

Kvalheim, Matthew. "A Generalization of the Hopf Degree Theorem." Proceedings of the American Mathematical Society 151, no. 01 (January 2023): 453–54.
<https://doi.org/10.1090/proc/16218>.

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A GENERALIZATION OF THE HOPF DEGREE THEOREM

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ABSTRACT. The Hopf theorem states that homotopy classes of continuous maps from a closed connected oriented smooth n -manifold M to the n -sphere are classified by their degree. Such a map is equivalent to a section of the trivial n -sphere bundle over M . A generalization of the Hopf theorem is obtained for sections of nontrivial oriented n -sphere bundles over M .

Throughout, let M be a compact connected oriented smooth n -dimensional manifold without boundary. The Hopf degree theorem is as follows [Mil97, p. 51].

Theorem (Hopf). A pair of continuous maps $f, g: M \rightarrow \mathbb{S}^n$ to the n -sphere are homotopic if and only if $\deg(f) = \deg(g)$.

Homotoping f and g to smooth maps, denoted by the same symbols, does not affect their degrees. Let $c \in \mathbb{S}^n$ be a regular value for both. Then $\deg(f) = I(f, c)$ and $\deg(g) = I(g, c)$ coincide with oriented intersection numbers. If instead c is viewed as the constant map $M \rightarrow \{c\} \subset \mathbb{S}^n$, rather than as a point in \mathbb{S}^n , these can be viewed as oriented intersection numbers of maps $M \rightarrow \mathbb{S}^n$ [Mil97, GP10, Hir94]. Such a map is equivalent to a section of the trivial sphere bundle $M \times \mathbb{S}^n \rightarrow M$. This motivates the following question.

Question. What can be said when the sphere bundle is nontrivial and $M \rightarrow \{c\}$ is replaced by a general continuous section?

A partial answer is given by the following generalization of the Hopf theorem, which is the special case that π is a trivial bundle and Z is constant.

Theorem 1. Let X, Y, Z be continuous sections of a smooth n -sphere bundle $\pi: E \rightarrow M$, where E is oriented. Then X, Y are homotopic through sections if and only if $I(X, Z) = I(Y, Z)$.

Remark 1. The proof works by reducing to the situation of the Hopf theorem.

Proof. Homotopy invariance of the oriented intersection number implies that $I(X, Z) = I(Y, Z)$ if X and Y are homotopic.

Conversely, assume that $I(X, Z) = I(Y, Z)$. It suffices to show that X and Y are homotopic through sections to some common other section. By approximation techniques we may assume (after preliminary homotopies through sections) that X, Y, Z are smooth and transverse (cf. [Hir94, p. 56, Ex. 3]). In particular, X, Y intersect Z only at finitely many points in M .

Finiteness and connectedness of M imply that these intersection points are contained in the interior of a compact set $B \subset M$ diffeomorphic to a ball in \mathbb{R}^n [MV94].

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2020 *Mathematics Subject Classification.* Primary 57R19; Secondary 55R25, 55N45.

Since B is smoothly contractible there is a fiber-preserving diffeomorphism

$$\pi^{-1}(B) \approx B \times \mathbb{S}^n$$

with respect to which sections over B may be viewed as \mathbb{S}^n -valued and $Z|_B$ is constant. After homotopies through sections we may assume that

- (1) $X|_B$ and $Y|_B$ coincide with $-Z|_B$ on a neighborhood of ∂B in B .

It follows that X, Y induce, by collapsing the boundary of B , self-maps

$$\mathbb{S}^n \approx B/\partial B \rightarrow \mathbb{S}^n$$

whose degrees coincide with $I(X, Z) = I(Y, Z)$. The Hopf theorem and its proof imply that the X -induced self-map is homotopic to the Y -induced self-map through maps sending $[\partial B] \in \mathbb{S}^n$ to the constant $-Z|_B \in \mathbb{S}^n$ [Mil97, pp. 50–51]. By (1) this homotopy extends by the constant one to yield a global homotopy of X, Y through sections. \square

Remark 2 (cf. [Mil97, p. 50, Remarks]). Theorem 1 easily generalizes to let M have a nonempty boundary ∂M : continuous sections X, Y disjoint from $Z|_{\partial M}$ are homotopic through sections disjoint from $Z|_{\partial M}$ if and only if $I(X, Z) = I(Y, Z)$.

REFERENCES

- [GP10] V Guillemin and A Pollack, *Differential topology*, AMS Chelsea Publishing, Providence, RI, 2010, Reprint of the 1974 original. MR 2680546
- [Hir94] M W Hirsch, *Differential topology*, Graduate Texts in Mathematics, vol. 33, Springer-Verlag, New York, 1994, Corrected reprint of the 1976 original. MR 1336822
- [Mil97] J W Milnor, *Topology from the differentiable viewpoint*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997, Based on notes by David W. Weaver, Revised reprint of the 1965 original. MR 1487640
- [MV94] P W Michor and C Vizman, *n-transitivity of certain diffeomorphism groups*, Acta Math. Univ. Comenianae **63** (1994), no. 2, 221–225.

ACKNOWLEDGMENTS

This work was funded by ONR N00014-16-1-2817, a Vannevar Bush Faculty Fellowship sponsored by the Basic Research Office of the Assistant Secretary of Defense for Research and Engineering.