Aaron Bornstein

History 338

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An Incomplete Truth:

The Role of Crisis and Politics in Early Twentieth Century Mathematics

FACULTY RECOMMENDATION

Comments on Aaron Bornstein's Seminar Paper for History 338

Aaron Bornstein's paper "Incomplete Truth: The Role of Crisis and Politics in Early Twentieth Century Mathematics" is a telling example of how broad the liberal arts are, though we sometimes overlook mathematics in favor of the more traditional fields of history, literature and philosophy. This is an extremely comprehensive, well-researched and presented study of the 20th century history of mathematics with special attention given to the conflicts among mathematicians. In its reach it affords all of us an opportunity to expand beyond our intellectual comfort zones, and it is certainly worthy of a Julia Rogers Research Prize.

The requirements for this course were that students write a 20-30 page paper based on primary sources. In Aaron's case the math itself became a primary source as did the selected correspondence of the mathematicians, some found in the Julia Rogers Library, others in the Library of Congress. The strength of the end result is Aaron's ability to provide an historical context for the battles of the mathematicians and to tell a compelling story of the conflict over set theory and its ramifications. There are numbers here, but there are also human beings and an historical narrative. He tells the story with sufficient attention to a festering problem in all nonfiction: that is how much to explain to outsiders. And he does this well, at least the students in my seminar thought so. As I was, they were impressed with the range of the paper--from Euclid to Turing in thirty pages.

Additionally Aaron came up with the counter intuitive conclusion (at least for mathematicians) that math does not always rely solely on the facts and is, in its incompleteness, like other disciplines in the uncertainties that in a sense are necessary for further development of the field. Yes, he writes, "the square root of two is always an irrational number, but to truly understand the impact of that statement, one must know of the Pythagorean School ...and of the blood that was spilt over what is now considered to be such a trivial mathematical fact."

There are many aspects of his paper worth applauding, but I would particularly draw attention to the ambitiousness of the design of his paper and his success in making a difficult subject readable and interesting.

Jean H. Baker

RESEARCH METHODS

As a History and Computer Science double major, Goucher's library services have served as a valuable asset for research and topic exploration. The purpose of my submission for the Julia Rogers research prize was to combine my diverse interests through the exploration of the development of mathematical foundations at the turn of the 20th century and its effect on the theory of computation. This process was accomplished by dividing the research into five distinct stages: preliminary reading, primary source analysis, topic outlining, drafting, and peer/faculty review.

From the collected works of L.E.J Brouwer and Constance Reid's biography of David Hilbert to William Dunham's *Journey Through Genius* and more, the Athenaeum's large collection of primary and secondary literature greatly supplemented the preliminary reading phase of my research. In addition to taking advantage of the Athenaeum's resources, I traveled to the Library of Congress where I located a valuable compilation of primary sources annotated by the Dutch mathematician Dirk Van Dalen. I also reached out to both Professor Robert Lewand and Professor Justin Brody in the Math and Computer Science department to guide my research. After gaining a solid appreciation of the theory, sources and history necessary for embarking on this project, I began the process of outlining my paper. The greatest challenge was providing an analysis of technical topics to readers with no formal mathematical background. I found that the use of parables and story-telling to explain complex concepts helped to overcome this challenge.

During the drafting of my paper, the invaluable expertise and advice of both Professor Baker and Professor Lewand helped me maintain focus. I am greatly indebted to them and my peers for pushing me to turn "Incomplete Truth: The Role of Crisis and Politics in Early Twentieth Century Mathematics" in to one of the strongest works of my undergraduate career.

Aaron T. Bornstein

ABSTRACT

The turn of the twentieth century revolutionized contemporary culture and science. Artists revolted against the longstanding establishment through subjective Post-Impressionist exploration. In the natural sciences, physicists began to realize that their classical understanding of the world was flawed. In the field of mathematics a philosophical debate reigned over the classical foundations of the discipline. As the first quarter of the century progressed, Formalists, led by David Hilbert clashed with Intuitionists, led by L.E.J Brouwer. The schism divided the foremost mathematicians of Europe and led to the integration of personal politics into the field of mathematics. Contemporary mathematicians consider the culmination of the debate to have had a shocking, yet anticlimactic conclusion. However this paper argues that Gödel's incompleteness theorems and the failure of Hilbert's reforms inspired Alan Turing, and thus contributed to the development of the modern theory of computation.

INTRODUCTION

On August 6th 1900, a sweltering Paris summer day, two hundred and fifty of the world's greatest mathematicians flocked to attend the lecture of the century. As the newly constructed Gare de Lyon clock tower struck nine, they gathered to hear the great Göttingen Professor David Hilbert inaugurate the Second International Congress of Mathematicians.[1] Over one thousand individuals had registered for the event in April, but by August, the summer heat, coupled with the overwhelming popularity of the affair, had deterred all but the most resolute. Hilbert would soon announce his program to address the twenty-three most significant problems facing the field of mathematics. These remarkable problems piqued the curiosity of the international mathematical community and directed the course of mathematical development for the next half century.

THE NEW CENTURY

Tracing the history of mathematical foundations cannot be conducted in a vacuum. The turn of the twentieth century brought about radical changes in all domains including the arts and natural sciences. The discovery of antibiotics and improvements in hospital sanitation led to a population boom. The explosive growth contributed to wide scale migration towards industrialized cities. At the turn of the nineteenth century, only one Western city, London, had a

[1] "ICM Paris 1900 *The Congress in Detail*. N.p., n.d. Web. 12 May 2013. <http://euler.us.es/~curbera/icm/PaginasH/1900-cd.htm>.

population that exceeded one million. By the year 1900, there were eight such cities; New York, Chicago, Philadelphia, Paris, Vienna, Berlin, Moscow and St. Petersburg.[2]

Many in the arts felt that the invention of photography relieved artisans of their age-old duty to memorialize their surroundings. The subsequent Impressionist and Post-Impressionist movements paralleled the radical change in nineteenth century mathematics. Just as Post-Impressionists were now free to express what they saw beyond their own field of vision, so too were mathematicians able to experiment and examine the foundations of their discipline. As Hilbert prepared his inauguration of the new mathematical era, the Post-Impressionist movement was firmly entrenched. Names such as Vincent Van Gogh, Georges Seurrat and Henri Matisse were well known throughout the intellectual circles of Europe.[3]

As in the arts, the closing of the nineteenth century paved the way for a revolution in scientific thought. In the same year that Hilbert was to give his centennial address, Max Planck published his quantum hypothesis.[4] Five years later, Albert Einstein confirmed that Planck's new theory was superior to Newton’s traditional model. For the first time since its compilation by Isaac Newton in his *Principia Mathematica*, the foundations of physics were in flux. Coinciding with the revolutions occurring in the arts and sciences, Hilbert now proposed that mathematicians begin to examine the meaning of mathematical truth in order to determine mathematical potential.

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[2] Robin W. Winks and R. J. Q. Adams. *Europe, 1890-1945: Crisis and Conflict*. New York: Oxford University Press, 2003. p2

[3] Ibid p5

[4] In order to explain black body radiation Max Planck composed his quantum hypothesis. All objects release thermal radiation. Hotter objects emit higher frequencies of light. Most objects that people interact with on a daily basis are cool enough that their thermal radiation is not visible to humans. However as objects are heated their radiation emission can reach the visible range, first red, red and yellow, then across all visible frequencies. The phenomenon is called Black Body radiation. Max Planck showed that Black Body radiation cannot be expressed with classical Newtonian physics.

MATHEMATICAL TRUTH

Mathematical truth is seemingly dissimilar to historical, scientific and artistic truth. It is neither transformed by paradigm shifts nor subjective interpretations. Mathematical truth by definition is foundational; the truth of any mathematical statement must be explicitly proven. The axiom serves as the core of mathematical proof. Axioms are a set of base assumptions -- outwardly unequivocal truths -- from which the rest of mathematics is derived. When a mathematical statement is proven, it becomes a theorem. Theorems are then used to construct new arguments, which in turn form new theorems. Unlike in history and the natural sciences, mathematical truth outwardly leaves no room for conjecture. If a statement cannot be proven, no matter how self-evident it may seem, it cannot be used. In the field of mathematics there is a story in which a philosopher, a physicist and a mathematician are on a train in Scotland. The philosopher looks out of the window, sees a black sheep standing in a field, and remarks, "How odd. All the sheep in Scotland are black!" "No, no, no!" says the physicist. "Only some Scottish sheep are black." The mathematician scoffs at his companions' primitive thinking and exclaims, "In Scotland, there is at least one sheep, at least one side of which appears to be black from here some of the time."

However, if two equally true, yet contradictory mathematical arguments arise from axioms in one of its critical bodies (geometry, algebra, arithmetic, etc.), a foundational crisis occurs. For over two millennia, mathematicians in times of crisis had to decide whether they could salvage the working axioms of a mathematical body, or whether they must scrap that sub-discipline from mathematics all together. At the turn of the twentieth century, having averted three foundational crises, mathematicians were confident that they could prove any argument. They believed that any crisis could be averted, as long as the correct set of axioms was maintained. When paradox arose from set theory, the newest field of mathematics at the time, mathematicians assumed that they could resolve the troubles that plagued their discipline. L.E.J Brouwer argued that discarding some of the classical axioms of the past could restore order. However, Hilbert believed that these classical axioms were too important. In 1922, he set out on a campaign to save mathematics from Browuer. By the conclusion of the fourth foundational crisis, the young German mathematician Kurt Gödel showed the world something revolutionary. He demonstrated that there is no absolute mathematical truth, that no set of axioms can prove every mathematical statement. As in the sciences and the arts, complete mathematical truth is unobtainable.

SET THEORY AND PARADOX

The development of mathematical truth is often intertwined with the ambitious politics of its creators. A prime case study of this phenomenon is the Intuitionist versus Formalist debate of the fourth foundational crisis, whose roots lay in the creation of set theory by Gregor Cantor in 1874.

The eldest of six children, Cantor was born in St. Petersburg in 1845 to Georg Waldemar Cantor, a successful German-Jewish stockbroker who converted to Lutheranism and Maria Anna Böhm a respected Catholic artist. Cantor's youth was shaped by art, math, and theology.[5] When he graduated from the Wiesbaden Gymnasium, his father encouraged him to apply his mathematical skills to the practical trade of engineering. Although Cantor's passion was mathematics, he enrolled at the University of Zürich to pursue an engineering degree at his father's behest. In June 1863, Cantor's father died. Using the money received from his inheritance, Cantor immediately transferred to the University of Berlin to study mathematics.[6]

[5] University of St Andrews in Scotland. "Cantor Biography." The MacTutor History of Mathematics Archive.
[6] Ibid

As a student at the University of Berlin, Cantor was greatly influenced by the work of his doctoral advisor Karl Weierstrass. During Cantor's youth, Weierstrass laid the axiomatic foundations for calculus through the use of abstract symbols.[7] Prior to Weierstrass, axioms were often defined using natural languages such as French, German, Latin, Greek and English.[8] The use of these natural languages often led to the misinterpretation of mathematical works and the construction of false proofs. Given the foundational nature of mathematics, every theorem dependent upon a false proof had to be discarded. Weierstrass's universal symbols minimized the amount of ambiguity in mathematical expression.

In Berlin, Cantor examined different groups of numbers for common properties. He soon realized how complex the relationships between different sets of numbers were, and began to compile proofs using his advisors symbols for a new mathematical system to model these relationships.[9] His work led him to believe that mathematics was ingrained with divinity.[10] Cantor's background in theology and his intense devotion to his Lutheran faith inspired him to apply his new system to express the infinite.

Before Cantor, the mathematical infinite was treated as an abstract concept, with no real physical meaning. Since the time of Zeno in Ancient Greece, mathematicians recognized the existence of the potential infinite in the case of the natural numbers (0, 1, 2, 3, 4 …n, n+1, etc.). They understood that there was no largest natural number, for someone could always add one to the greatest imaginable number and get an even greater number in return.

Cantor conceded that the infinite could not be counted. However his work with numerical relationships led him to believe the infinite could be enumerated. In order to do so, he

[7] Real Analysis
[8] Dunham, William. *Journey Through Genius: The Great Theorems of Mathematics.* New York: Wiley, 1990. p.250-251
[9] Ibid p251
[10] Ibid p252

constructed the concept of set cardinality.[11] According to Cantor, objects did not have to be counted in order to determine whether they were of the same quantity. He claimed, "two sets M and N are equivalent … if it is possible to put them, by some law, in such a relation that to one another that to every element of each one of them corresponds to one and only one element of the other."[12]

Cantor's new cardinality had widespread mathematical implications. For example, in order to determine whether there is the same number of students in a classroom as there are seats, a professor no longer has to count all the students and seats in the classroom. Instead, the professor can tell every student to take a seat. If students are left standing, then the professor can conclude that there are more students than seats. However, if there are vacant seats, then the professor knows that there are more seats than students. Still, if no students are standing and no seats are vacant, then it can safely be assumed that there is the same number of students as there are seats.

Using the set of all natural numbers $\mathbb{N}$, whose count he entitled $\aleph_{0,}$ Cantor employed set cardinality to enumerate infinitely sized numerical collections, such as the set of all integers and the set of all rational numbers. As he did so, Cantor wondered if there existed sets that were non-denumerable. Using his diagonal argument, Cantor was able to show that the set of all real numbers $\mathbb{R}$ was non-denumerable. To do so, he took an arbitrary list of **r** real numbers, and then demonstrated that the diagonal of the list did not have a one to one correspondence with the set of all natural numbers.

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[11] Ibid p254
[12] Ibid p253

Cantor hypothesized that the cardinality of the natural numbers was less than that of $\mathbb{R}$, which he termed $\aleph_{1.}$ [13] He argued that there existed a hierarchy of non-denumerable infinitely numerical sets. Each layer of the hierarchy consisted of the powerset, of the previous layer.[14] These observations highlighted a paradox deeply rooted in the core axioms of set theory. By Cantor's new definition, the infinite could now be enumerated by comparing set cardinality. Yet the powerset of all possible numbers contains a set larger than itself, which is paradoxical.

As a result of his belief in the divinity of mathematics, Cantor did not fear paradox.[15] When he published his findings and his set theory in 1874, he shocked the mathematical world. Mathematicians like Leopold Kronecker claimed that Cantor's use of the infinite was unnatural, while others, like Poincare, considered set theory to be a tumor that would have to be removed by future generations.[16] Attacks on Cantor's work and his character led to a mental breakdown in 1884. From his first hospitalization until his death in 1918, Cantor was unable to contribute to the field that he pioneered.[17]

While Cantor was not concerned about the contradictions of set theory the paradox he established led to foundational crisis. In set theory the new crisis would be explored and popularized by the pacifist, logician and analytical philosopher Bertrand Russell. Russell, unlike Cantor, was an atheist. His mother died when he was two, and his father, the son of Earl Russell, former Prime Minister of England, died when he was four. Bertrand and his older brother Frank were placed in the care of their paternal grandparents. Bertrand's childhood was a lonely one, for

[13] Modern Mathematicians do not assume the continuum hypothesis and therefore denote non-denumerable cardinalities as ב and not א

[14] In mathematics, the powerset a set S is the set of all subsets of S. For example If S is the set {1,2,3 }, then the subsets of S are: {} , {1}, {2}, {3}, {1,2}, {2, 3} ,{1,3}, {1, 2, 3}.

[15] Chaitin, Gregory. "*The Search for the Perfect Language*." Lecture from the Perimeter Institute for Theoretical Physics, Waterloo, Ontario, September 9, 2009.

[16] Dunham, William. *Journey Through Genius: The Great Theorems of Mathematics.* New York: Wiley, 1990. p279

[17] Ibid

in addition to his parents, his grandfather died when he was only six years old. Russell's passion for mathematics began at age eleven when his older brother Frank agreed to teach him Euclid's *Elements*. In his later autobiographies and interviews Russell often noted that in his childhood:

> There was a footpath leading across the fields to new Southgate, and I used to go there alone to watch the sunset and contemplate suicide. I did not however, commit suicide, because I wished to know more of mathematics.[18]

By age fifteen, Russell began to contemplate the validity of Christian religious dogma. He concluded that there was no free will. When Russell was eighteen, he read the autobiography of his godfather John Stuart Mill. For the rest of his life Russell would declare himself an atheist.[19]

The young atheist's interest in the paradoxes of set theory differed greatly from that of Cantor. Unlike Cantor, Russell did not consider mathematics to be divine. He felt that if the paradoxes of set theory could be addressed, then mathematics could serve as the foundation for rational expression, and so, with the help of his former Trinity College Professor Alfred Whitehead, Russell set out to explore the paradoxes of set theory.

From 1901 to 1910 Russell and Whitehead began to link the principles of Cantor's set theory to the foundations of other mathematical bodies. The team published its work in their *Principia Mathematica* in 1910. It was in this work that Russell made Cantor's set theory fundamental to the foundations of mathematics. As he continued to explore the field of set theory, Russell began to notice more paradoxes. The most famous one was published in 1903, three years after Hilbert's centennial address. Russell's Paradox states that the set of all sets that do not contain themselves, does not know whether it contains itself. The confusing uncertainty of Russell's self-swallowing set is better explained through its equivalent Barbershop Paradox.

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18 Russell, Bertrand. *The Autobiography of Bertrand Russell*. 1st American ed. Boston: Little, Brown, 1967. p38

19 Ibid p35-36.

In a small New England town there exists a local barbershop. Three times a week the town barber shaves all of the men in the town who do not shave themselves. However this barber has a peculiar pet peeve: he refuses to shave anyone in the town who has ever shaved himself. One day a city-slicker drives through the town and decides that he would like to have a shave. When he arrives at the barbershop he is refused service. In anger, he cries out to the clean-shaven barber, “How do you shave yourself every week if you only shave those who do not shave themselves?” As the New Yorker leaves, the barber begins to ponder the question. He realizes that he can only shave in the morning if, and only if, he does not shave. This drives the barber insane, for when he decides not to shave himself anymore; he realizes that he no longer shaves every man in the town who does not shave himself.

Just as the town barber did not know whether to shave himself, sets at the time of Russell were plagued by this peculiar ambiguity. Russell’s revelation to the mathematical community about the true significance of set theory, and the paradoxes that afflicted it, spawned the fourth foundational crisis.

BROUWER & WYEL: FOUNDATIONAL QUESTIONS AND THE INTUITIONIST ATTACK

At first, few mathematicians took the new crisis seriously. In 1904, in a speech at the University of Heidelberg, Hilbert told the mathematical community that the framework of his Centennial address was sufficient to confront the crisis stemming from set theory.[20] Hilbert reassured his fellow mathematicians that if they could compile a pure set of axioms, they would be able to deduce whether any mathematical statement was decidable, and all mathematical paradox would disappear. Confident that the mathematical community would make progress on

[20] Reid, Constance. *Hilbert*. New York: Copernicus, 1996. p99

the program outlined in his centennial address, Hilbert decided to apply his expertise to the axiomization of the new physics.

However not everyone was as confident about the stability of mathematics as Hilbert. At the University of Amsterdam, an energetic doctoral student named Luitzen Egbertus Jan Brouwer began to examine the works of Russell and Whitehead. In 1905, looking for a new challenging field to explore for his thesis that would set him apart from his peers, Brouwer submitted a proposal to his doctoral advisor Diederik Korteweg. [21] He intended to explore the newly discovered paradoxes stemming from set theory.[22] From his research, Brouwer rejected the concept of mathematics as a formal language of reason or logic. He claimed, "mathematical reasoning is not logical and that it is only out of a poverty of language that mathematical reasoning uses the connectives of logical reasoning."[23] Brouwer turned to Immanuel Kant and Rene Descartes to construct his own Intuitionist approach to mathematics. He was particularly influenced by Kant, who wrote, "All Human Knowledge begins with intuition, then passes to conception, and ends with ideas."[24] Brouwer argued that this Intuitionist idea should extend to mathematical truth. His intuitionism treated mathematics as a primordial mental activity, independent of language or formal expression. Brouwer concluded that the paradoxes of set theory were the result of a deeper misunderstanding about the origins of classical mathematics.

Classical mathematics at the time of Brouwer can be traced to the 4th Century BCE. Two and a half thousand years earlier, the Ancient Pythagorean cult set out to discover the origins of meaning through an exploration of mathematical truth. Since the time of Pythagoras,

[21] Diederik Korteweg was the most influential mathematician of Holland at the time.
[22] *1906 to D.J Korteweg*, Dalen, D. van *The Selected Correspondences of L.E.J. Brouwer.* London: Springer p25
[23] Ibid
[24] Kant Immanuel, *Critique of Pure Reason* p730

mathematical statements were considered to be either true or false. Augustus De Morgan, in the mid-nineteenth century, categorized this phenomenon as the principle of the excluded middle.

The principle of the excluded middle is the core of proof by contradiction, one of the most valuable tools in mathematics. In proof by contradiction, the mathematician assumes a statement is true or false. The mathematician then attempts to show that the use of the statement leads to a logical contradiction. If the statement leads to a contradiction, the mathematician can safely conclude that it is not mathematically valid. Many of the core theorems of contemporary mathematics stem from proof by contradiction. For example, the proof for the existence of irrational numbers that led to a war between the Pythagorean schools, mathēmatikoi and akousmatikoi, in the 5th century BCE, was proved by contradiction.

Unlike the Pythagoreans and those who followed them, Brouwer believed that mathematics was not discovered. He argued instead that mathematical truth originated in the sub-consciousness of the mathematician, and therefore only constructive mathematical proofs could be considered valid. He claimed that there existed a third possibility to the classical determinability of a mathematical expression. He explained:

> Suppose that A is the statement "There exists a member of the set S having the property P." If the set S is finite then it is possible – in principle- to examine each member of S and determine either that there is another member of S with the property P or that every member of S lacks the property P. However for infinite sets we cannot even in principle- examine each member of the set. Therefore whether or not S has the property P is unknown and therefore cannot be assumed.[25]

According to Brouwer, Cantor's set theory abstracted the principle of the excluded middle into the Axiom of Choice, which stated that the product of a collection of non-empty sets is itself non-empty. It was this Axiom of Choice, Brouwer argued, that led to Russell's paradoxes. Therefore, he reasoned, in order to resolve the foundational crisis of set theory, mathematics

[25] Reid, Constance. *Hilbert*. New York: Copernicus, 1996. p149

would have to be reformed, and all proofs dependent on the principle of the excluded middle would have to be discarded or replaced with constructive proofs.

In 1906 Brouwer submitted his doctoral dissertation to Kortweg for approval. He was disappointed to learn that Kortweg disapproved of his use of Kant and 'mystic philosophy' to describe mathematical truth. In this incident, Brouwer demonstrated his eagerness for confrontation and immediately sent Kortweg a second copy of his dissertation combined with excerpts from the works of Russell, Whitehead and Hilbert which addressed the philosophical questions of mathematics.[26] Kortweg was extremely offended by the gesture, but nonetheless he agreed to reexamine Brouwer's work.[27] In 1907 Brouwer was awarded his doctorate for his dissertation *Over de Grondslagen der Wiskunde* (On the Foundations of Mathematics). In spite of his approval Korteweg urged Brouwer to concentrate on more respectable mathematics in order to enhance his mathematical reputation.[28]

In 1909 Brouwer was presented with the post of privaatdocent, or unpaid lecturer, at the University of Amsterdam. For the first three years of his position, Brouwer set aside his Intuitionist program to work on the field of topology. Brouwer transformed the field of topology into a sub discipline of classical mathematics through his invariance of dimension proof, the fixed point theorem, the mapping degree theorem and the formal definition of dimension.[29] Brouwer's successes in the field of topology brought him international recognition, and in 1912 he was elected to the Dutch Royal Academy of Sciences. Later that year he was offered a full professor position in the field of set theory, function theory, and axiomatics. On the inauguration of his post, Brouwer delivered his first lecture on the rejection of the early Formalist approach to

[26] *1906 to D.J Kortweg*, Dalen, D. van *The Selected Correspondences of L.E.J. Brouwer.* London: Springer p25
[27] *DJ Korteweg response II.XI* Dalen, D. van *The Selected Correspondences of L.E.J. Brouwer.* London: Springer p31-35
[28] Ibid
[29] *Brouwer,* Stanford Online Encyclopedia of Philosophy.

the foundational crisis. *Intuitionism and Formalism* became the first widespread publication in English on intuitionism.

During this period of mathematical success, Brouwer became friends with some of the greatest mathematicians of Europe, including Herman Weyl, Felix Klein and Otto Blumenthal. In 1914, he accepted an offer from David Hilbert to join the editorial board of the prestigious mathematical journal *Mathematische Annalen*. At this time, Brouwer maintained a steady correspondence with Hilbert about subjects such as geometry and politics. When writing Brouwer always addressed Hilbert by his formal Gehemirat[30] title. On November 25, 1918, as WWI drew to a dramatic close, Brouwer wrote with affection to Hilbert addressing his letter for the first time to Herr Hilbert rather than the formal Herr Gehemirat. Brower wrote, "May the hale of your fatherland overcome the present crisis; and may the German lands soon blossom in exceptional ways in a world of justice.[31]

Later, during the Treaty of Versailles in 1919, he voiced to Hilbert his disgust at the behavior of the Allies towards post war Germany:

> I don't know whether these lines can bring you any consolation, but I set great store by declaring to you on the day of the signing of the Peace Treaty that seen from Holland, the Allied Powers have, through the peace extorted today, taken upon themselves a guilt, that is certainly not less than the combined guilt of those (whoever they actually were!), that started this war. [32]

His letter ended with the soon to be ironic statement, "At the end of the day, we scholars are after all in a fortunate situation, because such a large part of our realm of thoughts is completely independent of political nonsenses."[33]

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[30] Privy Councilor
[31] *To D. Hilbert 25.XI.1918* Dalen, D. van *The Selected Correspondences of L.E.J. Brouwer*. London: Springer160
[32] *To D.Hilbert 6-28- 1919* Ibid p161
[33] Ibid

By the end of 1919, Brouwer's relationship with Hilbert grew to the point where he was offered a professorship at the University of Göttingen. However Brouwer declined the offer in order to continue his work on Intuitionism.[34] Hilbert took Brouwer's refusal of his offer as a personal offense, and the relationship between the two mathematicians soured.[35]

In 1918, Brouwer began to reconstruct the axioms of mathematics under his own Intuitionist framework. He spent the next two years of his life carefully reconstructing the axioms of set theory, without all the classical assumptions based on the principle of the excluded middle and the axiom of choice. In 1919, Brouwer published his results in the international journal, the *Jahresbericht der Deutschen Mathematiker Vereinigung*. The article, *Intuitionist Set Theory,* was Brouwer's first international publication of his Intuitionist crusade. The article directly attacked the axiomatic framework laid out by Hilbert in his centennial address nineteen years earlier.

Brouwer wrote:

> The axiom of the *solvability of all problems* as formulated by Hilbert in 1900 is equivalent to the logical Principle of the Excluded Middle; therefore since there are no sufficient grounds for this axiom and since logic is based on mathematics - and not vice versa- the use of the Principle of the Excluded Middle is *not permissible* as part of a mathematical proof. The Principle of the Excluded Middle has only scholastic and heuristic value; so that theorems in their proof cannot avoid the use of this principle lack all mathematical content.[36]

In the year following the publishing of Brouwer's *Intuitionist Set Theory* and the initiation of the Grundlagenstreit[37], most distinguished mathematicians scoffed at Brouwer's radical attacks on Hilbert. Like Korteweg before them, they believed that Brouwer was overly pessimistic, and that the paradoxes of set theory were at most minimal annoyances. The majority believed that too

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[34] *Brouwer,* Stanford Online Encyclopedia of Philosophy.
[35] Stigt, Walter P. van. *Brouwer's Intuitionism*. Amsterdam: North-Holland, 1990. P.80
[36] *Intuitionist Set Theory* Brouwer: Mancosu, Paolo. From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s. New York: Oxford University Press, 1998. p23
[37] German word for the Foundational Debate.

much effort would be required to restructure mathematics after the loss of the Axiom of Choice and the Principle of the Excluded Middle. Brouwer's few early disciples were his young gifted students at the University of Amsterdam; they listened passionately to the lectures of their Professor as he began to reconstruct mathematics.

Then, in Zürich, a former student of Hilbert and one of the most distinguished mathematical writers of the time, Hermann Weyl, began to have doubts about the validity of his former mentor's program. Weyl was considered to be a Universal Mathematician. By the time of his death in 1951 he had contributed to modern number theory, algebra, geometry, analysis, mathematical physics, logic and epistemology.[38] Weyl had been interested in the foundational crisis since the publishing of the *Principia Mathematica* in 1910. He published his first book on the subject, *Das Kontinuum* in 1918, where he declared that Russell's paradoxes showed the foundations of set theory to be "a house built on sand."[39] Weyl believed that these paradoxes were symptoms of deeper uncertainty stemming from the core axioms of classical mathematics. However, he did not know how to address these issues without disfiguring classical mathematics in such a manner that it could ever be reassembled, and therefore supported Hilbert's Centennial framework.[40] When Weyl read Brouwer's *Intuitionist Set Theory*, he abandoned his previous concessions and declared the Intuitionist program as, "Die Revolution in der Mathematik."[41] Weyl began working on a defense of the Intuitionist school of thought. On June 5th 1920 Weyl wrote to Brouwer:

> Finally I have sent the long promised [work] off to you. It should not be viewed as a scientific publication but as a propaganda pamphlet, thence the size. I hope you will find

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[38] Mancosu, Paolo. From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s. New York: Oxford University Press, 1998. p65
[39] Ibid p70
[40] Ibid p66
[41] Ibid

> it suitable to rouse the sleepers, that is why I want to publish it. I would be grateful for your opinions and comments.[42]

When Brouwer received Weyl's letter he was ecstatic. He quickly sent back comments to Weyl:

> Your unreserved scientific assistance has given me an infinite pleasure. The reading of your manuscript was a continual delight and your exposition; it seems to me, will also be clear and convincing to the public.[43]

In 1921 Weyl finally published, *Über die neue Grundlagenkrise der Mathematik,* he wrote:

> The antimonies of set theory are usually treated as border conflicts concerning only the most remote provinces of the mathematical realm, and in no way endangering the inner soundness and security of the realm and its proper core provinces. The statements on these disturbances of the peace that authoritative sources have given (with the intention to deny or mediate) mostly due not have the character of a conviction born out of thoroughly investigated evidence that rests firmly on itself. Rather, they belong to the sort of one-half to three quarters honest attempts of self-delusion that are so common in political and philosophical thought.[44]

He continued:

> Indeed any sincere and honest reflection has to lead to the conclusion that these inadequacies in the border provinces of mathematics must be counted as Symptoms. They reveal what is hidden by the outwardly shining and frictionless operation in the center: namely, the inner groundlessness of the foundations that rests the super structure of the realm. I only know of two attempts to eradicate this evil. One is due to Brouwer, who, since 1907 has put forward certain guiding ideas about his envisioned form of set theory and analysis.[45]

With these words, Weyl publicly declared his allegiance to the Intuitionist school of thought, and declared war on those who ignored the significance of the foundational questions of mathematics. The great debate of the fourth foundational crisis had begun.

HILBERT: CLASSICAL FOUNDATIONS AND THE FORMALIST RESPONSE

Meanwhile, in Göttingen, Hilbert was extremely worried about the rapid growth of the Intuitionist reformation. He believed that a widespread adoption of the Intuitionist program,

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[42] *From H. Weyl 6.V.1920* Dalen, D. van. *The Selected Correspondences of L.E.J. Brouwer.* London: Springer p215

[43] *To H.Weyl – after 6.V.1920* Dalen, D. van. *The Selected Correspondences of L.E.J. Brouwer.* London: Springer p217

[44] *On the New Foundational Crisis of Mathematics* Weyl: Mancosu, Paolo. From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s. New York: Oxford University Press, 1998. p86

[45] Ibid

however tempting, was a mistake. Hilbert's experiences with two foundational crises led him to understand the risks of throwing the core principles of classical mathematics out with the bathwater (Das Kind mit dem Bade ausschütten). Hilbert was born in January 1862, in the small Prussian town of Königsberg. His father, Otto Hilbert, was a respected Prussian Judge who instilled in his son a belief in order from a young age.[46] When Hilbert was a young mathematician, the mathematical world was recovering from the second foundational crisis. The crisis stemmed from the groundlessness of the definitions of a limit in calculus. Weierstrass, Cantor's mentor, resolved the foundational crisis through the axiomatization of calculus with his symbolic language that inspired Cantor's set theory. The other foundational crisis of Hilbert's youth arose from the axioms of Euclidian geometry.

In 300 BCE, Euclid of Alexandria compiled the first set of axioms and theorems in order to construct the first solid foundational basis for mathematics. From his set of twenty-three definitions, five postulates, and five general assumptions, Euclid derived all of previous mathematics. The work's five postulates and five general assumptions formed the ten axioms of Euclid's *Elements*. The *Elements* was the first implementation of the axiomatic method.[47] For over two thousand years, Euclid's thirteen-volume collection of geometric and numeric rules served as the core of classical mathematics.

However Euclid's fifth postulate bothered many mathematicians. While the postulate seemed to be self-evident, it did not conform to the structure of Euclid's other axioms.[48] For over two millennia, mathematicians from Proclus to Johann Lambert tried in vain to derive the

[46] Reid, Constance. *Hilbert*. New York: Copernicus, 1996. p1-8
[47] Chaitin, Gregory. "The Search for the Perfect Language." Lecture from The Perimeter Institute for Theoretical Physics, Waterloo, Ontario, September 9, 2009. p28-9
[48] Euclids Fifth Postulate or Parallel Postulate states that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. Heath, Thomas Little. *The thirteen books of Euclid's Elements,*. 2d ed. New York: Dover Publications, 1956. Print.

fifth postulate from Euclid's other postulates. Then, in the early nineteenth century, three mathematicians, Bolyai, Gauss and Lobachevski, independently attempted to prove the soundness of the fifth parallel postulate. Each of the three mathematicians attempted to construct an equivalent geometrical system after omitting the parallel postulate. In their new geometries they defined triangles as three sided shapes, whose angles added up to less than 180 degrees. If their systems led to a contradiction, they would be able to show that the fifth parallel postulate was absolute. To their surprise, the more they worked with the axioms of their new system, the more they realized that there might not be any contradictions. In 1854, Reimann added to the confusion when he constructed a geometric system whose angles exceeded 180 degrees which also seemed to be consistent. Then, in 1865, Eugino Beltrami proved that all non-Euclidian geometry was as logically consistent as Euclid's *Elements*.[49] The question remained, if there were now three geometries, which geometry represented the real world? This chaos in mathematics led to the third foundational crisis.

In 1872 Göttingen Professor of Geometry Felix Klein proposed that Euclidean and non-Euclidean geometric planes could be considered special cases of cuts in conic sections.[50] It was David Hilbert, who in 1895 at the request of Klein, transferred to the University of Göttingen to work on the program of formalizing geometry. In 1899 Hilbert published the first all-encompassing set of geometric axioms since Euclid. His axiomatization of geometry inspired the mathematical community to invite Hilbert to give his centennial address in August 1900.

Hilbert's experience formalizing all three systems of geometry led him to believe that set theory could also be axiomatized. He affirmed that classical mathematics could be saved from

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[49] Dunham, William. *Journey Through Genius: The Great Theorems of Mathematics.* New York: Wiley, 1990.56

[50] Given a figure formed by the intersection of a plane and a right circular cone. Depending on the angle of the plane with respect to the cone, a conic section may be a circle, an ellipse, a hyperbola, or a parabola. Felix Klein argued that these shapes could be used to model the different geometric planes.

paradox through the correct selection of axioms. Until 1921, Hilbert did not pay considerable attention to Brower's work. In fact, he never even read Brouwer's *Intuitionist Set Theory*.[51] However when his former student Weyl gave Brouwer his support, Hilbert was irritated.[52] With the help of his assistant Paul Bernays, Hilbert set out to construct a foundational program that guaranteed the reliability of mathematics without sacrificing its classical roots.

In 1922, Hilbert published his *Neubegründung der Mathematik*[53] which he divided into two distinct sections. The first section was a response to the attacks from Brouwer's and Weyl's Intuitionist program. Hilbert defiantly proclaimed:

> What Weyl and Brouwer do amounts in principle to following the erstwhile path of Kronecker[54] they seek to ground mathematics by throwing overboard all phenomena that make them uneasy and by establishing a dictatorship of prohibitions a la Kronecker. But this means to dismember and mutilate our science, and if we follow such reformers, we run the danger of losing our most valuable treasures.[55]

The second section of *The New Grounding of Mathematics* outlined Hilbert and Bernay's: new program to reinforce the foundations of mathematics. In it he exclaimed "I should like to regain for mathematics the old reputation for incontestable truth, which it appears to have lost as a result of the paradoxes of set theory; but I believe that this can be done while fulfilling its accomplishments."[56] Hilbert's Formalist program set out to derive the foundations of set theory through a compilation of universally agreed upon axioms and proofs. The collection was to be expressed in terms of a perfect symbolic formal language, uninfluenced by the interpretation of the mathematician. According to Hilbert, it did not matter who performed an operation using his

[51]Hilbert preferred instead to get his information from conversations and lectures. Reid, Constance. *Hilbert*. New York: Copernicus, 1996. p150
[52] Ibid p148
[53] The New Grounding of Mathematics
[54] Kronecker had argued that all classical axioms that led to non -decidable mathematical statements should be removed from mathematics all together no matter how critical he is known for the quote," God made natural numbers; all else is the work of man."
[55] *Neubegründung der Mathematik* Hilbert: Mancosu, Paolo. From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s. New York: Oxford University Press, 1998. p200
[56] Ibid

axioms; if the calculations were performed correctly, the results would always be the same, and paradox would be averted.

Hilbert divided his approach to mathematical foundations into three layers. The first two layers of Hilbert's program mirrored the framework outlined in his centennial address. *Ordinary Mathematics* consisted of all the operations and proofs in classical mathematics that were performed through the use of foundational axioms. *Proper Mathematics* consisted of all the foundational axioms that were fundamental to performing operations. Hilbert believed the first two layers of his program to be sufficient enough to address the crisis stemming from set theory. However, merely averting one foundational crisis no longer satisfied Hilbert. He feared that if future foundational crises occurred, they too could lead to the loss of large portions of classical mathematics. Therefore, Hilbert's last layer attempted to prevent future foundational crises through the construction of M*etamathematics*. Metamathematics ensured that all the axioms of Hilbert's program would not lead to paradox if three things could be proven.

All proper mathematical axioms had to be:

1. **Consistent** - No two proper mathematical axioms led to statements that contradicted each other.
2. **Independent**- The removal of any one proper mathematical axiom makes it impossible to prove some of the fundamental theorems of mathematics
3. **Complete/Decidable**- Every ordinary mathematical statement can be decided from the proper mathematical axioms.

Hilbert's Formalist response to Brower and Weyl's Intuitionist program divided the majority of the mathematical community of the West into two camps, the Formalists and the Intuitionists.

DISCOURSE AND POLITICS: A CLASH OF THE TITANS

In 1925, Hilbert was diagnosed with pernicious anemia[57]. His condition greatly slowed his work.[58] Meanwhile, many of the younger mathematicians of Europe, such as Heyting, Freudenthal and Menger, set out to make their mark in mathematics by joining Weyl and Brouwer's Intuitionist revolution. By 1926, Brouwer's home in Laren had become an international center for mathematics.[59] Many great mathematicians from Caratheodory and Hurwicz to Newman and Hopf came to lecture and learn the new mathematics pouring out of the Netherlands. However, many of the more established mathematicians of Europe, such as John Von Neumann, Ernst Zermelo and Abraham Fraenkel, inspired by Hilbert's work on the axiomization of the new physics, sided with Hilbert's Formalist program and set out to reestablish the absoluteness of classical mathematics. By 1925, John Von Neuman had circumvented the paradox stemming from Cantor's use of cardinality through his use of well-ordered sets. Meanwhile, Bernay, Frankel and Zermelo began to experiment with different axioms to rid the rest of set theory of paradox.

As Hilbert recovered from his illness, the Formalist program also began to recover. By the end of 1927, Brower's work on intuitionism stalled, and the gaze of the mathematical community returned to Göttingen. In an attempt to regain support for intuitionism, Brouwer traveled to Berlin, where he gave a series of lectures promoting the Intuitionist program. Brouwer's lectures were well received in Berlin. The newspaper *Berliner Tageblatt* even proposed to publish a public debate between Brouwer and Hilbert. When this did not occur,

[57] Pernicious anemia is a condition in which the body is unable to produce enough healthy red blood cells since it does not have enough vitamin B12
[58] Reid, Constance. *Hilbert*. New York: Copernicus, 1996. p179
[59] Laren is a Dutch town in the North of Holland. It is the oldest town in the Gooi region of the Netherlands.

as a result of Germany's postwar economic situation, Brouwer was invited to speak at the University of Göttingen. During his lecture on *Whether Every Real Number has a Decimal Expansion*, one of Hilbert's disciples interjected, "You say [Herr Brouwer] that we cannot know whether in the decimal representation of π, ten 9's occur in succession. Maybe we cannot know, but God knows."[60] Annoyed, Brouwer responded curtly, "I do not have a pipeline to God." [61] Hilbert arose from the audience and shouted to Brouwer, "With your methods most of the results of modern mathematics would have to be abandoned, and to me the most important thing is not to get fewer results but to get more."[62] The auditorium erupted into thundering applause, and Brouwer left Göttingen humiliated by his adversary's unprofessionalism.[63]

However, in 1928 the Intuitionist school was graced with its first major breakthrough in the reconstruction of mathematics. Arend Heyting, one of Brouwer's most promising students, published the first formal compilation of Intuitionist logic that supported Brouwer's method of reconstructing mathematics. While Brouwer personally disagreed with Heyting's formalization of Intuitionistic principles, the work gave intuitionism much needed legitimacy.[64]

Brouwer's next battle with Hilbert took place on the international stage. After the signing of the Treaty of Versailles, the allied powers established two international intellectual organizations through the auspices of the League of Nations. The goal of the *Conseil Internationale de Recherches* and the *Union Mathematique Internationale* was to boycott and effectively undermine German scholastic dominance over Europe. Since his appointment as a fellow of the Dutch Royal Academy, Brouwer had fought to end the academic boycott of

[60] Reid, Constance. *Hilbert*. New York: Copernicus, 1996. p185
[61] Ibid
[62] Ibid
[63] Ibid
[64] Walter P. van Stigt (1990). *Brouwer's Intuitionism.* Amsterdam: North Holland p85

Germany. While many German academics such as Max Planck and Ludwig Bieberbach[65] appreciated Brouwer's efforts, Hilbert was personally offended by them. He believed that Brouwer was meddling in German affairs.[66]

In the summer of 1928, German academics were allowed, for the first time since 1919, to attend an international conference of mathematics in Bologna.[67] However, German mathematicians were still denied equal representation at the conference. Annoyed by the audacity of the Union Mathematique, Bieberbach encouraged all German Universities to boycott the conference until German mathematicians received full membership in the Union Mathematique. Brouwer urged the Dutch mathematical community to support their German brethren by also boycotting the conference. Again Brouwer's decision angered Hilbert. He interpreted Brouwer's actions as an attempt to bar him as head of the German delegation from speaking to an international audience. Hilbert immediately sent a letter to all his German colleagues in which he wrote:

> We are convinced that pursuing Herr Bieberbach's way will bring misfortune to German Sciences and will expose us all to justifiable criticism from well-disposed sides. … Our Italian colleagues have troubled themselves with the greatest idealism and expense of time and effort. … It appears under the present circumstances a command of rectitude and the most elementary courtesy to take friendly attitude towards the conference.[68]

He then proceeded to lead the German delegation, the largest at the conference, to Bologna. At the conference Hilbert used his role as the head of the delegation to address the mathematical community on the status of his Formalist program.

Hilbert announced that the work of Ackermann and von Neumann established the consistency of number theory, and that his program's much needed proof for analysis had been

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[65] Ludwig Bieberbach was head of the department of mathematics at the University of Berlin.
[66] Ibid
[67] Ibid
[68] Reid, Constance. *Hilbert*. New York: Copernicus, 1996. p188

constructed, “to the extent that the only remaining task consists in the proof of an elementary finiteness theorem that is purely arithmetical.”[69] He told the world that Brouwer’s Intuitionist program was not only dangerous, but that it was unnecessary, for the formalization of foundational mathematics would soon be at hand.[70] Hilbert was so confident that Von Neumann and Akermann’s work would rid set theory of paradox that he announced it was time for the mathematical community to begin to search for the solution to the Entscheidungsproblem, the proof for the determinability of all mathematics. Hilbert’s address at Bologna undermined the successes of the Intuitionist program.

Meanwhile, Brouwer’s refusal to attend the conference left his international reputation scarred, affording Hilbert an opportunity to end the Intuitionist vs Formalist debate. On October 1928, after his return to Göttingen from Bologna, Hilbert wrote to Brouwer:

> Because it is not possible for me, given the incompatibility of our views on fundamental questions, to cooperate with you, I have asked the members of the editorial board of Mathematische *Annalen* for the authorization, and received that authorization from Messrs. Blumenthal and Carathéodory to inform you that henceforth we will forgo your cooperation in the editing of the *Annalen*, and that consequently we will delete your name from the cover page.[71]

Hilbert, however, had not communicated with either or Carathéodory or Einstein, his two executive co-publishers before sending the letter. As such Hilbert’s action of removing Brouwer from the editorial board was not legally binding. Certainly the action surprised many of Hilbert’s closest friends. Eager to avoid a scandal, Carathéodory was sent to Amsterdam to persuade Brouwer not to take legal action against Hilbert.[72] Carathéodory, a lifelong friend of Brouwer’s, felt uncomfortable about his charge, but nonetheless set out for Holland.

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[69] *Hilbert Program,* Stanford Online Encyclopedia of Philosophy.
[70] Walter P. van Stigt (1990). *Brouwer's Intuitionism.* Amsterdam: North Holland p101
[71] Dalen, D. van. *The Selected Correspondences of L.E.J. Brouwer.* London: Springer p339-340
[72] Walter P. van Stigt (1990). *Brouwer's Intuitionism.* Amsterdam: North Holland p101

On October 30th, Carathéodory met with Brouwer. He told the Dutch mathematician that Hilbert was ill and would, "regret his decision in a few weeks."[73] Brouwer at first agreed to refrain from legal action against Hilbert. However, by November 2nd, he realized that not only was Hilbert not going to retract the letter of dismissal, but that his friends Blumenthal and Carathéodory were allowing the Göttingen professor to ruin his career. He therefore furiously wrote to both Carathéodory and Blumenthal:

> After careful consideration and extensive consultations I have to take the point of view, that the plea you directed to me, namely to treat Hilbert as of an unsound mind, could only be complied with, if it would have reached me in writing, and in fact jointly from Hilbert's wife and his family doctor.[74]

Meanwhile Brouwer began to write an appeal to his fellow editors of the *Mathematische Annalen*. On October 5th he forwarded his appeal to the editors of the *Annalen*. In it, he claimed that Hilbert was attempting to remove him from its editorial board, not because he posed any danger to mathematics. Rather, Hilbert had acted because he [Brouwer] had turned down a post at the University of Göttingen in 1919 because of their disagreements over the Formalist Intuitionist debate, and because of the politics surrounding the Bologna conference. Brouwer asserted that Hilbert's inability to cooperate was merely a pretext, since over his last six years at the *Mathematische Annalen,* he and Hilbert, "had never really cooperated."[75] He then lashed out at Blumenthal and Carathéodory for supporting Hilbert's unlawful action:

> Carathéodory and Blumenthal explain their cooperation in this undertaking by the fact that they estimate the advantages of it for Hilbert's state of health higher than my rights and honor and professional prospects and than the moral prestige of the *Mathematische Annalen*, that are about to be sacrificed.[76]

[73] Ibid
[74] Dalen, D. van. *The Selected Correspondence of L.E.J. Brouwer.* London: Springer p339-341
[75] Ibid p342
[76] Ibid p343

Blumenthal was enraged over Brouwer's comment on his character. He quickly responded to Brouwer's letter by circulating a response begging his fellow editors of the *Annalen* not to react to Brouwer until he had time to write an, "extensive exposition of some new events, which appear to me essential for judging the situation."[77]

In his subsequent explanation, Blumenthal came to Hilbert's defense. He claimed that while he was originally shocked by Hilbert's decision, he had come to the conclusion that Hilbert was correct and that Brouwer's confrontational personality, and indirectly his Intuitionist program, was dangerous to the future of mathematics. He argued that if the mathematical community did not act now to remove Brouwer from the board of the *Annalen*, the future of mathematics would be at risk when Hilbert died. Blumenthal ended his appeal by publishing the private letter Brouwer had sent him on November 2nd 1928.

Blumenthal and Hilbert's blatant personal attacks on Brouwer divided the mathematical community. Mathematicians such as Bernays, Courant and Von Neumann believed that Hilbert was justified in his decision, and that the greater goal of formalizing mathematics was more important than Brouwer's honor. Furthermore, they were disgusted that Brouwer had requested Hilbert's medical records as a prerequisite for quietly resigning from the *Annalen*. Others, such as Einstein and Carathéodory, were equally disgusted with Hilbert, since their reputations had been used as part of what Einstein termed, "a game of frogs and mice among mathematicians".[78] Carathéodory was particularly furious that Blumenthal had twisted his role in the affair, when he learned from Courant that Hilbert was using Brouwer's political views as a pretense to get rid of his opposition.[79] They, in turn, refused to sign Brouwer's dismissal notice, so Hilbert disbanded the *Mathematische Annalen* and created a new mathematical journal under his sole editorial

[77] Ibid p344
[78] Walter P. van Stigt (1990). *Brouwer's Intuitionism.* Amsterdam: North Holland p102
[79] Ibid

control. As a result of his removal from the board of the *Mathematische Annalen,* Brouwer suffered from severe bout of depression and resolved to isolate himself from the international mathematical community.[80] While Hilbert lost the support of Einstein and Carathéodory, the politics and viciousness of the *Annalen Affair* marked the end of Brouwer's work on the Intuitionist movement. Through the use of his position and his connections in Göttingen, and not through mathematic reasoning, Hilbert succeeded in purging all serious opposition to his Formalist program.

Without Brouwer's opposition, Hilbert's Formalist program thrived. In 1929, the budding German mathematician, Kurt Gödel showed that First Order Logic, one of the central tenets of Hilbert's program was consistent and complete.[81] On January 1st 1930, Zermel and Fraenkel finally published their completed formalization of set theory. Their new system rid the discipline of Russell and Cantor's paradoxes through a mathematically consistent implementation of the Axiom of Choice.[82] With the completeness of set theory restored, the Entscheidungsproblem remained the last obstacle in the way of Hilbert's formalism.

The success of the Formalist program brought Weyl back to Hilbert's sphere of influence. As Hilbert began to prepare for mandatory retirement at the age of 68, Weyl agreed to succeed Hilbert at the University of Göttingen.[83] Hilbert decided to give his retirement address in his home town of Königsberg. Confident of the future of formalism and the end of uncertainty in mathematics, Hilbert's last triumphant declaration of his thirty-five year career was broadcast to mathematicians across the world.

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[80] Ibid
[81] Chaitin, Gregory. "*The Search for the Perfect Language*." Lecture from the Perimeter Institute for Theoretical Physics, Waterloo, Ontario, September 9, 2009.
[82] Ibid
[83] Reid, Constance. *Hilbert*. New York: Copernicus, 1996. p190-191

> We must not believe those, who today, with philosophical bearing and deliberative tone, prophesize the fall of culture and accept the ignorabimus.[84] For us there is no ignorabimus, and in my opinion there is none whatever in natural science. In opposition to the foolish ignorabimus our slogan shall be:
>
> Wir müssen wissen. Wir werden wissen!
>
> *We Must Know. We shall Know!* [85]

With these words Hilbert made it known to world that the age of pure mathematical reason and absolute truth was soon to be at hand.

EPILOGUE: OUT OF THE ASHES OF AN INCOMPLETE TRUTH

The formalist paradise was short lived. In 1931, mere months after Hilbert's retirement, Gödel published his incompleteness theorems with the title, *"On Formally Undecidable Propositions of "Principia Mathematica" and Related Systems"*. He reasoned that if a system for the decidability of mathematics is simpler than all of mathematics, it will never be complete. Yet if a system is equally as complex as mathematics, it will always be equally inconsistent. Therefore, Gödel concluded that mathematical truth was innately incomplete. Gödel's proofs showed definitively that not all mathematics could be formalized, and that the Entscheidungsproblem could never be solved. Gödel's work demonstrated that Hilbert's program was unfeasible, because for a formal system of mathematics to be consistent, it cannot be complete. Gödel's theorems showed that no system could ever lead to the formalization of all mathematics.

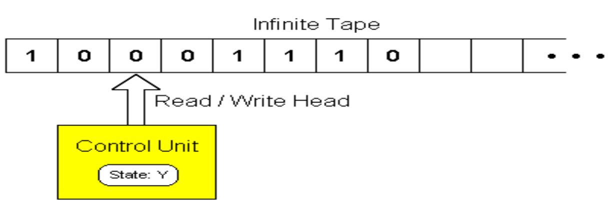
When Hilbert learned of Gödel's incompleteness theorems, he was devastated. His dream for a foundation of mathematics that would never again lead to crisis was unobtainable. As in the arts and the natural sciences, ignoramus et ignorabimus once again reigned supreme. Since

[84] Latin maxim of the 19th century coined by Emil du Bois-Reymond to express the limits of understanding nature in 1872. The phrase translates to "We do not know and will not know".
[85] Reid, Constance. *Hilbert*. New York: Copernicus, 1996. p196

Zermel and Fraenkel had resolved the crisis of set theory, the majority of the mathematical community decided that the remaining questions of foundational mathematics were best left undisturbed. They felt that Gödel's work only showed one rare example of incompleteness in mathematics, and so they shifted their attention to other fields of mathematical and physical analysis.

However one mathematician who had been raised in the midst of the foundational debate was still intrigued by Gödel's results. Alan Turing, after examining Gödel's proof on the unfeasibility of the Entscheidungsproblem, began to ponder how many mathematical statements could be considered un-decidable or incomputable. In order to explore the depths of incompleteness, he constructed a theoretical machine that could perform a simple set of serial operations. This hypothetical device manipulated symbols on a strip of infinite tape according to a set of inputted rules.

TURING'S MACHINE



[86]

Turing then modified his hypothetical device to emulate any of his other machines. Using reduction, Alan Turing demonstrated that any set of operations that could not run on his Universal Turing Machine was un-decidable. The new machine was revolutionary, for although its intended purpose was to examine the incomputable -- indirectly the machine turned the ashes of Hilbert's formalism into the phoenix of the modern computer.

AFTERWORD

[86] *Turing Machines* from science.slc.edu

Mathematics as a discipline is taught independently of social politics. In the natural sciences, students learn about the opposition Copernicus and Galileo faced during their attempts to show that the earth revolved around the sun. They learn how Columbus fought to prove that the world was flat, that Darwin's theories on evolution were and still are rejected by pockets of social conservatism. In literature, students are taught about the realities and evils of censorship. They are told how the spread of ideas is often impacted by the race, gender and political orientation of an author. They are educated about sociopolitical concepts such as McCarthyism and Samizdat. In the fields of history and other social sciences, students learn that their entire disciplines are intertwined with politics. They learn that they must critically examine authors and their biases in order to truly understand their words. Yet, only in the field of mathematics are students told that a proof is a proof. It does not matter that Gottfried Leibniz was German, or that Isaac Newton was British. It does not matter that personal disputes between two titans can divide entire scientific and mathematical communities. All that matters is that calculus is calculus, and that a derivative or a fluxion describes the rate of change of a continuous function.

Despite the strong desire to assert that mathematics is pure and above all politics, this is not always the case. Yes, the square root of two is always an irrational number, but to truly understand the impact of that statement, one must know of the Pythagorean School, of the divinity of Ancient Greek math, and of the blood that was spilt over what is now considered to be such a trivial mathematical fact. Without the integration of politics and ego into the mathematical world, it is uncertain whether the modern computer would exist today. Gödel and Turing's work on incompleteness would not have been possible if the principle of the excluded middle had been exorcised from mathematics. If Brouwer's Intuitionist program had been allowed to regain its strength, if the Formalist program had faltered, it is very likely that the technological revolution of the late twentieth century would not have occurred.

Therefore it is important that the study of mathematics be interwoven with the social and natural sciences. When Gödel showed the impossibility of obtaining complete mathematical truth, his colleagues were quick to abandon their epistemological quest. In history, art and the natural sciences, absolute truth is also unobtainable. However, although historians, artists, and scientists recognize the limitations of their disciplines, they constantly reexamine their field's foundations in order to derive deeper meaning from the truth at hand. Turing's reexamination of the unobtainability of mathematical truth, after his contemporaries became disillusioned with the study of foundations, led to the conception of the computer. It was because of Turing's persistence, that the battles surrounding the uncertainty, chaos and politics of early twentieth century mathematics gave birth to the modern information age.

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