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The equivalent absorption path length in an integrating cavity is examined. In an otherwise excellent paper, Tranchart et al. [Appl. Opt. 35, 7070 (1996)] made an important error in obtaining the expressions for the equivalent path length in an integrating cavity. This error has been propagated through several other publications in the literature. Since the equivalent path length is the sine qua non for obtaining an accurate absorption coefficient when using an integrating cavity, it is our intent here to give the correct formulas to prevent further errors when extracting absorption coefficients. © 2010 Optical Society of America

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### 1. Introduction
The paper by Tranchart et al. [1] provides some valuable insights into the theory of integrating cavities. However, in the discussion leading up to their Eq. (15), an important error is introduced. We found 12 references [2–13] to this paper, and 3 of them [3,6,13] make explicit use of the expressions that are in error.

### 2. Discussion
The authors are considering a spherical integrating cavity with diameter $D$, wall reflectivity $\rho$, and mean path length $L_{\text{eff}} = 2D/3$ that contains an absorbing gas with absorption coefficient $\alpha(\nu)$. The radiant flux (watts) is denoted by $\Phi$. In the discussion leading up to their Eq. (15), they state “…after a first pass through the absorbing gas the total flux falling on the wall is $[1 - \alpha(\nu) L_{\text{eff}}] \Phi$. The flux reflected diffusely by the coating (characterized by its reflectance $\rho$) and leaving the surface after a single reflection is $\rho [1 - \alpha(\nu) L_{\text{eff}}] \Phi$. This expression shows that a sphere containing an absorbing medium acts as if the coating had an effective reflectance given by $\rho [1 - \alpha(\nu) L_{\text{eff}}] \Phi$. They used this result to show that the fractional absorption in the cavity is

$$
\frac{\Delta \Phi}{\Phi_e(0)} = \frac{\alpha(\nu) L_{\text{eff}}}{1 - \rho [1 - \alpha(\nu) L_{\text{eff}}](1 - h)},
$$

where $\Phi_e(0)$ is the total flux exiting the detector port when $\alpha = 0$, and $h$ is the fraction of the cavity wall that is open for ports. They then state: “In the limiting case of a sphere with very small apertures ($h \ll 1$) filled with a gas characterized by a very small absorption coefficient $[\alpha(\nu) L_{\text{eff}} \ll 1]$, the fractional absorption is then enhanced by a factor of $1/(1 - \rho)$, and the equivalent absorption path length of the integrating sphere is of the order of

$$
L_{eq} = \frac{L_{\text{eff}}}{1 - \rho} = \frac{2D}{3(1 - \rho)}.
$$

"Mathematically, Eq. (1) follows correctly from their expression for the flux leaving the surface after a single reflection; we argue that Eq. (2) is incorrect.

In the derivation of Eq. (1), they provide an expression for the flux leaving the surface after a single
reflection:
\[ \rho [1 - \alpha(\nu) L_{\text{eff}}] \Phi. \]  
(3)

This is an approximation to the rigorous result
\[ \Phi \rho \exp[-\alpha(\nu) L_{\text{eff}}]. \]  
(4)

We point out, however, that these expressions for the flux leaving the surface after the first reflection are dependent on the symmetry of the incident radiation. Written generally, the flux after the first reflection is
\[ \Phi \rho \exp[-\alpha(\nu) L_{\text{eff}}] \]  
(5)

where \( \rho \) is the appropriate prefactor given in Fig. 1.

For collimated incident radiation, the appropriate prefactor is \( \rho \exp[-\alpha(\nu) D] \); for isotropic incident radiation, the correct prefactor is \( \rho(1 - h) \exp[-\alpha(\nu) L_{\text{eff}}] \). Throughout the rest of our argument, we assume isotropic incident radiation.

Given Eq. (5) and proceeding along the same lines as Berger et al. [14], we arrive at the exact expression for fractional absorption:
\[ \frac{\Delta \Phi}{\Phi_0} = 1 - \Phi_\alpha(\nu) \]  
(6)

This corresponds to the approximate expression given by Eq. (1), which is used in Tranchart et al. [1]. The approximation given by Eq. (1) is good for low absorption. Since we here assume that \( \alpha(\nu) L_{\text{eff}} \ll 1 \), we can expand Eq. (6) into a more useful form, yielding an equivalent path length of

\[ L_{\text{eq}} = \frac{L_{\text{eff}}}{1 - \rho(1 - h)(1 - \alpha(\nu) L_{\text{eff}})} \]
\[ \approx \frac{L_{\text{eff}}}{1 - \rho + \rho h + \rho \alpha(\nu) L_{\text{eff}}}. \]  
(7)

Great care must be taken, however, in simplifying Eq. (7). Clearly, as the absorption coefficient \( \alpha(\nu) \) of the medium increases, the equivalent absorption path length \( L_{\text{eq}} \) must decrease. Equation (2) shows no such dependence, and thus it cannot be correct if \( \alpha(\nu) \neq 0 \). Equation (7) does show the correct dependence. A claim that \( \alpha(\nu) L_{\text{eff}} \) is so small it can be neglected so that Eq. (7) will reduce to Eq. (2) is incorrect; \( \alpha(\nu) L_{\text{eff}} \) must be compared to \( 1 - \rho \), which is also small. Furthermore, to argue that \( \alpha(\nu) L_{\text{eff}} \) is negligible in comparison with \( 1 - \rho \) is equivalent to saying the absorption losses (that we wish to measure) in the medium are negligible in comparison with the energy losses at the cavity walls, i.e., one would be trying to measure a negligible signal on top of a large signal. Actually, the goal for maximum sensitivity is to obtain a diffuse reflectivity \( \rho \) so close to unity that \( 1 - \rho \) is negligible in comparison with \( \alpha(\nu) L_{\text{eff}} \).

Figure 2 demonstrates the error of approximating Eq. (7) by Eq. (2). It compares the relative difference between Eq. (2) and the accepted value of Eq. (7) as a function of absorption \( \alpha(\nu) \) and reflectivity \( \rho \). It can be seen that the relative error is minimized as \( \alpha(\nu) \) and \( \rho \) approach 0.001 m\(^{-1}\) and 0.9, respectively. Even so, the relative error of Eq. (2) with these values is 9%. With increasing reflectivity, the denominator of Eq. (7) becomes sensitive to small changes, so that, when the reflectivity approaches 0.999, the relative error of Eq. (2) can reach 500% and greater with even very small values of absorption.

It is instructive to give an \textit{ab initio} calculation using a very different approach. We follow the argument in Eqs. (1)–(6) of Fry et al. [15]. The flux \( \Phi \) incident on the cavity wall decays exponentially:
\[ \Phi(t) = \Phi_0 \exp \left[ -\frac{t}{\tau} \right], \]  
(8)

where \( \tau \) is the time for the flux to drop to 1/e of its initial value \( \Phi_0 \). Now, on average, after the first transit of the cavity and the first reflection,

![Fig. 1](Color online) For radiation incident upon the interior of the cavity that is collimated, the appropriate prefactor corresponding to the first reflection is \( \rho \exp[-\alpha(\nu) D] \). However, for isotropic incident radiation, the appropriate prefactor is \( \rho(1 - h) \exp[-\alpha(\nu) L_{\text{eff}}] \).

![Fig. 2](Color online) Relative percent error between Eq. (2) and the more accurate Eq. (7) as a function of absorption of the medium and reflectivity of the cavity walls. The inner diameter of the integrating cavity is 10 cm, whereas the fractional area open for ports is 0.01. The color mapping is linear from 0 to 150 relative percent error.)
\[
\Phi(t) = \rho(1 - h)\Phi_0 \exp[-\alpha(\nu)L_{\text{eff}}],
\]
where \(t = L_{\text{eff}}/c\) is the time for light to travel the mean path length \(L_{\text{eff}}\) in the cavity. On average, after the \(n\)th transit of the cavity and the \(n\)th reflection,

\[
\Phi(n\tau) = \rho^n(1 - h)^n\Phi_0 \exp[-n\alpha(\nu)L_{\text{eff}}] = \Phi_0 \exp[n\ln\rho + n\ln(1 - h) - n\alpha(\nu)L_{\text{eff}}].
\]

Now, setting \(t = \tau = n\tau\) and equating the resulting exponents on the right-hand side of Eqs. (8) and (10) gives

\[
-1 = n\ln\rho + n\ln(1 - h) - n\alpha(\nu)L_{\text{eff}}.
\]

Solving for \(n\) gives the number of reflections at which the flux \(\Phi\) drops to \(1/e\) of its initial value:

\[
n = \frac{1}{\alpha(\nu)L_{\text{eff}} - \ln\rho - \ln(1 - h)}.
\]

The corresponding time \(\tau\) is

\[
\tau = \frac{L_{\text{eff}}}{c} = \frac{L_{\text{eff}}}{c(\alpha(\nu)L_{\text{eff}} - \ln\rho - \ln(1 - h))}.
\]

The equivalent absorption path length is \(L_{\text{eq}} = c\tau\), hence

\[
L_{\text{eq}} = c\tau = \frac{L_{\text{eff}}}{\alpha(\nu)L_{\text{eff}} - \ln\rho - \ln(1 - h)} \approx \frac{L_{\text{eff}}}{1 - \rho + h + \alpha(\nu)L_{\text{eff}}},
\]

in good agreement with Eq. (7).

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