Duration and Convexity of Inverse Floating Rate Bonds

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Inverse floating rate bonds introduce unusual patterns of interest rate risk into a portfolio. To help assess and manage that risk, this paper presents duration and convexity measures for inverse floating rate bonds. Both the duration and the convexity of the inverse floater are shown to be weighted averages of the duration and convexity of an associated portfolio of fixed rate bonds and floating rate bonds. This portfolio of bonds mimics the inverse floater, and the weights are determined by the relative prices and the leverage ratio of the mimicking portfolio. The duration of inverse floating rate bond is greater than the duration of the associated fixed rate bond, and the convexity of inverse floating rate bond is greater than the convexity of the associated fixed rate bond. The time decay patterns of duration and convexity of inverse floating rate bond are shown to be dependent on the time decay patterns of the duration and convexity of the associated fixed rate bond and floating rate bond, as well as on the relative prices of the bonds and the leverage ratio. Finally both the duration and the convexity of the inverse floating rate bond are shown to increase dramatically with the leverage ratio.
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I. INTRODUCTION

Interest rate volatility has increased dramatically over the past twenty-five years. The FDIC improvement act of 1991 instructs regulators to take into account the interest rate risk exposure of a bank in determining its capital adequacy. The Federal Reserve Board and the Office of Thrift Supervision have come up with proposals to implement this legislation. As a result thousands of commercial banks and S & Ls across the country have formally or informally begun the process of interest rate risk measurement and management. These financial institutions are acutely aware of the potential negative effects of duration mismatches in their asset-liability structure. Financial innovation may help these institutions in aligning their asset durations closer to their liability durations. Since asset durations are typically much higher than liability durations for most depository institutions, issuance of inverse floating rate bonds (which have relatively large durations) as a liability can help mitigate the duration mismatch problem for these institutions.

An inverse floating rate bond, also known as an "inverse floater", is a variable rate bond whose coupon payments are reset periodically, to a pre-specified benchmark rate minus the current short term rate or some other reference rate. In contrast, a floating rate bond, also known as a floater, is a variable rate bond, with coupon payments reset periodically to the current short term rate or some other reference rate. The inverse floater can be used as either a hedging instrument or as a speculative instrument, because of its high and complicated duration and convexity profile. For example, holding or shorting...
the inverse floater can dramatically amplify or mute one’s exposure to anticipated interest rate changes, without the administrative complexity of rolling over futures or options contracts.

This paper derives measures of duration and convexity for inverse floating rate bonds. Our derivation relies on the specification of the cash flow stream of a floating rate bond and a fixed rate bond. This approach, while requiring a set of relative prices, avoids the need to specify each term of the present value function of the inverse floater, a difficult task given that the coupon payments of the inverse floater are dependent upon forward short term spot rates at each repricing date.

The paper is organized as follows. The next section presents the derivation of both duration and convexity of the inverse floater. Section 3 presents an illustration. Here we find that except for extreme cases the duration of the inverse floater is greater than its maturity, and the convexity of the inverse floater is greater than its maturity squared. The pattern of time decay in the duration and convexity of the inverse floater is described and contrasted with that of the associated fixed rate bond. We also show in section 3 the pattern of duration and convexity of the inverse floater under different leverage ratios. The leverage ratio will be defined in section 2. Section 4 concludes the paper.

II. The Derivation

As shown by Yawitz [1986] and Fabozzi [1995], \( N_{FL} \) floating rate bonds and \( N_{IF} \) inverse floating rate bonds can be created from \( N_{Fixed} \) fixed rate bonds, where \( N_{Fixed} = N_{FL} + N_{IF} \). The face value and maturity of all the three bonds are the same. Let the leverage ratio \( L \) represent the ratio of the number of floaters to the number of the
fixed rate bonds, which equals to $N_{FL} / N_{Fixed}$. In the absence of arbitrage opportunities, there must exist some relationship among the coupon payments as well as among the prices of the three bonds. We have the following equation for the coupon rates:

$$C_{Fixed} = L \cdot C_{FL} + (1 - L) \cdot C_{IF}$$ (1)

Where $C_{Fixed}$ is the coupon rate for the fixed rate bond, $C_{FL}$ is the coupon rate for the floater ($C_{FL}$ also known as the reference rate), and $C_{IF}$ is the coupon rate for the inverse floater. Rearranging equation (1), we have

$$C_{IF} = \left( \frac{1}{1 - L} \right) \cdot C_{Fixed} - \left( \frac{L}{1 - L} \right) \cdot C_{FL}$$ (2)

Here $\left( \frac{1}{1 - L} \right) \cdot C_{Fixed}$ is the so-called inverse floater benchmark rate.

The price relationship among the three bonds is given in the following equation:

$$P_{Fixed} = L \cdot P_{FL} + (1 - L) \cdot P_{IF}$$ (3)

where $P_{Fixed}$ is the price of the fixed rate bond, $P_{FL}$ is the price of the floating rate bond, and $P_{IF}$ is the price of the inverse floater.

Rewriting equation (3) expresses the price of the inverse floating rate bond as:

$$P_{IF} = \left( \frac{1}{1 - L} \right) \cdot P_{Fixed} - \left( \frac{L}{1 - L} \right) \cdot P_{FL}$$

As shown in equation (3), the inverse floater can be considered equivalent to a portfolio of $1/(1-L)$ long positions in a fixed rate bond and $L/(1-L)$ short positions in a floating rate bond.
Because the duration of a portfolio of bonds is equal to the weighted average of the durations of the bonds in the portfolio, the duration of the inverse floater can be given as:

$$D(1)_{IF} = \left( \frac{1}{1-L} \right) \cdot \frac{P_{Fixed}}{P_{IF}} \cdot D(1)_{Fixed} - \left( \frac{L}{1-L} \right) \cdot \frac{P_{FL}}{P_{IF}} \cdot D(1)_{FL}$$ (5)

Similarly, the convexity of a portfolio of bonds is equal to the weighted average of the convexities of the bonds in the portfolio. The convexity of the inverse floater can be given as:

$$D(2)_{IF} = \left( \frac{1}{1-L} \right) \cdot \frac{P_{Fixed}}{P_{IF}} \cdot D(2)_{Fixed} - \left( \frac{L}{1-L} \right) \cdot \frac{P_{FL}}{P_{IF}} \cdot D(2)_{FL}$$ (6)

From Equations (5) and (6), we see that to calculate the duration and the convexity of the inverse floater requires the duration and convexity of both the corresponding fixed rate bond and floating rate bond. We next give the derivation of these four measures.

Yawitz (1986) and Yawitz, Kaufold, Macirowski, and Smirlock [1987] show that a default free floating rate bond is equivalent to a zero coupon bond whose maturity is equal to the time period remaining to repricing. We assume that repricing occurs at the next coupon payment date, such that the duration of the zero coupon bond is equal its time to maturity, and the convexity of the zero coupon bond is equal to its time to maturity squared:

$$D(1)_{FL} = \frac{1}{m} - f$$ (7)

$$D(2)_{FL} = \left( \frac{1}{m} - f \right)^2$$ (8)

where $m$ is the number of coupon payments per year, therefore $1/m$ is the repricing or coupon payment interval in terms of years, and $f$ is the length of time expired since the
last coupon payment date. So right before the coupon payment date, the duration and convexity of a floating rate bond will both be zero implying no interest rate sensitivity for the floater. The duration and convexity of a floater will jump from zero to $1/m$ and $(1/m)^2$ respectively right after the coupon is paid.

By the same token, the price of a floating rate bond is equivalent to the price of a zero coupon bond whose maturity is equal to the time period remaining to repricing, and whose face value is equal to the sum of the next coupon payment and the face value (see Yawitz et al. [1987]). So right before coupon payment, the price of the floater is equal to the coupon payment plus its face value, and right after the coupon payment, the price of the floater is equal to its face value. The price drops by the amount of the coupon payment right after coupon is paid.

The calculation of the duration and convexity of a fixed rate bond is straightforward. Duration is the weighted average of the cash flow maturity dates, and convexity is the weighted average of the squares of cash flow dates, where the weights are the present values of the respective cash flows, divided by the fixed rate bond price. If we assume that the time to maturity of the fixed rate bond is $T$ years, the face value is $F$, the annual coupon rate is $C_{Fixed}$, and the annual interest rate is $r$, the following duration and convexity formulas can be obtained for the fixed rate bond. For the simplicity of demonstration we assume a flat term structure of interest rates in this paper.$^3$

$$D(1)_{Fixed} = \frac{\sum_{t=1}^{T} \frac{C_{Fixed}}{m} \left( \frac{T - f}{m} \right) e^{\left( \frac{T - f}{m} \right)}}{P_{Fixed}} + \frac{F \cdot (T - f)}{e^{(T - f)}}$$

(9)
At the coupon payment dates, the duration and the convexity of both the fixed rate bond and the floating rate bond rise abruptly. The rise occurs because the fixed rate bond’s price drops by the amount of the coupon payment, which results in a smaller denominator, while the numerator remains the same, both immediately before and after each coupon payment.

Substituting equation (7) and (9) into (5), we have equation (12) for the duration of the inverse floater. Substituting equation (8) and (10) into (6), we have equation (13) for the convexity of the inverse floater.

\[
D(1)_{IF} = \left(\frac{1}{1-L}\right) \cdot \left(1 - \sum_{t=1}^{m^-} \frac{C_{Fixed}}{m} \cdot \frac{\left(\frac{t}{m} - f\right)}{e^{\left(\frac{t}{m} - f\right)}} + \frac{F \cdot (T - f)}{e^r(T-f)}\right) \\
- \left(\frac{L}{1-L}\right) \cdot \frac{P_{FL}}{P_{IF}} \cdot \left(\frac{1}{m} - f_{FL}\right)
\]
\[ D(2)_{IF} = \left( \frac{1}{1-L} \right) \cdot \frac{1}{P_{IF}} \sum_{i=1}^{T} \left( \frac{C_{Fixed}}{m} \left( \frac{t-f}{m} \right)^2 \right) \left( \frac{F \cdot (T-f)^2}{e^{r(T-f)}} \right) \] 

\[ - \left( \frac{L}{1-L} \right) \cdot \frac{P_{FL}}{P_{IF}} \left( \frac{1}{m} - f \right)^2 \] 

Unlike the duration and convexity of the fixed rate bond and the floating rate bond, which always jump up at the coupon payment dates, the duration and convexity of inverse floaters can either jump up or jump down, depending upon the relative prices and the ratio \( L \). We demonstrate this in detail in the following example.

III. AN EXAMPLE

Consider a five year fixed rate bond created, from a floating rate bond and an inverse floater. The fixed rate bond has a 12 percent coupon rate that pays semi-annually. The floating rate bond has a maturity of five years, a reference rate equal to the six-month Treasury bill rate, and a repricing period of six months. If the leverage ratio \( L \) is 0.2, then the benchmark rate for the inverse floater is 15 percent. The maturity for the inverse floater is also five years. We assume that the annual interest rate is 10 percent, and the interest rate does not change as the bonds approach maturity. The face value of each bond is assumed to be $100.

When the bonds are first issued, the time to maturity \( T \) equals 5 years and \( f \) equals 0. Substituting \( m = 2 \) (i.e., semi-annual compounding), \( L = 0.2 \), and using the equations derived in the last section, we calculate the price, duration and convexity of the corresponding fixed rate bond and the inverse floater as follows:
\( P_{\text{Fixed}} = $106.7, \)
\( D(1)_{\text{Fixed}} = 3.94 \text{ years}, \)
\( D(2)_{\text{Fixed}} = 17.88 \text{ years squared}, \)
\( P_{IF} = $108.4, \)
\( D(1)_{IF} = 4.73 \text{ years}, \)
\( D(2)_{IF} = 21.95 \text{ years squared}. \)

We calculate the price, duration and convexity of the inverse floater right before the first coupon payment by substituting \( T = 4.5 \text{ years} \) and \( f = 0.5 \) into the equations. Right after the first coupon payment we will use \( T = 4.5 \text{ years} \) and \( f = 0. \)

We repeat the above calculation until \( T = 0 \) and \( f = 0. \) Figure 1 gives the pattern of the duration of the fixed rate bond as it approaches maturity.

Fig 1. Time decay in the duration of a fixed rate bond. When first issued, the time to maturity of the fixed rate bond is 5 years. The coupon rate is 12 percent paid semi-annually, and the face value is $100. The annual interest rate is 10 percent. We assume a static flat term structure. The price of the fixed rate bond is calculated using equation (11), and the duration of the fixed rate bond is calculated using equation (9).
The upper panel of Figure 2 gives the pattern of the duration of the inverse floater as time approaches maturity, given that the leverage ratio is 0.2. For comparison, the lower panel of Figure 2 gives the pattern of the duration of the inverse floater as time approaches maturity if the leverage ratio is 0.8.
Fig. 2. Time decay in duration of an inverse floater. Upper panel: $L = 0.2$; lower panel: $L = 0.8$. When first issued, the time to maturity of the inverse floater is 5 years. The corresponding fixed rate bond has a 12 percent annual coupon rate that pays semi-annually. The corresponding floating rate bond has a reference rate equal to the six month Treasury Bill rate, and the repricing period is six months. The leverage ratio in the upper panel is 0.2, and the leverage ratio in the lower panel is 0.8. The face value of all the bonds is $100. The annual interest rate is 10 percent. We assume a flat term structure, and further assume that it remains the same over time. The prices and durations are calculated using equations (3) through (11).
Comparing Figure 1 and Figure 2, we find that the duration of an inverse floater has two distinguishing characteristics when compared with other types of bonds.

First, when the inverse floater is issued, the duration is 4.73 years if the leverage ratio is 0.2, and 14.25 years if the leverage ratio is 0.8. In both cases, the duration of the inverse floater is higher than the 3.94 duration of the corresponding fixed rate bond. Also, the higher the leverage ratio, the higher the duration of the inverse floater. Because duration serves as an indicant of bond price volatility, the inverse floater's volatility is higher than that of other bonds, such as the associated fixed rate bond.

Second, the time decay in the duration of the inverse floater is different from that of the fixed rate bond. On the coupon payment dates, as shown in Figure 1, fixed rate bond duration increases with a discrete jump, while inverse floater duration can decrease or increase with a discrete jump, depending on the relative prices of the bonds and the leverage ratio. Equation (5) demonstrates that the duration of the inverse floater is determined by a linear combination of the corresponding fixed rate bond duration and the floating rate bond duration. After the coupon payment date, there is a jump in duration equal to the time interval between coupon payments for both the fixed rate bond and the floating rate bond. The jump in duration of the inverse floater will depend on the jumps in durations and relative prices of the fixed rate bond and the floating rate bond, as well as on the leverage ratio. As shown in Figure 2, when the leverage ratio is 0.2, in the early life of the bond, there is an upward jump of duration on coupon payment dates. As time approaches maturity, with the change of relative prices, the effect of floating rate bond duration starts to outweigh the effect of the fixed rate bond duration. The jump gets smaller and smaller, and eventually the net effect becomes a drop in duration on coupon
payment dates. But as shown in the lower panel of Figure 2, if the leverage ratio equals 0.8, there is a drop of duration on the coupon payment date throughout the life of the inverse floater. Again, as time approaches maturity, the relative prices change, and the drop amount becomes larger and larger. Given some different, but not extreme, parameters, the pattern could be slightly different. Furthermore, the duration decline between coupon payment dates for the fixed rate bond is equal to the time interval between coupon payment dates (0.5 year in this example). The duration decline between coupon payment dates for the inverse floater can be smaller or bigger than the fixed bond's drop in duration, depending on the parameters of the example.

Next, we calculate the duration of the inverse floater under a different leverage ratio $L$, keeping the time to maturity at 5 years. This is shown in Figure 3. When $L = 0$, the inverse floater is actually a fixed rate bond and its duration is 3.94 years. But as $L$ increases, the number of inverse floaters is larger and larger relative to the number of floating rate bonds and the duration of the inverse floater increases, indicating that its risk is rising (volatility).
Fig. 3. The duration of an inverse floater under different leverage ratios. Leverage ratio is defined as the ratio of the number of floating rate bonds to the number of fixed rate bonds. All other assumptions are the same as those in Figures 1 and 2. In this figure we keep the time to maturity at 5 years and calculate the duration under different leverage ratios.
The time decay in convexity of the above fixed rate bond and the inverse floater can be obtained in a similar way, by using the equation derived in section 2. This is shown in Figure 4 and Figure 5, respectively.

Fig 4. Time decay in convexity of fixed rate bond. When first issued, the time to maturity of the fixed rate bond is 5 years, the annual coupon rate for the fixed bond is 12 percent paid semi-annually, and the face value is $100. The annual interest rate is 10 percent. We assume a flat term structure which remains the same over time. The price of the fixed rate bond is calculated using equation (11), and the convexity of the fixed rate bond is calculated using equation (10).
Fig. 5. Time decay in convexity of inverse floater. Upper panel: \( L = 0.2 \); lower panel: \( L = 0.8 \). When first issued, the time to maturity of the inverse floater is 5 years. The corresponding fixed rate bond has a 12 percent annual coupon rate that pays semi-annually. The corresponding floating rate bond has a reference rate equal to the six month Treasury Bill rate, and the repricing period is six months. The leverage ratio in the upper panel is 0.2, and the leverage ratio in the lower panel is 0.8. The face value of all the bonds is $100. The annual interest rate is 10 percent. We assume a flat term structure which remains constant over time. The prices and durations are calculated using equations (3) through (11).
Again, the upper panel of Figure 5 gives the time decay in the convexity of the inverse floater when leverage ratio is 0.2, and the lower panel of the figure gives the time decay in the convexity of inverse floater when the leverage ratio is 0.8.

The convexity of the inverse floater also has two distinguishing characteristics, when compared with other types of bonds. First, as shown in Figure 5, when time to maturity $T$ is equal to 5 years, the convexity of the inverse floater is equal to 21.95 years squared if the leverage ratio is 0.2, and 70.72 years squared if the leverage ratio is 0.8. In both cases, the convexity of inverse floater is higher than the convexity of the corresponding fixed rate bond, which is 17.88 years squared, and the higher the leverage ratio, the higher the convexity of the inverse floater. Given the desirability of convexity for parallel term structure shifts\(^4\), the greater convexity of the inverse floater will lead to higher gains vis-a-vis other types of bonds.

Second, the time decay in the convexity of the inverse floater is quite different from that of a fixed rate bond. As shown in Figure 4, on coupon payment dates the fixed rate bond’s convexity increases with a discrete jump while the inverse floater’s convexity can decrease or increase by a discrete jump, depending on the relative prices of the three bonds and the leverage ratio $L$, following equation (6). As we can see in Figure 5, for both leverage ratios, there is a jump in convexity in the early life of the inverse floaters. As time approaches maturity, with the change of relative prices, the jumps become smaller and smaller, and eventually there are declines in convexity on the coupon payment dates. For higher leverage ratios, as in the lower panel of Figure 5, the rate of change in the jump’s amount is larger.
Finally Figure 6 plots the convexity of the inverse floater at $T = 5$ under different leverage ratios $L$. As we can see, when $L = 0$, the inverse floater is actually a fixed rate bond, and its convexity is 17.88 years squared. As $L$ gets larger, the number of inverse floaters is larger relative to the number of floating rate bonds, and the convexity of the inverse floater rises. Thus, while price volatility, expressed as duration, becomes larger for the inverse floater as the leverage ratio rises, the convexity also becomes higher.

![Convexity of an inverse floater under different leverage ratios](image)

Fig. 6. The convexity of an inverse floater under different leverage ratios. The leverage ratio is defined as the ratio of the number of floating rate bonds to the number of fixed rate bonds. All other assumptions are the same as those in Figures 1 and 2. In this figure, we keep the time to maturity at 5 years, and calculate the convexities under different leverage ratios.
IV. CONCLUSIONS

This paper presents a methodology to compute the duration and the convexity of an inverse floating rate bond. Both duration and convexity are derived as weighted averages of the duration and convexity of a floating rate bond and a fixed rate bond. Because the duration and convexity of fixed rate and floating rate bonds are relatively easy to assess, the risk profile of an inverse floater is computationally tractable. The duration and convexity of an inverse floating rate bond are greater than that of the corresponding fixed rate bond. The time decay pattern of duration and convexity of an inverse floater depends on the time decay pattern of the duration and convexity of the corresponding fixed rate and floating rate bonds, as well as on the relative prices of the bonds and the leverage ratio $L$. Both the duration and the convexity of the inverse floating rate bond increase dramatically as the leverage ratio increases.
Endnotes

1. Unlike Fabozzi (1995), who defines \( L \) equal to the ratio of the number of floaters to that of the inverse floaters, we define \( L \) as the ratio of the number of floaters to that of the fixed rate bonds. Using our definition, the value \( L \) ranges between 0 and 1.

2. We assume that all the three bonds are priced with reference to the same term structure. We further assume that the prices given in the equations are not quoted prices, but rather are adjusted to include accrued interest.

3. If we do not assume flat term structure, the derivation can be obtained similarly, but the results will be altered slightly. Since the focus of this paper is to demonstrate the characteristics of the duration of inverse floaters, not to study the term structure of interest rates, we make this simple assumption.

4. For a parallel term structure shift, a barbell portfolio with higher convexity outperforms a bullet portfolio with lower convexity. Also see Ingersoll, Skelton and Weil (1978) and Fabozzi (1995).
References


