certain other situations – for example, when we consider the product $h(x)\,\Phi(x)$ with $\Phi(x) > 0$ for all $x \in E$. Thus, the above proof could be modified to get results similar to (i) for other appropriate combinations of h and Φ.

Proof of Corollary 1 (i) Suppose a is an optimizer of $\max_E (h+\Phi)$. As h is assumed to be convex and differentiable, $h'(a)$ is the only element in $\partial\, h(a)$. The stated assertion comes from Theorem 3, Item (i).

(ii) The strong commutativity part comes from (i). Also, the maximum value is

$$h(a)+\Phi(a) = \langle c, a\rangle + \Phi(a) = \langle \lambda(c), \lambda(a)\rangle + \Phi(a),$$

where the second equality comes from Theorem 4.

(iii) When $h(x) = \langle c, x\rangle$ for all x, and a solves $\min_E (h+\Phi)$, we consider the problem $\max_E (-h-\Phi)$ and apply (ii) by observing that $-\Phi$ is a spectral function. Also, the minimum value is

$$-\Big(\langle -c, a\rangle - \Phi(a)\Big) = -\langle \lambda(-c), \lambda(a)\rangle + \Phi(a) = \langle \widetilde{\lambda}(c), \lambda(a)\rangle + \Phi(a).$$

(iv) Suppose a solves $\mathrm{VI}(G,E)$ so that $\langle G(a), x-a\rangle \geq 0$ for all $x \in E$. Then, a solves the problem $\min_E h$, where $h(x) := \langle G(a), x\rangle$ for all $x \in \mathcal{V}$. By Item (iii), a and $-G(a)$ strongly operator commute. $\square$

Remark 3. We note that strong operator commutativity of a and b implies the operator commutativity of a and $\pm b$. Hence, Items $(ii)-(iv)$ in Corollary 1 improve known operator commutativity relations ([13], Theorem 2 and Proposition 8) for linear h and variational inequalities. We also note that this Corollary is similar to Theorem 1.3 in [6], which is applicable to *simple* Euclidean Jordan algebras.

We now provide some illustrative examples.

Example 1. This example shows that in Theorem 1, differentibility alone is not enough to give strong operator commutativity. In the Euclidean Jordan algebra $\mathcal{R}^2$ spectral sets are just permutation invariant sets. So the set $E = \{(1,0),\,(0,1)\}$ is spectral. For the function $h(x,y) := \frac{1}{2}x^2 - x + x(y^2+y)$, we have $h(1,0) = -\frac{1}{2}$ and $h(0,1) = 0$. Also, $h'(x,y) = (x-1+y^2+y, 2xy+x)$. So, $h'(1,0) = (0,1)$ and $h'(0,1) = (1,0)$. We note that the elements $(1,0)$ and $(0,1)$ operator commute in $\mathcal{R}^2$, but not strongly. Thus, if a denotes either a minimizer or a maximizer of h on E, then a and $h'(a)$ do not strongly operator commute.

Example 2. Consider two $n\times n$ complex Hermitian matrices C and A with eigenvalues $c_1 \geq c_2 \geq \cdots \geq c_n$ and $a_1 \geq a_2 \geq \cdots \geq a_n$. In the algebra $\mathcal{H}^n$, consider the spectral set

$$E := \{UAU^* : U \in \mathcal{C}^{n\times n} \text{ is unitary}\}.$$

As this is also compact, the linear function $\langle C, X\rangle$ attains its maximum on this set at some matrix

D in E. By Corollary 1, Item (ii), C and D strongly operator commute and

$$\max_{X \in E} \langle C, X \rangle = \langle C, D \rangle = \langle \lambda(C), \lambda(D) \rangle = \langle \lambda(C), \lambda(A) \rangle = \sum_{i=1}^{n} c_i a_i.$$

Thus we get the classical result of Fan, namely,

$$\max \left\{ \operatorname{tr}(CUAU^*) : U \in \mathcal{C}^{n \times n} \text{ is unitary} \right\} = \sum_{i=1}^{n} c_i a_i.$$

Example 3. In $\mathcal{V}$, an element c is said to be an *idempotent* if $c^2 = c$. It is known that zero and one are the only possible eigenvalues of such an element. If c has exactly k nonzero eigenvalues (namely, ones), then we say that c has *rank* k. Every idempotent of rank k is of the form $e_1 + e_2 + \cdots + e_k$ for some Jordan frame $\{e_1, e_2, \ldots, e_n\}$. Now, consider the set of all idempotents of rank k, where $1 \leq k \leq n$. This set is a spectral set in $\mathcal{V}$; it is also known to be compact. Now, for any $c \in \mathcal{V}$, we maximize $\langle c, x \rangle$ over this spectral set. By Corollary 1, the maximum is attained at some a which strongly operator commutes with c. So, this maximum $= \langle c, a \rangle = \langle \lambda(c), \lambda(a) \rangle = \lambda_1(c) + \lambda_2(c) + \cdots + \lambda_k(c)$ since $\lambda(a) = (1, 1, \ldots, 1, 0, 0, \ldots, 0)$. Thus, *for any $c \in \mathcal{V}$, the sum of the largest k eigenvalues equals the maximum of $\langle c, x \rangle$ over the set of all idempotents of rank k.* This is a well-known *variational principle*, see [1]. We remark that Theorem 1 falls short of justifying this principle. For a broader result in the setting of certain hyperbolic systems, see [2], Corollary 5.6.

Example 4. In $\mathcal{V}$, let K be a closed convex cone that is also a spectral set. For example, $K = \lambda^{-1}(Q)$, where Q is a permutation invariant closed convex cone in $\mathcal{R}^n$ [9]. For a function $f : \mathcal{V} \to \mathcal{V}$, consider the cone complementarity problem, $\mathrm{CP}(f, K)$, which is to find $x \in \mathcal{V}$ such that

$$x \in K,\ y := f(x) \in K^*, \text{ and } \langle x, y \rangle = 0,$$

where K^* denotes the dual of K in $\mathcal{V}$. We specialize Corollary 1, Item (iv) to get: If a solves $\mathrm{CP}(f, K)$, then a strongly operator commutes with $-f(a)$. This means that

$$a \in K,\ \ b := f(a) \in K^*,\ \text{ and } 0 = \langle a, b \rangle = \langle \lambda(a), \widetilde{\lambda}(b) \rangle,$$

where $\widetilde{\lambda}(b) = -\lambda(-b)$.

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