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Title of Dissertation: Higher order regularity, long term dynamics, and data assimilation for magnetohydrodynamic flows

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ABSTRACT

Title of Dissertation: HIGHER ORDER REGULARITY, LONG TERM DYNAMICS, AND DATA ASSIMILATION FOR MAGNETOHYDRODYNAMIC FLOWS

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First we consider various inviscid equations of fluid dynamics and show that if the initial data is analytic in the space variables, then the resulting flows extend as analytic functions of both space and time variables, with explicit estimates of the analyticity radius. We then consider higher order regularity of the viscous magnetohydrodynamic equations for an incompressible conductive fluid. We establish the Gevrey regularity of solutions when the initial data is in a Sobolev class, of possibly negative order, in two and three spatial dimensions. In particular, we show that solutions evolving from singular initial data instantaneously become analytic, with the analyticity radius eventually expanding in time. This in turn allows us to establish decay in higher order Sobolev norms. Finally, using a recently developed data assimilation algorithm based on linear feedback control, we show that when the initial data is unknown, sparse measurement data is sufficient for accurate reconstruction of magnetohydrodynamic flows. This algorithm convergences exponentially in time to the reference solution and moreover, the reconstruction is exact on the attractor.

HIGHER ORDER REGULARITY, LONG TERM DYNAMICS, AND DATA ASSIMILATION FOR MAGNETOHYDRODYNAMIC FLOWS

by

Joshua Hudson

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Preface

The content of this document is organized as follows: Chapter 1 is devoted to introducing the notation and setting used in the following chapters. The reader familiar with fluid dynamics may simply use Chapter 1 to get acquainted with the notation, or may find it sufficient to skip this chapter and consult the list of abbreviations when needed.

In Chapter 2 we show that solutions to a large class of inviscid equations, in Eulerian variables, extend as holomorphic functions of time, with values in a Gevrey class (thus they are space-time analytic), and are solutions of complexified versions of the said equations. The class of equations we consider includes those of fluid dynamics such as the Euler, surface quasi-geostrophic, Boussinesq and magnetohydrodynamic equations, as well as other equations with analytic nonlinearities, under periodic boundary conditions. The initial data are assumed to belong to a *Gevrey class*, i.e., they are analytic in the space variables. Our technique follows that of the seminal work of Foias and Temam (1989), where they introduced the so-called Gevrey class technique for the Navier-Stokes equations to show that the solutions of the Navier-Stokes equations extend as holomorphic functions of time, in a complex neighborhood of $(0, T)$, with values in a Gevrey class of functions (in the space variable). We show that their technique extends to a wide class of *inviscid* models.

In Chapter 3 we restrict our attention to the *viscous* magnetohydrodynamic (MHD) equations, either on the whole space ($\mathbb{R}^d$) or with space periodic boundary conditions. We obtain local (for initial data of arbitrary size) and global (for initial data with small norm in critical homogeneous Sobolev spaces) existence in Gevrey norms for solutions of the 2D and 3D MHD equations. Then, using a new uniform bound on both the curl of the velocity and the curl of the magnetic field in L_1, as well as an adequate version of the Leray energy inequality, we get eventual smoothness and decay for all higher order derivatives of the solutions.

In Chapter 4 we consider data assimilation for the 2D MHD equations (for which global existence is known) under periodic boundary conditions. We propose several continuous data assimilation (downscaling) algorithms based on feedback control. We show that for sufficiently large values of the control parameter and fine enough resolution, and assuming that the observed data is error-free, the solution of the controlled system converges at an exponential rate (in L^2 and H^1 norms) to the reference solution, independently of the initial data chosen for the controlled system. Furthermore, we show that a similar result holds when controls are placed on only the horizontal (or vertical) variables, or on a *single* Elsässer variable, under more restrictive conditions on the control parameter and resolution. Finally, using the data assimilation system, we show the existence of *abridged* determining modes, nodes and volume elements.

In Chapter 4 we study the (numerical) efficacy of the algorithms described in Chapter 4 for the 2D magnetohydrodynamic equations, as well as some additional feedback control data assimilation algorithms. We find that the algorithms work with much less resolution in the data than required by the rigorous estimates. We also obtain numerical evidence that it may be possible to approximate a general

MHD flow using velocity measurements alone.

We defer the techical proofs of each chapter to the Appendix. Chapter 1 contains material from [20], [19], and [21]. The material from Chapter 2 was taken entirely from [20]. Chapter 3 was mostly taken from [19], but also contains material from [21], and [83] in the introduction. The content of Chapter 4 and Chapter 5 can be found in [21] and [83] respectively.

Dedication

To the 103 teachers I have had throughout my life, for each of their contributions to making me into the man I have become.

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This work has been made possible by the contributions of many people. First and foremost, I would like to thank my wife, Chelsea Hudson, for all of her support in all of its forms: from weathering my impromptu lectures in the car, to making my health and happiness a priority whenever I did not.

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List of Abbreviations and Symbols

NSE	Navier-Stokes equations
MHD	Magnetohydrodynamic
Ω	the spatial domain, either $\mathbb{R}^d$ or $[0, L]^d$
$\lvert \cdot \rvert$	absolute value, Euclidean norm, or Frobenius norm of a scalar, vector or matrix respectively
$\\| \cdot \\|$	$\\|\boldsymbol{u}\\| = \int_\Omega \lvert\boldsymbol{u}(\mathbf{x})\rvert^2 d\mathbf{x}$
$\\| \cdot \\|_{L^p}$	$\\|\boldsymbol{u}\\|_{L^p} = \left(\int \lvert\boldsymbol{u}\rvert^p\right)^{\frac{1}{p}}$
$\\| \cdot \\|_p$	$\\|\boldsymbol{u}\\|_p \\|\boldsymbol{u}\\|_{L^p}$
A	$(-\Delta)$
Λ	$A^{1/2}$
$\\|\cdot\\|_{\dot{H}^\alpha_p}$	$\\|\boldsymbol{u}\\|_{\dot{H}^\alpha_p} = \\|\Lambda^\alpha \boldsymbol{u}\\|_p$
$\\| \cdot \\|_{G(\beta,\alpha,p)}$	$\\|\boldsymbol{u}\\|_{G(\beta,\alpha,p)} = \\|\Lambda^\alpha e^{\beta\Lambda} \boldsymbol{u}\\|_{L^p}$

Chapter 1: Notation and setting for incompressible fluid dynamics

In the following chapters, we will consider several evolutionary (incompressible) fluid dynamic models including the incompressible Euler equations, the surface quasi-geostrophic equation (SQG), the Boussinesq equations and the magnetohydrodynamic equations (MHD). In each case, the equations will be considered on a spatial domain $\Omega = [0, L]^d,\ d \in \mathbb{N}$, and supplemented with the space periodic boundary condition (with spatial period L), i.e., the phase space will consist of scalar-valued or vector-valued functions, which are periodic in the space variable $\mathbf{x}$ with period L in all spatial directions. In Chapter 3, we will also allow for $\Omega = \mathbb{R}^d$.

For notational simplicity in the periodic setting, we will often assume

$$L = 2\pi, \text{ and therefore, } \kappa_0 := \frac{2\pi}{L} = 1.$$

The inner product on $L^2(\Omega) := \left\{ \boldsymbol{u} : \Omega \to \mathbb{R}^d,\ \int_\Omega |\boldsymbol{u}(\mathbf{x})|^2 d\mathbf{x} < \infty \right\}$ is denoted $\langle \cdot\,, \cdot \rangle$ and the corresponding L^2-norm will be denoted as $\| \cdot \|$. As usual, the Euclidean length of a vector in $\mathbb{R}^d(\text{ or } \mathbb{C}^d)$ is denoted by $|\cdot|$, and for a matrix M, we denote $|M|^2 := \sum_{i,j} |M_{i,j}|^2$. We also define $\|(\boldsymbol{v}, \boldsymbol{w})\|^2 := \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2$ given a pair $\boldsymbol{v}, \boldsymbol{w}$ in $L^2(\Omega)$, and extend the definition of other norms to pairs $(\boldsymbol{v}, \boldsymbol{w})$ in a similar way.

For a function $\boldsymbol{u} : \Omega \to \mathbb{R}^d (\text{ or } \mathbb{C}^d)$, its Fourier coefficients are defined by

$$\widehat{\boldsymbol{u}}(\mathbf{k}) = \frac{1}{(2\pi)^d} \int_\Omega \boldsymbol{u}(\mathbf{x}) e^{-\imath \kappa_0 \mathbf{k}\cdot\mathbf{x}} d\mathbf{x} \; (\mathbf{k} \in \mathbb{Z}^d).$$

Then by the Parseval identity,

$$\|\boldsymbol{u}\|^2 = (2\pi)^d \sum_{\mathbf{k}\in\mathbb{Z}^d} |\widehat{\boldsymbol{u}}(\mathbf{k})|^2.$$

In the case that $\Omega = \mathbb{R}^d$, we have

$$\widehat{\boldsymbol{u}}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} \boldsymbol{u}(\mathbf{x}) e^{-\imath 2\pi \boldsymbol{\xi}\cdot\mathbf{x}} d\mathbf{x} \; (\boldsymbol{\xi} \in \mathbb{R}^d),$$

and by the Plancherel theorem,

$$\|\boldsymbol{u}\|^2 = \int_{\mathbb{R}^d} |\widehat{\boldsymbol{u}}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi}.$$

In all the models we consider, if the space average of the initial data is zero, then the space average remains zero for all future times under the evolution. Therefore, we will always make the additional assumption that the elements of the phase space have space average zero (over the spatial domain Ω). In terms of the Fourier coefficients, this amounts to the condition $\widehat{\boldsymbol{u}}(0) = 0$ (which is then preserved under the evolution).

We will denote

$$\dot{L}^2(\Omega) = \left\{ \boldsymbol{u} \in L^2(\Omega) : \int_\Omega \boldsymbol{u}(\mathbf{x}) \, d\mathbf{x} = 0, \text{ or equivalently, } \widehat{\boldsymbol{u}}(0) = 0 \right\}.$$

In the case of incompressible fluid dynamics, the phase space H is given by

$$H = \left\{ \boldsymbol{u} \in \dot{L}^2(\Omega), \; \nabla \cdot \boldsymbol{u} = 0 \right\},$$

where the derivative is understood in the distributional sense. We will also define the usual Sobolev spaces,

$$H^k = \left\{ \boldsymbol{u} \in L^2(\Omega) : \|\partial^\alpha \boldsymbol{u}\|_{L^2} < \infty, \text{ for all multi-indices } \alpha,\ |\alpha| \leq k \right\}.$$

Note that the space $(-\Delta)(H \cap H^2) \subset H$. The Stokes operator, A, with domain $\mathcal{D}(A) = H \cap H^2$, is defined to be $A = (-\Delta)|_{\mathcal{D}(A)}$. A is positive and self adjoint with a compact inverse, and therefore admits a unique, positive square root, denoted $\Lambda = A^{1/2}$. The domain, V, of Λ is characterized by

$$V = \{\boldsymbol{u} \in H : \|\Lambda \boldsymbol{u}\|^2 = (2\pi)^d \sum_{\mathbf{k} \in \widetilde{\mathbb{Z}}^d} |\mathbf{k}|^2 |\widehat{\boldsymbol{u}}(\mathbf{k})|^2 < \infty\},$$

where $\widetilde{\mathbb{Z}}^d = \mathbb{Z}^d \setminus \{0\}$. The spectrum of A is comprised of eigenvalues $0 < 1 = \lambda_1 \leq \lambda_2 \leq \cdots$, where, for each i, $\lambda_i \in \{|\mathbf{k}|^2 : \mathbf{k} \in \widetilde{\mathbb{Z}}^d\}$. The set of eigenvectors $\{\mathbf{e}_i\}_{i=1}^\infty$, where $\mathbf{e}_i$ is an eigenvector corresponding to the eigenvalue λ_i, form an orthonormal basis of H. We will denote $H_N = \operatorname{span}\{\mathbf{e}_1, \cdots, \mathbf{e}_N\}$.

It is easy to see that the dual V' of V is given by

$$V' = \{\boldsymbol{v} \in \mathscr{D} : \widehat{\boldsymbol{v}}(\mathbf{k}) = \overline{\widehat{\boldsymbol{v}}(-\mathbf{k})},\ \widehat{\boldsymbol{v}}(0) = 0,\ \sum_{\mathbf{k} \in \widetilde{\mathbb{Z}}^d} \frac{|\widehat{\boldsymbol{v}}(\mathbf{k})|^2}{|\mathbf{k}|^2} < \infty\},$$

where $\mathscr{D}$ denotes the space of distributions. The duality bracket between V and V' is given by

$${}_V\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{V'} = \sum_{\mathbf{k} \in \widetilde{\mathbb{Z}}^d} \widehat{\boldsymbol{u}}(\mathbf{k}) \overline{\widehat{\boldsymbol{v}}(\mathbf{k})}, \quad \boldsymbol{u} \in V, \boldsymbol{v} \in V'.$$

In Chapter 3 we will consider the existence times of solutions in general L^p spaces,

$$L^p := \{\boldsymbol{u} : \Omega \to \mathbb{R}^d \ : \ \|\boldsymbol{u}\|_p < \infty\}, \quad \|\boldsymbol{u}\|_p := \|\boldsymbol{u}\|_{L^p} = \left(\int |\boldsymbol{u}|^p \right)^{\frac{1}{p}}.$$

In addition, we define the homogeneous Sobolev spaces,

$$\dot{H}_p^\alpha := \{\boldsymbol{u} : \Omega \to \mathbb{R}^d \;\;:\;\; \|\boldsymbol{u}\|_{\dot{H}_p^\alpha} < \infty,\; \nabla \cdot u = 0\}, \quad \|\boldsymbol{u}\|_{\dot{H}_p^\alpha} := \|\Lambda^\alpha \boldsymbol{u}\|_p,$$

where Λ^α is defined as the Fourier multiplier with symbol $|\boldsymbol{\xi}|$, i.e.

$$\widehat{\Lambda \boldsymbol{u}}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^\alpha\, \widehat{\boldsymbol{u}}(\boldsymbol{\xi}).$$

We will find it useful in Chapter 2 to define the so-called Wiener algebra,

$$\mathcal{W} := \{\boldsymbol{u} \in H : \|\boldsymbol{u}\|_{\mathcal{W}} := \sum_{\mathbf{k}} |\widehat{\boldsymbol{u}}(\mathbf{k})| < \infty\}. \tag{1.1}$$

Clearly, from the expression of $\boldsymbol{u}$ in terms of its Fourier series,

$$\boldsymbol{u}(\mathbf{x}) = \sum_{\mathbf{k} \in \widetilde{\mathbb{Z}}^d} \widehat{\boldsymbol{u}}(\mathbf{k}) e^{\imath \mathbf{k} \cdot \mathbf{x}},$$

it immediately follows that

$$\|\boldsymbol{u}\|_{L^\infty} \leq \|\boldsymbol{u}\|_{\mathcal{W}}.$$

In addition, we have the elementary inequality

$$\|\boldsymbol{u}\|_{L^\infty} \leq \|\boldsymbol{u}\|_{\mathcal{W}} \leq \tfrac{2\pi^{d-1}}{L^d} \tfrac{2s-d+1}{2s-d} \|\Lambda^s \boldsymbol{u}\|,\;\; s > \frac{d}{2}. \tag{1.2}$$

We will be using (1.2) with $s = r - \frac{1}{2}$ for a number $r > \frac{d}{2}$, and so for readability we will define $c_r = \frac{2\pi^{d-1}}{L^d} \frac{2(r-\frac{1}{2})-d+1}{2(r-\frac{1}{2})-d} = \frac{1}{\pi 2^{d-1}} \frac{2r-d}{2r-1-d}$.

1.0.1 Standard Inequalities

We recall some standard inequalities. Here $\epsilon > 0$, $a, b \geq 0$, and u, v, and w are divergence-free periodic functions, with sufficient regularity to make all the norms involved finite.

We will frequently use the following forms of Young's inequality and Hölder's inequality:

$$ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2 \tag{1.3}$$

$$\left|\int_\Omega uvw\,dxdy\right| \leq \|u\|_{L^2}\|v\|_{L^4}\|w\|_{L^4} \tag{1.4}$$

We also recall the following version of Poincaré's inequality, valid for periodic functions with zero space average on Ω:

$$\|\nabla \boldsymbol{u}\|_{L^2} \geq 2\pi\|\boldsymbol{u}\|_{L^2}. \tag{1.5}$$

In Chapter 3 we will make use of the standard interpolation inequalities,

$$\|\boldsymbol{u}\|_q \leq \|\boldsymbol{u}\|_1^\theta \|\boldsymbol{u}\|_2^{1-\theta}, \qquad \frac{1}{q} = \theta + (1-\theta)/2, \qquad 1 \leq q \leq 2, \tag{1.6}$$

$$\|\boldsymbol{u}\|_{\dot{H}_p^\alpha} \leq \|\boldsymbol{u}\|_{\dot{H}_p^{\alpha_1}}^\theta \|\boldsymbol{u}\|_{\dot{H}_p^{\alpha_2}}^{1-\theta}, \quad \alpha = \theta\alpha_1 + (1-\theta)\alpha_2, \quad \theta \in (0,1), \quad \alpha_1, \alpha_2 \in \mathbb{R}, \tag{1.7}$$

and the Sobolev inequality,

$$\|\boldsymbol{u}\|_q \leq C\|\boldsymbol{u}\|_{\dot{H}_p^\delta}, \qquad 0 \leq \delta < \frac{d}{p}, \qquad \delta = \frac{d}{p} - \frac{d}{q}, \tag{1.8}$$

as well as the following, which is a product of (1.7) and (1.8):

$$\|\boldsymbol{u}\|_{L^p} \lesssim \|\Lambda^s \boldsymbol{u}\| \leq \|\boldsymbol{u}\|^{1-s}\|\Lambda \boldsymbol{u}\|^s, \ p = \frac{2d}{d-2s}, \ 1 \leq d \leq 4, \tag{1.9}$$

where $a \lesssim b$ has the same meaning as when we write $a \leq Cb$ (where C is used to denote a general constant). We will also need a generalization of the Gagliardo-Nirenberg inequality,

$$\|\boldsymbol{u}\|_{\dot{H}_p^\alpha} \leq C\|\boldsymbol{u}\|_{\dot{H}_{q_1}^{\alpha_1}}^\theta \|\boldsymbol{u}\|_{q_2}^{1-\theta}, \quad \theta \in (0,1), \quad \alpha = \theta\alpha_1, \quad \frac{1}{p} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}, \tag{1.10}$$

and the following consequence of the Calderon-Zygmund theorem:

$$\|\nabla \times \boldsymbol{u}\|_{\dot{H}_q^\delta} \sim \|\boldsymbol{u}\|_{\dot{H}_q^{1+\delta}}, \qquad 1 < q < \infty, \qquad \delta \geq 0. \tag{1.11}$$

The following inequality due to Ladyzhenskaya will be used to prove some of the results in Chapter 4:

$$\|u\|_{L^4}^2 \leq c_L \|u\|_{L^2} \|\nabla u\|_{L^2} \tag{1.12}$$

The next two inequalities are extensions of the Brezis-Gallouet inequality and are due to Titi [124]. They will also become useful in Chapter 4:

$$\left| \int_\Omega u \partial_i v w \, dxdy \right| \leq c_B \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \|w\|_{L^2} \left(1 + \ln \left(\frac{\|\nabla w\|_{L^2}}{2\pi \|w\|_{L^2}} \right) \right)^{1/2}, \tag{1.13}$$

$$\left| \int_\Omega u \partial_i v \Delta w \, dxdy \right| \leq c_T \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \|\Delta w\|_{L^2} \left(1 + \ln \left(\frac{\|\Delta z\|_{L^2}}{2\pi \|\nabla z\|_{L^2}} \right) \right)^{1/2}, \tag{1.14}$$

where in (1.14), z can be u or v.

1.0.2 Gevrey Classes

Let $0 < \beta < \infty$ and $\alpha > 0$. We define the Gevrey norm by

$$\|\boldsymbol{u}\|_{G(\beta,\alpha,p)} = \|\Lambda^\alpha e^{\beta\Lambda} \boldsymbol{u}\|_{L^p}.$$

In Chapter 3, we establish bounds on higher order derivatives, and we do this by showing that the Gevrey norm of a solution is bounded. The reason this works is that Gevrey regularity is closely related to space analyticity, where for a function $\boldsymbol{u} \in C^\infty(\mathbb{R}^d)$ to be real analytic we mean for each bounded domain $\Omega \in \mathbb{R}^d$ there is a $M > 0$ and a $\delta > 0$ such that

$$|\partial^\alpha \boldsymbol{u}(\boldsymbol{x})| \leq M \frac{\alpha!}{\delta^{|\alpha|}}$$

for all $\boldsymbol{x} \in \Omega$ and for all multi-indices α. The following proposition explains this connection (for more information, see [110]).

Proposition 1.0.1 *For any $\alpha > 0$,*

$$\bigcup_{\beta>0} G_{\beta,\alpha,2} \subset C^{\omega}(\mathbb{R}^d),$$

where $C^{\omega}(\mathbb{R}^d)$ denotes the class of real analytic functions on $\mathbb{R}^d$. In particular if $\boldsymbol{u} \in G_{\frac{\beta}{c^2},\alpha,2}$ then $\boldsymbol{u}$ is real analytic with radius $\sqrt{\frac{\beta}{d}}$ uniformly on all of $\mathbb{R}^d$.

The Gevrey norm is characterized by the decay rate of higher order derivatives, namely, if $\|\boldsymbol{u}\|_{G(\beta,\alpha,2)} < \infty$ for some $\beta > 0$, then we have the higher derivative estimates

$$\|\boldsymbol{u}\|_{H^{\alpha+n}} \leq \left(\frac{n!}{\beta^n}\right) \|\boldsymbol{u}\|_{G(\beta,\alpha,2)} \text{ where } n \in \mathbb{N}. \tag{1.15}$$

In particular, $\boldsymbol{u}$ in (1.15) is analytic with (uniform) analyticity radius β (see Theorem 4 in [101] and Theorem 5 in [110]). We can use (1.15) to obtain the following estimate, which will be used in the proof of Theorem 3.2.6 (for details, see [8]).

Proposition 1.0.2 *Suppose that* $\sup_{t\in[0,T]} \|\boldsymbol{u}\|_{G(\sqrt{t},\alpha_0,p)} < \infty$. *Then*

$$\|\boldsymbol{u}(t)\|_{\dot{H}^{\alpha}_p} \leq Ct^{-\frac{1}{2}(\alpha-\alpha_0)}\|\boldsymbol{u}(t)\|_{G(\sqrt{t},\alpha_0,p)}, \qquad \alpha > \alpha_0.$$

This property is where the "higher" in the phrase "all higher order derivatives" is derived, in that $\alpha > \alpha_0$ is required in the conclusion of the proposition.

1.0.3 Gronwall Inequality

The following generalization of the Grönwall Lemma will be useful, which was first shown by Foias et al. in [65]. For a proof of an even more general version due to Jones and Titi, see [64].

Proposition 1.0.3 (Generalized Gronwall Inequality) *Let $\psi : [0, \infty) \to \mathbb{R}$ be a locally integrable function such that for some $T > 0$ the following two conditions hold:*

$$\liminf_{t\to\infty} \frac{1}{T} \int_t^{t+T} \psi(s)ds > 0, \tag{1.16a}$$

$$\limsup_{t\to\infty} \frac{1}{T} \int_t^{t+T} \psi^-(s)ds < \infty, \tag{1.16b}$$

where $\psi^-(t) := \max\{0, -\psi(t)\}$. Then if $Y : [0, \infty) \to [0, \infty)$ is absolutely continuous and for almost all t,

$$\frac{d}{dt}Y + \psi Y \leq \varphi, \tag{1.16c}$$

where $\varphi(t) \to 0$ as $t \to \infty$, then $Y(t) \to 0$ as well. Furthermore, if $\varphi \equiv 0$ then $Y(t) \to 0$ at an exponential rate as $t \to \infty$.

Chapter 2: Space and time analyticity for inviscid equations of fluid dynamics

2.1 Introduction

It is well-known that solutions to a large class of dissipative equations are analytic in space and time [8, 23, 28, 60, 68, 69, 79, 106, 110]. In fluid dynamics, space analyticity radius has a physical interpretation. It denotes a length scale below which the viscous effects dominate and the Fourier spectrum decays exponentially, while above it, the inertial effects dominate [50]. This fact concerning exponential decay can be used to show that the Galerkin approximation converges at an exponential rate to the exact solution [49]. Other applications of the analyticity radius occur in establishing sharp temporal decay rates of solutions in higher Sobolev norms [17, 110], establishing geometric regularity criteria for solutions, and in measuring the spatial complexity of fluid flows [78, 93]. Likewise, time analyticity also has several important applications including establishing backward uniqueness of trajectories [40], parameterizing turbulent flows by finitely many space-time points [94] and numerical determination of the attractor [62].

Space and time analyticity of inviscid equations, particularly the Euler equations, has received considerable attention recently, as well as in the past. Space analyticity for Euler, in the Eulerian variables, was considered for instance in [10, 11, 95, 96, 101], while in [4, 9, 48] real analyticity in the time (and space) variable is established using harmonic analysis tools. In the above mentioned works, the initial data are taken to be analytic in the space variable. By contrast, in a recent work [43], it is shown that the Lagrangian trajectories are real analytic (in time), even though the initial velocity fields are slightly more regular than Lipschitz in the space variable. Similar results also appear elsewhere; see for instance in [74, 118, 120] and the references therein. Additionally, the contrast between the analytic properties in the Eulerian and Lagrangian variables has been considered recently in [41].

In this chapter, we show that solutions of the Euler, as well as the inviscid versions of the SQG, Boussinesq, MHD, and similar equations with analytic nonlinearities, with analytic initial data, extend as solutions of the complexified versions of the equations, as holomorphic functions of time, with values in a suitable Gevrey class of functions in the space variable. Since belonging to a Gevrey class is equivalent to a function being (complex) analytic, this immediately establishes that the solutions extend as holomorphic functions of both space and time. In contrast to, for instance, the results in [4, 48], we not only obtain holomorphic extensions (as opposed to real in time analyticity in [4, 48]) but also obtain *explicit estimates on the domain of (time) analyticity,* while in [4, 48], the region is given implicitly in terms of the *flow map* generated by the solutions. Our approach follows [69], in which such results are obtained for the Navier-Stokes equations. We also make use of the ideas introduced in [101] and [95].

It should be noted that unlike their "real" counterparts, the complexified inviscid models are not known to conserve "energy", which is due to the fact that the complexified nonlinear terms do not in general possess cancellation properties akin to their real counterparts. Yet, as in [101], the mild dissipation generated due to working in a Gevrey class setting is enough for local existence for the complexified versions of these inviscid models.

The chapter is organized as follows: in Sections 2.2-2.6, we respectively consider the Euler equations, the inviscid surface quasi-geostrophic equations, the inviscid Boussinesq equations, the inviscid magnetohydrdynamic equations and an equation with an anlytic nonlinearity.

2.1.1 Complexification.

In order to extend the solutions of the equations to complex times, we need to complexify the associated phase spaces and operators. Accordingly, let $\mathcal{L}$ be an arbitrary, real, separable Hilbert space with (real) inner-product $\langle \cdot\,, \cdot \rangle$. The complexified Hilbert space $\mathcal{L}_{\mathbb{C}}$ and the associated inner-product is given by

$$\mathcal{L}_{\mathbb{C}} = \{\boldsymbol{u} = \boldsymbol{u}_1 + \imath \boldsymbol{u}_2 : \boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathcal{L}\},$$

and for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{L}_{\mathbb{C}}$ with $\boldsymbol{u} = \boldsymbol{u}_1 + \imath \boldsymbol{u}_2, \boldsymbol{v} = \boldsymbol{v}_1 + \imath \boldsymbol{v}_2$, the complex inner-product is given by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathbb{C}} = \langle \boldsymbol{u}_1, \boldsymbol{v}_1 \rangle + \langle \boldsymbol{u}_2, \boldsymbol{v}_2 \rangle + \imath [\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle - \langle \boldsymbol{u}_1, \boldsymbol{v}_2 \rangle].$$

Observe that the complex inner-product $\langle \cdot\,, \cdot \rangle_{\mathbb{C}}$ is linear in the first argument while it is conjugate linear in the second argument. If A is a linear operator on $\mathcal{L}$ with domain

$\mathcal{D}(A)$, we extend it to a linear operator $A_{\mathbb{C}}$ with domain $\mathcal{D}(A_{\mathbb{C}}) = \mathcal{D}(A) + \imath\mathcal{D}(A)$ by

$$A_{\mathbb{C}}(\boldsymbol{u}_1 + \imath \boldsymbol{u}_2) = A\boldsymbol{u}_1 + \imath A\boldsymbol{u}_2, \boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathcal{D}(A).$$

Henceforth, we will drop the subscript notation from the complexified operators and inner-products and denote $A_{\mathbb{C}}$ and $\langle \cdot\,, \cdot \rangle_{\mathbb{C}}$ respectively as A and $\langle \cdot\,, \cdot \rangle$, but will retain the subscript in the notation of the corresponding complexified Hilbert spaces.

2.2 Incompressible Euler Equations

The incompressible Euler equations, on a spatial domain $\Omega = [0, 2\pi]^d$, $d = 2, 3$, are given by

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{2.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{2.1b}$$

$$\boldsymbol{u}(\mathbf{x}, 0) = \boldsymbol{u}_0(\mathbf{x}), \quad \text{in } \Omega, \tag{2.1c}$$

where $\boldsymbol{u} = \boldsymbol{u}(\mathbf{x}, t)$ denotes the fluid velocity at a location $\mathbf{x} \in \Omega$ and time $t \in \mathbb{R}_+ := [0, \infty)$ and $p = p(\mathbf{x}, t)$ is the fluid pressure. Since its introduction in [53], it has been the subject of intense research both in analysis and mathematical physics; see [13, 15] for a survey of recent results on 2.1. We supplement (2.1) with the space periodic boundary condition with space period 2π, i.e., the functions $\boldsymbol{u}$ and p are periodic with period 2π in all spatial directions.

We will also denote

$$B(\boldsymbol{u}, \boldsymbol{v}) = \mathbb{P}\,(\boldsymbol{u} \cdot \nabla \boldsymbol{v}) = \mathbb{P}\,\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{v}), \tag{2.2}$$

where $\mathbb{P} : \dot{L}^2(\Omega) \to H$ is the Leray-Helmholtz orthogonal projection operator onto the closed subspace H of $\dot{L}^2(\Omega)$. From (1.9), it readily follows that if $\boldsymbol{u}, \boldsymbol{v} \in V$, then $\|\boldsymbol{u} \otimes \boldsymbol{v}\| < \infty$ and consequently, $B(\boldsymbol{u}, \boldsymbol{v}) \in V'$.

The functional form of the incompressible Euler equations is given by

$$\frac{d}{dt}\boldsymbol{u} + B(\boldsymbol{u}, \boldsymbol{u}) = 0. \tag{2.3}$$

We will consider the complexified Euler equation given by

$$\frac{d\boldsymbol{u}}{d\zeta} + B_{\mathbb{C}}(\boldsymbol{u}, \boldsymbol{u}) = 0, \boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{2.4}$$

where, for $\boldsymbol{u} = \boldsymbol{u}_1 + \imath\boldsymbol{u}_2, \boldsymbol{v} = \boldsymbol{v}_1 + \imath\boldsymbol{v}_2 \in H_{\mathbb{C}}$, the complexified nonlinear term is given by

$$B_{\mathbb{C}}(\boldsymbol{u}, \boldsymbol{v}) := B(\boldsymbol{u}_1, \boldsymbol{v}_1) - B(\boldsymbol{u}_2, \boldsymbol{v}_2) + \imath[B(\boldsymbol{u}_1, \boldsymbol{v}_2) + B(\boldsymbol{u}_2, \boldsymbol{v}_1)].$$

As before, we will drop the subscript and write $B = B_{\mathbb{C}}$.

In the following, let $r > \frac{d+1}{2}$ be fixed, and we will consider the corresponding Gevrey norm, $\|\cdot\|_{G(\beta,r,2)}$, as defined in Section 1.0.2.

Theorem 2.2.1 *Let $\beta_0 > 0$ be fixed, and let $\boldsymbol{u}_0$ be such that $\|\boldsymbol{u}_0\|_{G(\beta_0,r,2)} < \infty$. The complexified Euler equation (2.4) admits a unique solution in the region*

$$\mathcal{R} = \left\{\zeta = se^{\imath\theta} : \theta \in [0, 2\pi), 0 < s < \frac{C\beta_0}{2^r c_r \|\boldsymbol{u}_0\|_{G(\beta_0,r,2)}}\right\}. \tag{2.5}$$

For the Euler equations in the real setting, it is well known that if the Beale-Kato-Majda condition [14], $\int_0^T \|\nabla \times \boldsymbol{u}\|_{G(L^\infty,r,2)} < \infty$, is satisfied, and there exists β_0 such that $\|\Lambda^r e^{\beta_0\Lambda}\boldsymbol{u}_0\| < \infty$, then there exists a continuous function $\beta(t) > 0$ on $[0, T]$ such that [10, 95, 96, 101]

$$\sup_{[0,T]} \|\Lambda^r e^{\beta(t)\Lambda}\boldsymbol{u}(t)\| < \infty.$$

In this case, $\boldsymbol{u}$ extends as a holomorphic function satisfying (2.4) in a neighborhood of $(0,\infty)$ in $\mathbb{C}$. More precisely we have the following:

Corollary 2.2.2 *If there exists a continuous function $\beta(\cdot) > 0$ on $[0,T]$ such that*

$$M := \sup_{t\in[0,T]} \|\boldsymbol{u}(t)\|_{G(\beta(t),r,2)} < \infty, \tag{2.6}$$

where $\boldsymbol{u}$ is a solution of (2.3), then $\boldsymbol{u}(\cdot)$ extends as a holomorphic function in a complex neighborhood of $(0,T)$, satisfying (2.4).

Proof. Let $\beta_0 = \inf_{t\in[0,T]} \beta(t) > 0$. Then by Theorem 2.2.1, $\boldsymbol{u}$ extends as a holomorphic function in a complex neighborhood of $(0,\varepsilon)$, where $\varepsilon = \frac{C\beta_0}{2^r c_r M}$. The proof follows by reapplying Theorem 2.2.1 with $\boldsymbol{u}_0 = \boldsymbol{u}(t_0)$, for each $t_0 \in \{\frac{\varepsilon}{2}, \frac{2\varepsilon}{2}, \frac{3\varepsilon}{2}, \dots\} \cap [0,T]$. □

Before proceeding with the proof of the theorem, we will need the following estimate of the nonlinear term.

Proposition 2.2.3 *Let $\boldsymbol{u} \in H_{\mathbb{C}}$ with $\|\Lambda^{1/2}\boldsymbol{u}\|_{G(\beta,r,2)} < \infty$. Then,*

$$|\langle B(\boldsymbol{u},\boldsymbol{u}), \Lambda^{2r} e^{2\beta\Lambda}\boldsymbol{u}\rangle| \lesssim 2^r c_r \|\boldsymbol{u}\|_{G(\beta,r,2)} \|\Lambda^{1/2}\boldsymbol{u}\|^2_{G(\beta,r,2)}. \tag{2.7}$$

Proof. See the appendix. □

2.2.1 Proof of Theorem 2.2.1.

Proof. Recall that for each $N \in \mathbb{N}$, $H_N = \operatorname{span}\{\mathbf{e}_1, \cdots, \mathbf{e}_N\} \subset H_{\mathbb{C}}$, where $\{\mathbf{e}_i\}_{i=1}^{\infty}$ is the complete, orthonormal system (in $H_{\mathbb{C}}$) of eigenvectors of A. Denote the orthogonal projection on H_N by P_N. The Galerkin system corresponding to (2.4) is given by

$$\frac{d\boldsymbol{u}_N}{d\zeta} + P_N B(\boldsymbol{u}_N, \boldsymbol{u}_N) = 0,\ \ \boldsymbol{u}_N(0) = P_N \boldsymbol{u}_0,\ \ \boldsymbol{u}_N(\zeta) \in H_N. \tag{2.8}$$

The Galerkin system is an ODE with a quadratic nonlinerity. Therefore it admits a unique solution in a neighborhood of the origin in $\mathbb{C}$. We will obtain *a priori* estimates on the Galerkin system in $\mathcal{R}$ (defined in (2.5)) independent of N. This will show that the Galerkin system corresponding to (2.4) has a solution for all $\zeta \in \mathcal{R}$ and forms a normal family on the domain $\mathcal{R}$. We can then pass to the limit through a subsequence by (the Hilbert space-valued version of) Montel's theorem to obtain a solution of (2.4) in $\mathcal{R}$. Since we will obtain estimates independent of N, henceforth we will denote by $\boldsymbol{u}(\cdot)$ a solution to (2.8), i.e., we will drop the subscript N.

Fix $\theta \in [0, 2\pi)$, and let

$$\zeta = se^{i\theta},\ \ s > 0.$$

We assume that the initial data $\boldsymbol{u}_0$ satisfies $\|\boldsymbol{u}_0\|_{G(\beta_0,r,2)} < \infty$ for some $\beta_0 > 0$. Fix $\delta > 0$, to be chosen later and define the time-varying norm

$$|\boldsymbol{u}(\zeta)| = \|\boldsymbol{u}(\zeta)\|_{G(\beta_0-\delta s,r,2)}.$$

The corresponding (time-varying) inner product will be denoted by $((,))$, i.e.,

$$\begin{aligned}
&((\boldsymbol{u}, \boldsymbol{v})) \\
&\quad = \langle \Lambda^r e^{(\beta_0-\delta s)\Lambda}\boldsymbol{u}, \Lambda^r e^{(\beta_0-\delta s)\Lambda}\boldsymbol{v}\rangle \\
&\quad = \langle \boldsymbol{u}, \Lambda^{2r} e^{2(\beta_0-\delta s)\Lambda}\boldsymbol{v}\rangle.
\end{aligned}$$

Taking the inner-product of (2.4) (in $H_{\mathbb{C}}$) with $\Lambda^{2r}e^{2(\beta_0-\delta s)\Lambda}\boldsymbol{u}$, then multiplying by $e^{i\theta}$, and finally taking the real part of the resulting equation, we readily obtain

$$\frac{1}{2}\frac{d}{ds}|\boldsymbol{u}(\zeta)|^2 + \delta\,|\Lambda^{1/2}\boldsymbol{u}(\zeta)|^2 = -Re\left(e^{i\theta}((B(\boldsymbol{u}(\zeta),\boldsymbol{u}(\zeta)),\boldsymbol{u}(\zeta)))\right)$$
$$\leq |((B(\boldsymbol{u}(\zeta),\boldsymbol{u}(\zeta)),\boldsymbol{u}(\zeta)))|.$$

For $s < \frac{\beta_0}{\delta}$, using Proposition 2.2.3, we obtain

$$\frac{1}{2}\frac{d}{ds}|\boldsymbol{u}|^2 + \delta\,|\Lambda^{1/2}\boldsymbol{u}|^2 \lesssim 2^r c_r|\boldsymbol{u}||\Lambda^{1/2}\boldsymbol{u}|^2. \tag{2.9}$$

Now choose

$$\delta = C2^r c_r\|\boldsymbol{u}_0\|_{G(\beta_0,r,2)}.$$

From (2.9), we see that $|\boldsymbol{u}|$ is non-increasing and

$$|\boldsymbol{u}(\zeta)| \leq \|\boldsymbol{u}_0\|_{G(\beta_0,r,2)}\ \forall\ \zeta = se^{i\theta},\ 0 < s < \frac{\beta_0}{\delta}.$$

In particular, this means

$$\sup_{\zeta\in\mathcal{R}}\|\Lambda^r\boldsymbol{u}(\zeta)\| \leq \|\boldsymbol{u}_0\|_{G(\beta_0,r,2)}.$$

As remarked above, the proof is now complete by invoking Montel's theorem. $\square$

2.3 Surface Quasi-geostrophic Equations

We consider the inviscid, two-dimensional (surface) quasi-gesotrophic equation (henceforth SQG) on $\Omega = [0, 2\pi]^2$, given by

$$\partial_t \eta + \boldsymbol{u} \cdot \nabla \eta = 0, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{2.10}$$

$$\boldsymbol{u} = [-R_2\eta, R_1\eta]^T, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{2.11}$$

$$\eta(0) = \eta_0, \quad \text{in } \Omega. \tag{2.12}$$

Here $\boldsymbol{u}$ is the velocity field, η is the temperature, and the operators $R_i = \partial_i \Lambda^{-1}, i = 1, 2$, are the usual Riesz transforms.

Observe that by the definition of $\boldsymbol{u}$, it is divergence-free. Also, without loss of generality, we will take $\boldsymbol{u}$ and η to be mean-free, i.e.,

$$\int_\Omega \boldsymbol{u} = 0, \int_\Omega \eta = 0.$$

The SQG was introduced in [42] and variants of it arise in geophysics and meteorology (see, for instance [112]). Moreover, the critical SQG is the two-dimensional analogue of the three-dimensional Navier-Stokes equations. Existence and regularity issues for the viscous and inviscid cases were first extensively examined in [113]. This equation, particularly the dissipative case with various fractional orders of dissipation, has received considerable attention recently; see [31,44,91,92], and the references therein. As in section 2.2, our focus here is time analyticity of the inviscid SQG, with values in an appropriate Gevrey class.

As before, for $r > \frac{3}{2}, \beta > 0$, we define

$$\|\eta\|_{G(\beta,r,2)} = \|\Lambda^r e^{\beta\Lambda}\eta\| \text{ and } \|\boldsymbol{u}\|_{G(\beta,r,2)} = \|\Lambda^r e^{\beta\Lambda}\boldsymbol{u}\|.$$

Note that because $\boldsymbol{u}$ is the Riesz transform of η, we have $\|\eta\|_{G(\beta,r,2)} \sim \|\boldsymbol{u}\|_{G(\beta,r,2)}$.

Theorem 2.3.1 *Let η_0 be such that $\|\eta_0\|_{G(\beta_0,r,2)} < \infty$ for some $\beta_0 > 0$. The complexified inviscid SQG equation*

$$\frac{d\eta}{d\zeta} + B(\boldsymbol{u},\eta) = 0, \eta(0) = \eta_0, \text{ where } B(\boldsymbol{u},\eta) = \boldsymbol{u}\cdot\nabla\eta, \tag{2.13}$$

admits a unique solution in the region

$$\mathcal{R} = \left\{\zeta = se^{i\theta} : \theta \in [0,2\pi), 0 < s < \frac{C\beta_0}{2^r c_r \|\eta_0\|_{G(\beta_0,r,2)}}\right\}. \tag{2.14}$$

Proof. Proceeding in a similar manner as in Proposition 2.2.3, we obtain

$$\begin{aligned}
&|\langle B(\boldsymbol{u},\eta), \Lambda^{2r} e^{2\beta\Lambda}\eta\rangle| \\
&\quad \lesssim 2^r c_r \left(\|\eta\|_{G(\beta,r,2)}\|\Lambda^{1/2}\boldsymbol{u}\|_{G(\beta,r,2)}\|\Lambda^{1/2}\eta\|_{G(\beta,r,2)} + \|\boldsymbol{u}\|_{G(\beta,r,2)}\|\Lambda^{1/2}\eta\|_\beta^2\right) \\
&\quad \lesssim 2^r c_r \|\eta\|_{G(\beta,r,2)}\|\Lambda^{1/2}\eta\|_\beta^2,
\end{aligned} \tag{2.15}$$

where the last inequality follows by noting that u is the Riesz transform of η.

Fix $\theta \in [0,2\pi)$. Let

$$\zeta = se^{i\theta}, s > 0.$$

The initial data η_0 satisfies $\|\eta_0\|_{G(\beta_0,r,2)} < \infty$ for some $\beta_0 > 0$. Fix $\delta > 0$, to be specified later, and define the time-varying norm

$$|\eta| = \|\eta(\zeta)\|_{G(\beta_0-\delta s,r,2)},$$

and the corresponding (time-varying) inner product, $((,))$, as we did in the proof of Theorem 2.2.1.

Taking the inner-product of (2.13) (in $H_{\mathbb{C}}$) with $\Lambda^{2r}e^{2(\beta_0-\delta s)\Lambda}\eta$, multiplying by $e^{i\theta}$ and taking the real part, we obtain

$$\frac{1}{2}\frac{d}{ds}|\eta(\zeta)|^2+\delta\,|\Lambda^{1/2}\eta(\zeta)|^2=Re\left(-e^{i\theta}((B(\boldsymbol{u}(\zeta),\eta(\zeta)),\eta(\zeta)))\right).$$

Using (2.15), we deduce

$$\frac{1}{2}\frac{d}{ds}|\eta|^2+\delta\,|\Lambda^{1/2}\eta|^2\lesssim 2^r c_r|\eta|\,|\Lambda^{1/2}\eta|^2. \tag{2.16}$$

Now choose

$$\delta=C2^r c_r\|\eta_0\|_{G(\beta_0,r,2)}.$$

From (2.16), we see that $|\eta|$ is non-increasing and

$$|\eta(\zeta)|\leq\|\eta_0\|_{G(\beta_0,r,2)}\ \forall\ \zeta=se^{i\theta},\ 0<s<\frac{\beta_0}{\delta}.$$

In particular, this means

$$\sup_{z\in\mathcal{R}}\|\eta(z)\|\leq\|\eta_0\|_{G(\beta_0,r,2)}.$$

This establishes a uniform bound on the Galerkin system and the proof is complete by invoking Montel's theorem as before. $\square$

2.4 Inviscid Boussinesq Equations

The inviscid Boussinesq system (without rotation) in the periodic domain $\Omega := [0, 2\pi]^d, d = 2, 3$, for time $t \geq 0$ is given by

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \eta\, g\mathbf{e}, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{2.17a}$$

$$\partial_t \eta + (\boldsymbol{u} \cdot \nabla)\eta = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{2.17b}$$

$$\nabla \cdot \boldsymbol{u} = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{2.17c}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \eta(0) = \eta_0, \qquad \text{in } \Omega, \tag{2.17d}$$

equipped with periodic boundary conditions in space. Here $\mathbf{e}$ denotes the unit vector in $\mathbb{R}^d$ pointing upward and g denotes the (scalar) acceleration due to gravity. The unknowns are the fluid velocity field $\boldsymbol{u}$, the fluid pressure p, and the function η, which may be interpreted physically as the temperature. The Boussinesq system arises in the study of atmospheric, oceanic and astrophysical turbulence, particularly where rotation and stratification play a dominant role [112, 116]. We will follow the notation for the norms as in Section 2.2 and Section 2.3

Theorem 2.4.1 *Let $(\boldsymbol{u}_0, \eta_0)$ be such that $\|(\boldsymbol{u}_0, \eta_0)\|_{G(\beta_0,r,2)} < \infty$ for some $\beta_0 > 0$, where $\|(\boldsymbol{u}_0, \eta_0)\|^2_{G(\beta_0,r,2)} = \|\boldsymbol{u}_0\|^2_{G(\beta_0,r,2)} + \|\eta_0\|^2_{G(\beta_0,r,2)}$. The complexified inviscid Boussinesq equations (2.17) admit a unique solution $(\boldsymbol{u}(\zeta), \eta(\zeta))$ in the region*

$$\mathcal{R} = \left\{\zeta = se^{i\theta} : \theta \in [0, 2\pi),\ 0 < s < \min\left\{\frac{C\beta_0}{2^r c_r \|(\boldsymbol{u}_0, \eta_0)\|_{G(\beta_0,r,2)}}, \frac{2\ln 2}{g}\right\}\right\}. \tag{2.18}$$

Proof. We proceed as in Section 2.2 and Section 2.3 by taking the inner product of the complexified versions of (2.17a) and (2.17b) with $\Lambda^{2r}e^{2(\beta_0-\delta s)\Lambda}u$ and $\Lambda^{2r}e^{2(\beta_0-\delta s)\Lambda}\eta$ respectively, then multiplying by $e^{i\theta}$ and taking the real part. Using (2.7) and (2.15) and adding the results, for $(\boldsymbol{u}(\zeta),\eta(\zeta)),\zeta=se^{i\theta}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}(|\boldsymbol{u}|^2+|\eta|^2)+\delta(|\Lambda^{1/2}\boldsymbol{u}|^2+|\Lambda^{1/2}\eta|^2)\\
&\lesssim 2^r c_r(|\boldsymbol{u}|+|\eta|)(|\Lambda^{1/2}\boldsymbol{u}|^2+|\Lambda^{1/2}\eta|^2)+g|\eta||\boldsymbol{u}|\\
&\leq 2^r c_r(|\boldsymbol{u}|+|\eta|)(|\Lambda^{1/2}\boldsymbol{u}|^2+|\Lambda^{1/2}\eta|^2)+\frac{g}{2}(|\boldsymbol{u}|^2+|\eta|^2).
\end{aligned} \tag{2.19}$$

Thus, as long as

$$(|\boldsymbol{u}|+|\eta|)\lesssim\frac{\delta}{2^r c_r}, \tag{2.20}$$

by the Gronwall inequality, we have

$$|\boldsymbol{u}|^2+|\eta|^2\leq e^{Tg}(\|\boldsymbol{u}_0\|^2_{G(\beta_0,r,2)}+\|\eta_0\|^2_{G(\beta_0,r,2)}),0<s\leq T. \tag{2.21}$$

Using the fact that $(a+b)\leq\sqrt{2(a^2+b^2)}$, as long as (2.20) holds, from (2.21) we have

$$(|\boldsymbol{u}|+|\eta|)\leq e^{\frac{Tg}{2}}\sqrt{2}\|(\boldsymbol{u}_0,\eta_0)\|_{G(\beta_0,r,2)}. \tag{2.22}$$

Now choose

$$\delta=C2^r c_r\|(\boldsymbol{u}_0,\eta_0)\|_{G(\beta_0,r,2)}.$$

For all $0<s<T=\min\{\frac{\beta_0}{\delta},\frac{2\ln 2}{g}\}$, (2.20) is satisfied and consequently, (2.22) holds.

$\square$

2.5 Inviscid Magnetohydrodynamic Equations

The inviscid incompressible magnetohydrodynamic system in the periodic domain $\Omega := [0, 2\pi]^d, d = 2, 3$, for time $t \geq 0$ is given by the following system

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \tfrac{1}{S}(\boldsymbol{b} \cdot \nabla)\boldsymbol{b} + \nabla(\tfrac{1}{\rho_0} p + \tfrac{|\boldsymbol{b}|^2}{2S}) = 0, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{2.23a}$$

$$\partial_t \boldsymbol{b} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{b} - (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} = 0, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{2.23b}$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad \nabla \cdot \boldsymbol{b} = 0, \quad \text{in } \Omega \times \mathbb{R}_+, \tag{2.23c}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{b}(0) = \boldsymbol{b}_0, \quad \text{in } \Omega, \tag{2.23d}$$

equipped with periodic boundary conditions in space. Here, $\boldsymbol{u}$ represents the fluid velocity field, $\boldsymbol{b}$ the magnetic field and p the fluid pressure. The constant ρ_0 is the fluid density, and $S = \rho_0 \mu_0$, where μ_0 is the permeability of free space.

We will discuss the magnetohydrodynamic equations in more detail in the later chapters. Simply stated, the magnetohydrodynamic equations govern the evolution of an electrically conductive fluid under the influence of a magnetic field, and so are useful in the design of fusion reactors, or the study of solar storms and other natural phenomenon. See [47] for more on the derivation of (2.23), and [46, 103] for some applications of the magnetohydrodynamic equations (MHD). The existence and uniqueness of solutions to the incompressible MHD for the viscous case is the subject of Chapter 3 (see also [97, 108]). For now, we consider the inviscid case, for which the existence and uniqueness of solutions is treated in [12, 32]. The space analyticity of solutions of (2.23) is discussed in [30], whereas in the present work we give criteria for solutions to be holomorphic functions of both the time and space variables.

By rewriting the equations in terms of the Elsässer variables (which are defined via the transformations $\boldsymbol{v} = \boldsymbol{u} + \frac{1}{\sqrt{S}}\boldsymbol{b}$, $\boldsymbol{w} = \boldsymbol{u} - \frac{1}{\sqrt{S}}\boldsymbol{b}$), we obtain the equivalent system

$$\partial_t \boldsymbol{v} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{v} + \nabla\mathcal{P} = 0, \qquad \text{in } \Omega\times\mathbb{R}_+, \tag{2.24a}$$

$$\partial_t \boldsymbol{w} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{w} + \nabla\mathcal{P} = 0, \qquad \text{in } \Omega\times\mathbb{R}_+, \tag{2.24b}$$

$$\nabla\cdot\boldsymbol{v} = 0, \quad \nabla\cdot\boldsymbol{w} = 0, \qquad \text{in } \Omega\times\mathbb{R}_+, \tag{2.24c}$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_0, \quad \boldsymbol{w}(0) = \boldsymbol{w}_0, \quad \text{in } \Omega, \tag{2.24d}$$

where $\mathcal{P} = \frac{1}{\rho_0}p + \frac{|\boldsymbol{v}-\boldsymbol{w}|^2}{8}$.

Theorem 2.5.1 *Let $(\boldsymbol{v}_0, \boldsymbol{w}_0)$ be such that $\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_{G(\beta_0,r,2)} < \infty$ for some $\beta_0 > 0$. The complexified inviscid magnetohydrodynamic equations (2.24) admit a unique solution $(\boldsymbol{v}(\zeta), \boldsymbol{w}(\zeta))$ in the region*

$$\mathcal{R} = \left\{ \zeta = se^{i\theta} : \theta\in[0,2\pi), 0 < s < \frac{C\beta_0}{2^r c_r \|(\boldsymbol{v}_0,\boldsymbol{w}_0)\|_{G(\beta_0,r,2)}} \right\}.$$

Proof. Proceeding as in the previous sections and using (2.7), for a fixed $\theta\in[0,2\pi)$, for $(\boldsymbol{v}(\zeta), \boldsymbol{w}(\zeta)), \zeta = se^{i\theta}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\{|\boldsymbol{v}|^2 + |\boldsymbol{w}|^2\} + \delta(|\Lambda^{1/2}\boldsymbol{v}|^2 + |\Lambda^{1/2}\boldsymbol{w}|^2) \\
&\lesssim 2^r c_r (|\boldsymbol{v}| + |\boldsymbol{w}|)(|\Lambda^{1/2}\boldsymbol{v}|^2 + |\Lambda^{1/2}\boldsymbol{w}|^2) \\
&\lesssim 2^r c_r |(\boldsymbol{v},\boldsymbol{w})|(|\Lambda^{1/2}\boldsymbol{v}|^2 + |\Lambda^{1/2}\boldsymbol{w}|^2).
\end{aligned} \tag{2.25}$$

Now choose

$$\delta = C2^r c_r \|(\boldsymbol{v}_0,\boldsymbol{w}_0)\|_{G(\beta_0,r,2)}.$$

From (2.25), we see that $|(\boldsymbol{v},\boldsymbol{w})|^2$ is non-increasing and

$$|(\boldsymbol{v}(\zeta),\boldsymbol{w}(\zeta))| \le \|(\boldsymbol{v}_0,\boldsymbol{w}_0)\|_{G(\beta_0,r,2)}\ \forall\ \zeta = se^{i\theta},\ 0 < s < \frac{\beta_0}{\delta}.$$

In particular, this means

$$\sup_{z\in\mathcal{R}} \|(\boldsymbol{v}(z),\boldsymbol{w}(z))\| \le \|(\boldsymbol{v}_0,\boldsymbol{w}_0)\|_{G(\beta_0,r,2)}.$$

$\square$

2.6 Analytic Nonlinearity

In this section, we consider the more general case of an analytic nonlinearity on our basic spatial domain $\Omega := [0,2\pi]^d$. Again, we consider an equation without viscous effects (see [60] for the dissipative version). For simplicity of exposition, we consider only the case of a scalar equation here. A vector-valued version, i.e. the case of a system, can be handled in precisely the same way, although notationally it becomes more cumbersome. Let

$$F(z) = \sum_{n=1}^{\infty} a_n z^n$$

be a real analytic function in a neighborhood of the origin. The "majorizing function" for F is defined to be

$$F_M(s) = \sum_{n=1}^{\infty} |a_n| s^n, \quad s < \infty. \tag{2.26}$$

The functions F and F_M are clearly analytic in the open balls (in $\mathbb{R}^d$ and $\mathbb{R}$ respectively) with center zero and radius

$$R_M = \sup\{s : F_M(s) < \infty\}. \tag{2.27}$$

We assume that $R_M > 0$. The derivative of the function F_M, denoted by F'_M, is also analytic in the ball of radius R_M. Therefore, for any fixed $r > 0$, the function $\widetilde{F}$, defined by

$$\widetilde{F}(s) = \sum_{n=1}^{\infty} |a_n| n^{r+\frac{3}{2}} (c_r)^{n-1} s^{n-1}, \quad s \in \mathbb{R}, \tag{2.28}$$

is analytic in the ball of radius R_M/c_r. Moreover,

$$\widetilde{F}(s) \geq 0 \text{ for } \ s \geq 0 \text{ and } \widetilde{F}(s_1) < \widetilde{F}(s_2) \text{ for } 0 \leq s_1 < s_2. \tag{2.29}$$

We will consider an inviscid equation of the form

$$\partial_t u = TF(u), \quad u(0) = u_0, \tag{2.30}$$

where T is given by

$$\widehat{Tu}(\mathbf{k}) = m_T(\mathbf{k})\widehat{u}(\mathbf{k}), |m_T(\mathbf{k})| \leq C|\mathbf{k}|, \mathbf{k} \in \widetilde{\mathbb{Z}}^d.$$

We will assume that (2.30) preserves the mean free condition under evolution. Here, the phase space $H = \dot{L}^2(\Omega)$. As before, we fix $r > \frac{d+1}{2}$ and define

$$\|u\|_{G(\beta,r,2)} = \|\Lambda^r e^{\beta\Lambda} u\|.$$

The following proposition is elementary.

Proposition 2.6.1 *For $x_1, \cdots, x_n \in \mathbb{R}_+$ and any $r > 0$, we have*

$$(x_1 + \cdots + x_n)^r \leq n^r (x_1^r + \cdots + x_n^r).$$

Proof. Without loss of generality, assume $x_1 = \max\{x_1, \cdots, x_n\} > 0$. Let $\xi_i = \frac{x_i}{x_1}$ and note that $0 \leq \xi_i \leq 1$. Then,

$$(\sum_{i=1}^n x_i)^r = x_1^r (\sum_{i=1}^n \xi_i)^r \leq x_1^r (\sum_{i=1}^n 1)^r = n^r x_1^r \leq n^r \sum_{i=1}^n x_i^r.$$

$\square$

We will need the following estimate of the nonlinear term to proceed.

Proposition 2.6.2 *Let $u \in H_{\mathbb{C}}$ with $\|\Lambda^{1/2}u\|_{G(\beta,r,2)} < \infty$. Then*

$$|\langle TF(u), \Lambda^{2r}e^{2\beta\Lambda}u\rangle| \lesssim \widetilde{F}(\|u\|_{G(\beta,r,2)})\|\Lambda^{1/2}u\|^2_{G(\beta,r,2)}. \tag{2.31}$$

Proof. See the appendix. $\square$

Theorem 2.6.3 *Let $r > \frac{d+1}{2}$ and $\beta_0 > 0$ be fixed and u_0 be such that $\|u_0\|_{G(\beta_0,r,2)} < \infty$. Then, the complexified equation (2.30) admits a unique solution in the region*

$$\mathcal{R} = \left\{ z = se^{i\theta} : \theta \in [0, 2\pi), 0 < s < \frac{C\beta_0}{\widetilde{F}(\|u_0\|_{G(\beta_0,r,2)})} \right\}.$$

Proof. Fix $\delta > 0$, to be chosen later and, as before, define the time-varying norm

$$|u(\zeta)| = \|u(\zeta)\|_{G(\beta_0-\delta s,r,2)}.$$

Recall that the corresponding (time-varying) inner product is denoted by $((,))$, i.e.,

$$\begin{aligned} &((u, v)) \\ &\quad = \langle \Lambda^r e^{(\beta_0-\delta s)\Lambda}u, \Lambda^r e^{(\beta_0-\delta s)\Lambda}v\rangle \\ &\quad = \langle u, \Lambda^{2r}e^{2(\beta_0-\delta s)\Lambda}v\rangle. \end{aligned}$$

Multiplying (2.30) by $e^{i\theta}$, taking the real part and then the inner-product with $\Lambda^{2r}e^{2(\beta_0-\delta s)\Lambda}u$, we readily obtain

$$\frac{1}{2}\frac{d}{ds}|u(\zeta)|^2 + \delta\,|\Lambda^{1/2}u(\zeta)|^2 = -((\ Re(e^{i\theta}F(u(\zeta))), u(\zeta)\ )),\ \zeta = se^{i\theta}.$$

Using Proposition 2.6.2, we obtain

$$\frac{1}{2}\frac{d}{ds}|u|^2 + \delta\,|\Lambda^{1/2}u|^2 \lesssim \widetilde{F}(|u|)|\Lambda^{1/2}u|^2. \tag{2.32}$$

Now choose

$$\delta = C\widetilde{F}(\|\boldsymbol{u}_0\|_{G(\beta_0,r,2)}).$$

From (2.32), and the fact that $\widetilde{F}(\cdot)$ is strictly increasing (2.29), we see that $|u|$ is non-increasing and

$$|u(\zeta)| \le \|u_0\|_{G(\beta_0,r,2)}\ \forall\ \zeta = se^{i\theta},\ 0 < s < \frac{\beta_0}{\delta}.$$

In particular, this means

$$\sup_{z\in\mathcal{R}} \|u(z)\| \le \|u_0\|_{G(\beta_0,r,2)}.$$

As before, the proof is now complete by invoking Montel's theorem. □

Remark 2.6.4 One can extend the method of this section to handle a nonlinearity of the form

$$F(u) = T_0 G(T_1 u, \cdots, T_n u),$$

where G is an analytic function of n-variables and T_i are Fourier multipliers with symbol m_i satisfying

$$|m_i(\mathbf{k})| \lesssim |\mathbf{k}|^{\alpha_i}\ \forall\ \mathbf{k} \in \widetilde{\mathbb{Z}}^d, 0 \le i \le n, \sum_{i=0}^{n} \alpha_i \le 1.$$

Using the exact same technique, one can in fact also consider the case of systems, in which case Theorem 2.2.1 becomes a special case.

Chapter 3: Eventual regularity and decay of solutions to the magnetohydrodynamic equations

3.1 Introduction to magnetohydrodynamics

The magnetohydrodynamic equations are a set of equations that model fluid dynamics when the fluid is conductive and coupled with an evolving magnetic field. A typical example of such a fluid is plasma, but any fluid which can hold a charge could also be described by the MHD, for instance salt water, liquefied metals, or the liquid part of the Earth's core. When an electrically conductive fluid is in the presence of a magnetic field, as the fluid moves, the magnetic field responds and in turn induces motion in the fluid. As a result of this interaction, the evolution equations for the fluid velocity and the magnetic field are coupled through quadratic nonlinear terms, the same type of nonlinearity present in the Navier-Stokes equations (NSE).

We consider the MHD equations for a fluid and magnetic field with zero space average, under incompressibility assumptions. In this chapter, we will consider the spatial dimension, d, to be either 2 or 3, and that the fields either fill all of $\mathbb{R}^d$ or satisfy periodic boundary conditions on the bounded domain $[0, L]^d$. In Chapter 4 and Chapter 5 we will restrict ourselves to $d = 2$ and the periodic case. Let $\boldsymbol{u}$, $\boldsymbol{b}$, and p represent the fluid velocity, magnetic field, and fluid pressure, respectively. The system can be written as (see, e.g., [47]):

System 3.1.1 (MHD)

$$\partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} + \tfrac{1}{\rho} \nabla \left(p + \tfrac{1}{2\mu_0} |\boldsymbol{b}|^2 \right) = \tfrac{1}{\rho \mu_0} (\boldsymbol{b} \cdot \nabla) \boldsymbol{b} + \boldsymbol{f}, \tag{3.1a}$$

$$\partial_t \boldsymbol{b} - \lambda \Delta \boldsymbol{b} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{b} + \nabla q = (\boldsymbol{b} \cdot \nabla) \boldsymbol{u} + \boldsymbol{g}, \tag{3.1b}$$

$$\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{b} = 0, \tag{3.1c}$$

Here, $\nu > 0$ is the kinematic fluid viscosity, ρ_0 is the fluid density, $\mu_0 := 4\pi \times 10^{-7} H/m$ is the permeability of free space, $\lambda = (\mu_0 \sigma)^{-1} > 0$ is the magnetic diffusivity, and σ is the electrical conductivity of the fluid. We impose initial conditions $\boldsymbol{u}(0, x, y) = \boldsymbol{u}_0(x, y)$ and $\boldsymbol{b}(0, x, y) = \boldsymbol{b}_0(x, y)$ in an appropriate function space, and allow for time-dependent forcing functions, denoted above by $\boldsymbol{f}$ and $\boldsymbol{g}$.

We will non-dimensionalize the system so that we can later reformulate it in terms of the Elsässer variables. Let U be a reference velocity and use L as a reference length. We denote the dimensionless fluid Reynolds number and the dimensionless magnetic Reynolds number by $Re := UL/\nu$ and $Rm := UL/\lambda$, respectively. In non-dimensional form, the system can be written as:

$$\partial_t \boldsymbol{u} - \tfrac{1}{Re} \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} - (\boldsymbol{b} \cdot \nabla) \boldsymbol{b} = -\nabla \mathcal{P} + \boldsymbol{f}, \tag{3.2a}$$

$$\partial_t \boldsymbol{b} - \tfrac{1}{Rm} \Delta \boldsymbol{b} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{b} - (\boldsymbol{b} \cdot \nabla) \boldsymbol{u} = \boldsymbol{g}, \tag{3.2b}$$

$$\nabla \cdot \boldsymbol{b} = 0, \quad \nabla \cdot \boldsymbol{u} = 0. \tag{3.2c}$$

with the initial conditions $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and $\boldsymbol{b}(0) = \boldsymbol{b}_0$, where $\mathcal{P}$ is the (non-dimensionalized) sum of the fluid and magnetic pressures, and $\boldsymbol{u}$, $\boldsymbol{b}$, $\boldsymbol{u}_0$, $\boldsymbol{b}_0$, $\boldsymbol{f}$, and $\boldsymbol{g}$ have been replaced by their appropriate non-dimensional versions. Note the bilinearity in $(\boldsymbol{u}, \boldsymbol{b})$ on the left-hand side of (3.2b) allows for the important fact that the four non-linear terms in (3.2) can be written with coefficients ± 1. We will denote the non-dimensionalized spatial domain by

$$\Omega := \mathbb{R}^d,$$

or in the case of a bounded domain,

$$\Omega := [0, 1]^d \subset \mathbb{R}^d.$$

Next, in order to simplify our calculations we will reformulate the MHD equations in terms of new variables which we call $\boldsymbol{v}$ and $\boldsymbol{w}$, in such a way as to symmetrize the system.

We assume, without loss of generality, that $\frac{1}{Re} \geq \frac{1}{Rm}$, and denote the Elsässer variables [52] by $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{b}$ and $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{b}$ (if $\frac{1}{Re} < \frac{1}{Rm}$ then we would denote $\boldsymbol{w} = \boldsymbol{b} - \boldsymbol{u}$ and proceed similarly).

Then we can derive evolution equations for $\boldsymbol{v}$ and $\boldsymbol{w}$ by considering both the sum and difference of (3.2a) and (3.2b) and obtain the following system:

System 3.1.2

$$\partial_t \boldsymbol{v} - \alpha \Delta \boldsymbol{v} - \beta \Delta \boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\, \boldsymbol{v} = -\nabla \mathcal{P} + \boldsymbol{f}, \tag{3.3a}$$

$$\partial_t \boldsymbol{w} - \alpha \Delta \boldsymbol{w} - \beta \Delta \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{w} = -\nabla \mathcal{P} + \boldsymbol{g}, \tag{3.3b}$$

$$\nabla \cdot \boldsymbol{v} = 0, \quad \nabla \cdot \boldsymbol{w} = 0, \tag{3.3c}$$

subject to the initial conditions $\boldsymbol{v}(0) = \boldsymbol{v}_0 := \boldsymbol{u}_0 + \boldsymbol{b}_0$ *and* $\boldsymbol{w}(0) = \boldsymbol{w}_0 := \boldsymbol{u}_0 - \boldsymbol{b}_0$.

Here we relabeled the forcing terms as $\boldsymbol{f} := \boldsymbol{f} + \boldsymbol{g}$ and $\boldsymbol{g} := \boldsymbol{f} - \boldsymbol{g}$, and we denote $\alpha := \frac{1}{2}(\frac{1}{Re} + \frac{1}{Rm})$ and $\beta := \frac{1}{2}(\frac{1}{Re} - \frac{1}{Rm})$. It will be important to note that $\alpha - \beta = \frac{1}{Rm} > 0$ and that $\alpha > 0$ and $\beta \geq 0$ (this last inequality is true by the assumption that $\frac{1}{Re} \geq \frac{1}{Rm}$, however if $\frac{1}{Re} < \frac{1}{Rm}$ then we would arrive at the above system except with a different sign on the pressure, and $\beta = \frac{1}{Rm} - \frac{1}{Re}$, so still we have $\beta \geq 0$, and in general we will have $\alpha - \beta = \min\{\frac{1}{Re}, \frac{1}{Rm}\}$).

Note that $\boldsymbol{v}$ and $\boldsymbol{w}$ solve this system with initial conditions $\boldsymbol{v}_0 = \boldsymbol{u}_0 + \boldsymbol{b}_0$ and $\boldsymbol{w}_0 = \boldsymbol{u}_0 - \boldsymbol{b}_0$ if and only if $\boldsymbol{u} = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{w})$ and $\boldsymbol{b} = \frac{1}{2}(\boldsymbol{v} - \boldsymbol{w})$ are solutions to the MHD equations with initial conditions $\boldsymbol{u}_0$ and $\boldsymbol{b}_0$, and by the triangle inequality, $\|\boldsymbol{v}\| + \|\boldsymbol{w}\| < \infty$ if and only if $\|\boldsymbol{u}\| + \|\boldsymbol{b}\| < \infty$, for any norm $\|\cdot\|$.

3.1.1 Known Results

The global well-posedness of these equations under periodic boundary conditions was established for the 2D case and local well-posedness for the 2D and 3D cases by Sermange and Temam in 1989 [119]. In 1999, Kim showed that locally solutions of the MHD equations with periodic boundary conditions have Gevrey regularity, and thus are immediately smooth, and are analytic in time [90].

In the half-space, Han and He [80] found algebraic decay rates of L^p norms for space derivatives of solutions up to order two as well as one order of time derivative, for small initial data in 3D or large initial data in 2D.

In the whole space, the subject of local well-posedness in various spaces and global existence in critical spaces has been studied by many authors. In [125], Wang discusses the decay of the global solutions in 3D arising from initial conditions in a critical frequency domain of negative "derivative" order, the decay being in terms of the negative order norm (like a negative order homogeneous Sobolev norm). Agapito and Schonbek [2] showed the decay of L^p norms of solutions with small initial data, and showed the L^2 decay of the velocity of solutions arising from the MHD equations without the presence of magnetic diffusion. They also found that solutions arising from initial conditions which are only assumed to be in L^2 converge to 0 in L^2 at a nonuniform rate in 2D and 3D, and that the nonuniform rate is optimal.

Recently, Biswas and Bae showed the local well-posedness and Gevrey regularity for a large class of dissipative equations in the whole space, also obtaining small data global existence results, and showed the eventual decay of solutions of the NSE in higher order Sobolev norms [8].

In this chapter, we follow the approach of Biswas and Bae and obtain local well-posedness and small data results with Gevrey regularity for the MHD equations, with initial conditions in a large range of homogeneous Sobolev spaces (including spaces of negative Sobolev order). Furthermore, we obtain the decay of the L^p norms of all higher order derivatives (not just first and second order) at algebraic rates when the initial data is in L^2 for $p \geq 2$. We are able to extend these results to the range $1 < p < 2$ by additionally requiring that the curl of the initial data be in L^1.

It is interesting to note that the result of Agapito and Schonbek says the L^2 norm can decay at a nonuniform rate only when the initial data is only in L^2 while we show that for the same initial data, the L^2 norms of all derivatives of positive order will decay at algebraic rates.

3.2 Results

3.2.1 Main Idea

We will need to first establish the local (in time) existence of solutions, the notion of existence being the finiteness of a time varying Gevrey norm, when the initial data is in certain homogeneous Sobolev spaces. We also obtain global existence for the case that the initial data is in a critical homogeneous Sobolev space (critical in the sense of a scaling invariance) and is sufficiently small in the corresponding norm.

To establish these existence results, we use the integral form of our simplified equations (called the mild-formulation) and define a self-map on a certain subset of $C([0,T],\dot{H}^{\alpha}_{p})$ in such a way that a fixed point of this map will give rise to a solution of the MHD equations. Then, if the initial data is in a non-critical space $\dot{H}^{\alpha}_{p}$, we will show that the self-map is a strict contraction when the time of existence, T, is sufficiently smaller than a power of the norm of the initial data, and so it will have a fixed point. This also gives us an estimate of the existence time.

If the initial data is in a critical space then whether or not the self-map is a contraction is not determined explicitly by the time of existence T. However, we can show that we will have a contraction anyway as $T \to 0$, thus we still have local existence. Furthermore, we can instead let $T = \infty$ and obtain a contraction by forcing the size of the norm of the initial data to be sufficiently small, thus obtaining the small data global existence result.

With the existence results established, we then focus on showing the decay of higher order derivatives. To start, we use the Leray energy inequality for the case that $p \geq 2$ and show that solutions will eventually become arbitrarily small in the critical space at a time t_0, and thus will have global existence after such a time. Using the properties of the Gevrey norm (which we then know will not blow-up after time t_0) and the Sobolev inequalities, we can show that the L^p norms of derivatives higher than a certain order will decay algebraically in time, for $p \geq 2$.

The Sobolev inequalities we use in establishing this result do not allow us to extend to the case that $1 < p < 2$. To circumvent this, with an additional assumption on the initial data, we can establish a uniform (in time) bound on the L^1 norm of the curl of the solutions, and, using various other inequalities and interpolating between $p = 1$ and $p = 2$, we can extend the results.

3.2.2 Assumptions and Definitions

Before we proceed further, we will make some additional assumptions. For the rest of this chapter, we will assume that there is no external forcing ($\boldsymbol{f} \equiv \boldsymbol{g} \equiv 0$), as we are currently interested in finding the rate at which solutions will decay when no external force is present. For simplicity, we will also assume that $Re = Rm = 1$.

Now we apply the Leray Projection $\mathbb{P}$ to (3.3) (i.e. we will project onto the space of divergence free vector fields) which cancels the gradient term on the right hand side. We can then define a classical solution as follows:

Definition 3.2.1 (Classical Solution) *A classical solution of the MHD equations is a pair* $(\boldsymbol{v}, \boldsymbol{w})$ *such that* $\boldsymbol{v}(\cdot, \boldsymbol{x}), \boldsymbol{w}(\cdot, \boldsymbol{x}) \in C^1(0, T)$ *for all* $\boldsymbol{x} \in \Omega$, $\boldsymbol{v}(t, \cdot), \boldsymbol{w}(t, \cdot) \in C^2(\Omega) \cap \dot{L}^2$ *for all* $t \in [0, T]$, $\boldsymbol{v}(0) = \boldsymbol{v}_0$, $\boldsymbol{w}(0) = \boldsymbol{w}_0$, *and*

$$\partial_t \boldsymbol{v} - \Delta \boldsymbol{v} + \mathbb{P}(\boldsymbol{w} \cdot \nabla)\boldsymbol{v} = 0, \tag{3.4}$$

$$\partial_t \boldsymbol{w} - \Delta \boldsymbol{w} + \mathbb{P}(\boldsymbol{v} \cdot \nabla)\boldsymbol{w} = 0. \tag{3.5}$$

Furthermore, $(\boldsymbol{v}, \boldsymbol{w})$ *is called a strong solution if* $\boldsymbol{v}, \boldsymbol{w} \in C\left([0, T], H \cap H^1\right)$ *and* (3.4) *and* (3.5) *are satisfied for almost all* t.

However, we will need to introduce a weaker notion of solution:

Definition 3.2.2 (Mild Solution) *A pair* $(\boldsymbol{v}, \boldsymbol{w}) \in C([0, T], \dot{H}_p^\alpha)$ *satisfying*

$$\boldsymbol{v}(t) = e^{-t\Lambda^2}\boldsymbol{v}_0 - \int_0^t e^{-(t-s)\Lambda^2}\mathbb{P}(\boldsymbol{w} \cdot \nabla)\boldsymbol{v} ds, \tag{3.6}$$

$$\boldsymbol{w}(t) = e^{-t\Lambda^2}\boldsymbol{w}_0 - \int_0^t e^{-(t-s)\Lambda^2}\mathbb{P}(\boldsymbol{v} \cdot \nabla)\boldsymbol{w} ds, \tag{3.7}$$

will be called a mild solution when $\boldsymbol{v}(0) = \boldsymbol{v}_0$, $\boldsymbol{w}(0) = \boldsymbol{w}_0$, *and we are considering the case that* $\boldsymbol{v}_0, \boldsymbol{w}_0 \in \dot{H}_p^\alpha$ *(we arrive at these equations by integrating* (3.4) *and* (3.5) *in time).*

Also, to establish a uniform bound in L^1 (which we will use to prove the eventual decay of solutions in certain Sobolev spaces), we'll need to consider the vorticity formulation. Defining the "vorticities" $\boldsymbol{\tau} := \nabla \times \boldsymbol{v}$ and $\boldsymbol{\psi} := \nabla \times \boldsymbol{w}$, we can write the vorticity formulation by taking the curl of (3.3a) and (3.3b), obtaining:

System 3.2.3 (Vorticity Formulation)

$$\partial_t \boldsymbol{\tau} - \Delta \boldsymbol{\tau} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{\tau} = \nabla \boldsymbol{v} * \nabla \boldsymbol{w}, \tag{3.8}$$

$$\partial_t \boldsymbol{\psi} - \Delta \boldsymbol{\psi} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{\psi} = \nabla \boldsymbol{v} * \nabla \boldsymbol{w}, \tag{3.9}$$

with $\boldsymbol{\tau}(0) = \nabla \times \boldsymbol{v}(0)$, $\boldsymbol{\psi}(0) = \nabla \times \boldsymbol{\psi}(0)$,

where $A * B := -\epsilon_{ijk} A_{kl} B_{lj} \boldsymbol{e}_i = \epsilon_{ijk} A_{jl} B_{lk} \boldsymbol{e}_i$ for any matrices A and B (ϵ_{ijk} is the permutation symbol). Note that $|A * B| \leq |A||B|$, where the first $|\cdot|$ is the regular euclidean norm of the vector $A*B$ and the second we are defining by $|A| := \sqrt{\sum A_{ij}^2}$.

3.2.3 Statements of the Results

Theorem 3.2.4 *Let $p \geq 1$, and $\frac{d}{p} > \alpha \geq \frac{d}{p} - 1$. If $\boldsymbol{v}_0, \boldsymbol{w}_0 \in \dot{H}_p^\alpha$, then there exists a $T > 0$ and a mild solution $(\boldsymbol{v}, \boldsymbol{w})$ on $[0,T]$ such that $(\boldsymbol{v}, \boldsymbol{w}) \in C([0,T], \dot{H}_p^\alpha) \times C([0,T], \dot{H}_p^\alpha)$, and for some $\beta > 0$,*

$$\max\{\sup_{0<t<T} \|\boldsymbol{v}(t), \boldsymbol{w}(t)\|_{G(\sqrt{t},\alpha,p)}, \sup_{0\leq t\leq T} t^{\frac{\beta}{2}} \|\boldsymbol{v}(t)\|_{G(\sqrt{t},\alpha+\beta,p)}\} \leq C\|\boldsymbol{v}_0, \boldsymbol{w}_0\|_{\dot{H}_p^\alpha}. \tag{3.10}$$

Furthermore, if $\alpha > \frac{d}{p} - 1$, there is a $\lambda > 0$ such that $T \geq \frac{C}{\|(\boldsymbol{v}_0,\boldsymbol{w}_0)\|_{\dot{H}_p^\alpha}^{1/\lambda}}$. Otherwise, $\alpha = \frac{d}{p}$ and there is an $\epsilon > 0$ such that if $\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\| \leq \epsilon$, then we may take $T = \infty$.

The next theorem establishes the persistence of solutions of the vorticity formulation in L^1, and is essential to proving the decay results of Theorem 3.2.6 when $1 < p < 2$. The corresponding result for the NSE was first shown by Constantin and Fefferman [39]. We are able to extend their arguments to the MHD, and obtain the following result:

Theorem 3.2.5 *Let $\boldsymbol{\tau}$ and $\boldsymbol{\psi}$ be solutions of the vorticity formulation, with $\boldsymbol{\tau}_0, \boldsymbol{\psi}_0 \in L^1$ and $\boldsymbol{v}_0, \boldsymbol{w}_0 \in L^2$, where $\boldsymbol{\tau}_0 = \nabla \times \boldsymbol{v}_0$ and $\boldsymbol{\psi}_0 = \nabla \times \boldsymbol{w}_0$. Then the L^1 norms of $\boldsymbol{\tau}$ and $\boldsymbol{\psi}$ remain finite for all time. In fact,*

$$\sup_{t \geq 0} \|\boldsymbol{\tau}(t)\|_1 \leq 4\left(\|\boldsymbol{v}_0\| \|\boldsymbol{w}_0\|\right)^{\frac{1}{2}} + \|\boldsymbol{\tau}_0\|_1. \tag{3.11a}$$

$$\sup_{t \geq 0} \|\boldsymbol{\psi}(t)\|_1 \leq 4\left(\|\boldsymbol{v}_0\| \|\boldsymbol{w}_0\|\right)^{\frac{1}{2}} + \|\boldsymbol{\psi}_0\|_1. \tag{3.11b}$$

The following theorem establishes explicit algebraic decay estimates of all higher derivative orders in L^p norms when $p > 1$.

Theorem 3.2.6 *Suppose $d = 3$ and let $p \geq 2$. For all initial conditions $\boldsymbol{v}_0, \boldsymbol{w}_0 \in L^2$ with corresponding solutions $\boldsymbol{v}, \boldsymbol{w}$, there is a time t_0 such that for all $t > t_0$ and $\alpha > \frac{3}{p} - 1$, we have*

$$\|\Lambda^\alpha \boldsymbol{v}(t)\|_p \leq C(t - t_0)^{-\frac{1}{2}(\alpha - \frac{3}{p} + 1)},$$

with a similar inequality holding for $\boldsymbol{w}$.

Furthermore, if $\boldsymbol{\tau}_0, \boldsymbol{\psi}_0 \in L^1$ we may take $1 < p < 2$.

The theorem is stated for $d = 3$ but the main ideas and techniques carry over to the case when $d = 2$. The difference is that the critical spaces are different, so some extra results are needed when $d = 2$. Specifically, when $p = 2$ the critical space is L^2 (since the critical value for α is $\frac{2}{2} - 1 = 0$) so we need to show $\liminf_{t \to \infty} \|(\boldsymbol{v}(t), \boldsymbol{w}(t))\| = 0$, which is shown in [2] for instance. A uniform bound in the $\dot{H}_p^{-\alpha}$ norm for $\alpha \in (0, 1)$ is also needed, and will be shown in a future work.

3.3 Proofs of the Results

3.3.1 Proof of Existence Results

Given any $T > 0$ and $\beta > 0$, define $\Sigma \subset C([0,T], \dot{H}_p^{\alpha}) \times C([0,T], \dot{H}_p^{\alpha})$ by

$$\Sigma = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \|(\boldsymbol{v}, \boldsymbol{w})\|_{\Sigma} \leq \infty\},$$

where

$$\|\boldsymbol{v}\|_{\Sigma'} := \sup_{0\leq t\leq T} t^{\frac{\beta}{2}} \|\boldsymbol{v}(t)\|_{G(\sqrt{t},\alpha+\beta,p)},$$

$$\|\boldsymbol{v}\|_{\Sigma} := \max\left(\sup_{0\leq t\leq T} \|\boldsymbol{v}(t)\|_{G(\sqrt{t},\alpha,p)}, \|\boldsymbol{v}\|_{\Sigma'}\right).$$

The space Σ will be the main setting in the proof of Theorem 3.2.4.

To simplify our notation, we will define the operators S, G, and I by

$$S(\boldsymbol{v}, \boldsymbol{w})(t) = e^{-t\Lambda^2}\boldsymbol{v}_0 - \int_0^t e^{-(t-s)\Lambda^2}\mathbb{P}(\boldsymbol{w}(s)\cdot\nabla)\boldsymbol{v}(s)ds$$

$$G(\boldsymbol{v})(t) = e^{-t\Lambda^2}\boldsymbol{v}_0$$

$$I(\boldsymbol{v}, \boldsymbol{w})(t) = \int_0^t e^{-(t-s)\Lambda^2}\mathbb{P}(\boldsymbol{w}(s)\cdot\nabla)\boldsymbol{v}(s)ds.$$

Now, let $N = \|\,(G(\boldsymbol{v}), G(\boldsymbol{w}))\,\|_{\Sigma}$, and consider the set

$$E = \{(\boldsymbol{v}, \boldsymbol{w}) \in \Sigma \mid \|(\boldsymbol{v}, \boldsymbol{w}) - (\,G(\boldsymbol{v}), G(\boldsymbol{w})\,)\,\|_{\Sigma} \leq N\}.$$

The following proposition gives a bound on N, and can be proved using similar arguments to those found in [8].

Proposition 3.3.1 *If $\boldsymbol{v}, \boldsymbol{w} \in H_p^{\alpha}$ then $N < C_{\beta}\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_{\dot{H}_p^{\alpha}}$, and if $\alpha = \frac{d}{p} - 1$, then $N \to 0$ as $T \to 0$*

We next show that S is a self-map on Σ and a contraction mapping when restricted to E, and therefore there will be a fixed point, i.e. a mild solution. But first we need the following estimate:

Proposition 3.3.2 *For $\alpha \geq \frac{d}{p} - 1$, there is a $\lambda \geq 0$ such that $\|I(\boldsymbol{v}(t), \boldsymbol{w}(t))\|_\Sigma \leq CT^\lambda \|\boldsymbol{v}(t)\|_{\Sigma'} \|\boldsymbol{w}(t)\|_{\Sigma'}$. Furthermore, $\lambda = 0$ if and only if $\alpha = \frac{d}{p} - 1$.*

The proof for the case $p = 2$ can be found in [24] (see Proposition 4.6 for $\alpha > \frac{d}{p} - 1$ and Proposition 4.8 for $\alpha = \frac{d}{p} - 1$). For the case $p \neq 2$, see [8].

Proof of Theorem 3.2.4 Since $\|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_{\alpha,p} < \infty$ we have that $N < \infty$ since $N < \|(\boldsymbol{v}_0, \boldsymbol{w}_0)\|_{\alpha,p}$ by *Proposition 3.3.1*. Now, by *Proposition 3.3.2*, $(\boldsymbol{v}, \boldsymbol{w}) \mapsto (\, I(\boldsymbol{v}, \boldsymbol{w})\,,\, I(\boldsymbol{w}, \boldsymbol{v})\,)$, which we call $\mathbf{I}$, is a self map on Σ, and hence so is $\mathbf{S}$ which is the mapping $(\boldsymbol{v}, \boldsymbol{w}) \mapsto (\, S(\boldsymbol{v}, \boldsymbol{w})\,,\, S(\boldsymbol{w}, \boldsymbol{v})\,)$. What remains to be shown is that $\mathbf{S}$ is a contraction mapping on E.

Let $(\boldsymbol{v}, \boldsymbol{w}), (\boldsymbol{v}', \boldsymbol{w}') \in E$. Then

$$\begin{aligned}
\mathbf{S}(\boldsymbol{v}, \boldsymbol{w}) - \mathbf{S}(\boldsymbol{v}', \boldsymbol{w}') &= -\mathbf{I}(\boldsymbol{v}, \boldsymbol{w}) + \mathbf{I}(\boldsymbol{v}', \boldsymbol{w}') \\
&= -\mathbf{I}(\boldsymbol{v}, \boldsymbol{w}) + \mathbf{I}(\boldsymbol{v}', \boldsymbol{w}') - \mathbf{I}(\boldsymbol{v}, -\boldsymbol{w}') + \mathbf{I}(-\boldsymbol{v}, \boldsymbol{w}') \\
&= -\mathbf{I}(\boldsymbol{v}, \boldsymbol{w} - \boldsymbol{w}') + \mathbf{I}(\boldsymbol{v}' - \boldsymbol{v}, \boldsymbol{w}')
\end{aligned}$$

so,

$$\|\mathbf{S}(\boldsymbol{v},\boldsymbol{w})-\mathbf{S}(\boldsymbol{v}',\boldsymbol{w}')\|_\Sigma \leq \|\mathbf{I}(\boldsymbol{v},\boldsymbol{w}-\boldsymbol{w}')\|_\Sigma+\|\mathbf{I}(\boldsymbol{v}'-\boldsymbol{v},\boldsymbol{w}')\|_\Sigma$$

$$= \|I(\boldsymbol{v},\boldsymbol{w}-\boldsymbol{w}')\|_\Sigma+\|I(\boldsymbol{w}-\boldsymbol{w}',\boldsymbol{v})\|_\Sigma+\|I(\boldsymbol{v}'-\boldsymbol{v},\boldsymbol{w}')\|_\Sigma+\|I(\boldsymbol{w}',\boldsymbol{v}'-\boldsymbol{v})\|_\Sigma$$

$$\leq CT^\lambda\left(\|\boldsymbol{v}\|_{\Sigma'}\|\boldsymbol{w}-\boldsymbol{w}'\|_{\Sigma'}+\|\boldsymbol{w}'\|_{\Sigma'}\|\boldsymbol{v}-\boldsymbol{v}'\|_{\Sigma'}\right)\ \text{(by } \textit{Proposition 2.2.}\text{)}$$

$$\leq CT^\lambda N\left(\|\boldsymbol{w}-\boldsymbol{w}'\|_{\Sigma'}+\|\boldsymbol{v}-\boldsymbol{v}'\|_{\Sigma'}\right)$$

$$= CT^\lambda N\|(\boldsymbol{v}-\boldsymbol{v}',\boldsymbol{w}-\boldsymbol{w}')\|_{\Sigma'}.$$

So if $CT^\lambda N<\frac{1}{2}$, then $\|\mathbf{S}(\boldsymbol{v},\boldsymbol{w})-\mathbf{S}(\boldsymbol{v}',\boldsymbol{w}')\|_\Sigma \leq \frac{1}{2}\|(\boldsymbol{v}-\boldsymbol{v}',\boldsymbol{w}-\boldsymbol{w}')\|_{\Sigma'}$, and $\mathbf{S}$ is a strict contraction on E.

If $\lambda>0$, then it suffices to take T such that $T\leq \frac{C}{\|(\boldsymbol{v}_0,\boldsymbol{w}_0)\|^{1/\lambda}_{\dot{H}^\alpha_p}}$ which will imply that $T\leq \frac{C}{N^{1/\lambda}}$ since $N<\|(\boldsymbol{v}_0,\boldsymbol{w}_0)\|_{\dot{H}^\alpha_p}$. If $\lambda=0$ (which happens if and only if $\alpha=\frac{n}{p'}-1$ and $p>1$), then we obtain a strict contraction using the fact that $N\to 0$ as $T\to 0$, by *Proposition 3.3.1*, giving a local existence time, or we can take $T=\infty$ if $\|(\boldsymbol{v}_0,\boldsymbol{w}_0)\|_{\dot{H}^\alpha_p}\leq\frac{1}{2C}$, giving the global existence for small data result.

In all the above cases, the solution is in E, so $\|(\boldsymbol{v},\boldsymbol{w})\|_\Sigma\leq 2N$. □

3.3.2 Proof of Decay Results

By the Leray energy inequalities (A.10) and (A.11), for solutions $\boldsymbol{v}$ and $\boldsymbol{w}$ of the simplified MHD with initial conditions $\boldsymbol{v}_0,\boldsymbol{w}_0\in L^2$, we have

$$\sup_{t\geq 0}\|\boldsymbol{v}(t)\|\leq\|\boldsymbol{v}_0\| \quad\text{and}\quad \sup_{t\geq 0}\|\boldsymbol{w}(t)\|\leq\|\boldsymbol{w}_0\|, \tag{3.12}$$

$$\liminf_{t\to\infty}\|\boldsymbol{v}(t)\|_{\dot{H}^1_2}=\liminf_{t\to\infty}\|\boldsymbol{w}(t)\|_{\dot{H}^1_2}=0, \tag{3.13}$$

Now we can prove Theorem 3.2.6, proceeding as is done for the Navier-Stokes equations in [8].

Proof of Theorem 3.2.6 We will show the result for $\boldsymbol{v}$, the proof for $\boldsymbol{w}$ being identical.

First let $p \geq 2$ and set $\alpha_0 = \frac{3}{2} - 1 = \frac{1}{2}$ (so that $\dot{H}_p^{\alpha_0}$ is the critical space when p=2), and let ϵ be the small data constant given by Theorem 1.1. Then from (3.13), $\exists\, t_0$ such that $\|\boldsymbol{v}(t_0)\|_{\dot{H}_2^1} < \left(\epsilon / \|\boldsymbol{v}_0\|^\theta\right)^{\frac{1}{1-\theta}}$ (where $\theta = \frac{1}{2}$), and so using (1.7) followed by (3.12),

$$\|\boldsymbol{v}(t_0)\|_{\dot{H}_2^{\alpha_0}} \leq \|\boldsymbol{v}(t_0)\|^\theta \|\boldsymbol{v}(t_0)\|_{\dot{H}_2^1}^{1-\theta} \leq \|\boldsymbol{v}_0\|^\theta \|\boldsymbol{v}(t_0)\|_{\dot{H}_2^1}^{1-\theta} < \epsilon.$$

Now Theorem 3.2.4 guarantees the solution is a global solution starting from time t_0, and satisfies

$$\sup_{t > t_0} \|\boldsymbol{v}\|_{G(\sqrt{t-t_0}, \alpha_0, 2)} < \infty$$

and therefore $\forall t > t_0$ and $\forall \alpha > \alpha_0$, by Proposition 1.0.2 and (3.10),

$$\|\boldsymbol{v}(t)\|_{\dot{H}_2^\alpha} \leq C \|\boldsymbol{v}(t_0)\|_{\dot{H}_2^{\alpha_0}} (t - t_0)^{-\frac{1}{2}(\alpha - \alpha_0)}. \tag{3.14}$$

Now, if $p > 2$, then by (1.8) and (3.14), $\forall t > t_0$ and $\forall \alpha > \max\{\alpha_0 - \delta, 0\}$ where $\delta = \frac{3}{2} - \frac{3}{p}$,

$$\|\boldsymbol{v}(t)\|_{\dot{H}_p^\alpha} = \|\Lambda^\alpha \boldsymbol{v}(t)\|_p \leq C \|\boldsymbol{v}(t)\|_{\dot{H}_2^{\alpha+\delta}} \leq C \|\boldsymbol{v}(t_0)\|_{\dot{H}_2^{\alpha_0}} (t - t_0)^{-\frac{1}{2}(\alpha + \delta - \alpha_0)}.$$

So, all that remains to show is the result for $1 < p < 2$, and for this we need the additional assumption that $\nabla \times \boldsymbol{v}_0, \nabla \times \boldsymbol{w}_0 \in L^1$. Then by (3.11), $\sup_{t \geq 0} \|\boldsymbol{\tau}(t)\|_1 \leq \infty$. Therefore, by (1.6), $\|\boldsymbol{\tau}(t)\|_p \leq \|\boldsymbol{\tau}(t)\|_1^\theta \|\boldsymbol{\tau}(t)\|_2^{1-\theta}$, and so using (1.11) with $\delta = 0$,

$$\|\boldsymbol{v}(t)\|_{\dot{H}_p^1} \leq C \|\boldsymbol{\tau}(t)\|_p \leq C \|\boldsymbol{\tau}(t)\|_1^\theta \|\boldsymbol{\tau}(t)\|_2^{1-\theta} \leq C \|\boldsymbol{\tau}(t)\|_1^\theta \|\boldsymbol{v}(t)\|_{\dot{H}_2^1}^{1-\theta}.$$

Again by (3.13), $\exists\, t_0$ such that $\|\boldsymbol{v}(t_0)\|_{\dot{H}^1_2} < \left(\epsilon / C \|\boldsymbol{\tau}(t_0)\|_1^{\theta}\right)^{\frac{1}{1-\theta}}$, so

$$\|\boldsymbol{v}(t_0)\|_{\dot{H}^1_p} \leq C\|\boldsymbol{\tau}(t_0)\|_p \leq C\|\boldsymbol{\tau}(t_0)\|_1^{\theta}\|\boldsymbol{v}(t_0)\|_{\dot{H}^1_2}^{1-\theta} < \epsilon, \qquad p \in (1,2). \tag{3.15}$$

Choosing $p = 3/2$ (so $\theta = 1/6$), the critical $\alpha_1 = \frac{3}{3/2} - 1 = 1$, and we have $\|\boldsymbol{v}(t_0)\|_{\dot{H}^{\alpha_1}_{3/2}} < \epsilon$, so we can apply the small data result of Theorem 3.2.4 and proceeding as before, we get

$$\|\boldsymbol{v}(t)\|_{\dot{H}^{\alpha}_p} = \|\Lambda^{\alpha}\boldsymbol{v}(t)\|_p \leq C\|\boldsymbol{v}(t)\|_{\dot{H}^{\alpha+\delta}_{3/2}} \leq C\|\boldsymbol{v}(t_0)\|_{\dot{H}^{\alpha_1}_{3/2}}(t-t_0)^{-\frac{1}{2}(\alpha+\delta-\alpha_1)},$$

for all $\frac{3}{2} \leq p < 2$ and $\alpha > \alpha_1 - \delta$, with $\delta = 2 - \frac{3}{p}$.

Lastly, we need to consider the case where $1 < p < \frac{3}{2}$.

By (3.14) and (1.11) with $q = 2$,

$$\lim_{t\to\infty} \|\boldsymbol{\tau}(t)\|_{\dot{H}^{\alpha}_2} \leq \lim_{t\to\infty} \|\boldsymbol{v}(t)\|_{\dot{H}^{\alpha+1}_2} = 0, \qquad \forall\, \alpha \geq 0.$$

By (1.10),

$$\|\boldsymbol{\tau}\|_{\dot{H}^{(\theta\alpha)}_p} \leq C\|\boldsymbol{\tau}\|_{\dot{H}^{\alpha}_2}^{\theta}\|\boldsymbol{\tau}\|_q^{1-\theta},$$

where $\theta := \frac{2}{p}\frac{p-q}{2-q}$ and $1 < q < 2$ is chosen so that $0 < \theta < 1$.

Then using (3.15)

$$\liminf_{t\to\infty} \|\boldsymbol{\tau}(t)\|_{\dot{H}^{(\alpha\theta)}_p} = 0,$$

and therefore since $\alpha_c := \frac{3}{p} - 1 > 1$ for $1 < p < \frac{3}{2}$, we have

$$\liminf_{t\to\infty} \|\boldsymbol{v}(t)\|_{\dot{H}^{\alpha_c}_p} = 0,$$

by choosing $\alpha \geq 0$ so that $\theta\alpha + 1 = \alpha_c$. The rest follows as before, using Theorem 3.2.4. $\square$

3.4 Conclusion and Future Work

Although we know that solutions with initial data in $\dot{H}_p^{\alpha}$ will immediately have Gevrey regularity for a short period of time (called an existence time, T), and if the initial data is in L^2 then the solutions will eventually become Gevrey regular at some time t_0, this does not preclude the possibility of a solution blowing-up in the interim (i.e. between T and t_0.)

We established some estimates for the existence time T in Theorem 1.1 when the initial data is in a non-critical homogeneous Sobolev space, in terms of the Gevrey norm, and these existence times are identical to those obtained by Biswas and Bae for the NSE. However, Robinson obtains better existence times in terms of the homogeneous Sobolev norms when $\alpha > \frac{3}{2}$ and $p = 2$ for the NSE [114]. Can the same be obtained for the MHD equations? Also, can the same improved rates be obtained in terms of the Gevrey norm, even for the NSE?

In [90] it was shown that solutions of the MHD equations are time analytic under periodic boundary conditions when the initial data is sufficiently regular (H_2^1). Can we show solutions are time analytic with values in a Gevrey class of functions when the space domain is all of $\mathbb{R}^d$ *and* for rougher initial data?

When considering the equations with a nonzero forcing term the solutions will not in general decay to 0. In this case one can ask if there is an attractor and the question has been pursued for the bounded domain case or for periodic boundary conditions. Also, the size of the attractor (size in the sense of the Hausdorff dimension) when it exists has been studied. What can we say about the attractor for the MHD equations in the whole space when there is a nonzero forcing term?

Chapter 4: Continuous data assimilation for the 2D magnetohydrodynamic equations using one component of the velocity and magnetic fields

4.1 Introduction

In the study of solar storms, space weather forecasting, earth's geodynamo, and other areas, predicting the motion of fluids with magnetic properties is a central concern. The governing equations are often taken to be the magnetohydrodynamic (MHD) equations, or some modification of them. These equations are notoriously difficult to solve both analytically and computationally. Moreover, accurately initializing the system is challenging due to the sparsity of the available data. Fortunately, data is often given not just at a single time, but can be streaming in (e.g., from devices monitoring space plasma dynamics), or given in history (e.g., from surface geomagnetic observations, which in the earth can be traced back up to 7000 years [25, 38, 115]). This situation is similar to the problem of weather prediction on earth. Therefore the techniques of data assimilation, which were developed in weather prediction, have been applied to the MHD equations in recent years (see, e.g., [29, 34, 71, 72, 77, 102, 107, 117, 121, 122]). It has also been speculated in [1] that data assimilation for magnetohydrodynamics may be useful in liquid sodium experiments modeling the Earth's core.

Data assimilation has been the subject of a very large body of work. Classically, these techniques are based on linear quadratic estimation, also known as the Kalman Filter. The Kalman Filter has the drawback of assuming that the underlying system and any corresponding observation models are linear. It also assumes that measurement noise is Gaussian distributed. This has been somewhat generalized via modifications, such as the Extended Kalman Filter and the Unscented Kalman Filter. For more about the Kalman Filter and its modifications, see, e.g., [45, 89, 100], and the references therein. Recently, a promising new approach to data assimilation was pioneered by Azouani, Olson, and Titi in [6, 7] (see also [33, 81, 111] for early ideas in this direction). This new approach is based on feedback control at the PDE level. The first works in this area assumed noise-free observations, but [16] adapted the method to the case of noisy data, and [66] adapted it to the case where measurements are obtained discretely in time and may be contaminated by systematic errors. Computational experiments on this technique were carried out in the cases of the 2D Navier-Stokes equations [75], the 2D Bénard convection equations [5], and the 1D Kuramoto-Sivashinsky equations [99, 104]. In [99], several nonlinear versions of this approach were proposed and studied. In addition to the results discussed here, a large amount of recent literature has built upon this idea; see, e.g., [3, 22, 54, 56–59, 73, 76, 84, 85, 98, 105, 109].

In the present work, we adapt the approach of [6, 7, 56] to the 2D MHD equations. In Theorem 4.3.1, we show that solutions of the feedback-controlled system converge exponentially in the L^2-norm to solutions of the MHD system when feedback control is applied to all variables (here, we use Elsässer variables for simplicity). This convergence holds under certain conditions on the spacing of the data and the weight given to the feedback control. Moreover, in Theorems 4.3.2 and 4.3.3, we establish *abridged data assimilation*, i.e., we show that feedback control need be applied to only a reduced set of the variables (horizontal variables or a single Elsässer variable, respectively) to obtain exponential convergence, at the cost of more restrictive conditions on the data resolution h and control weight μ. In Theorem 4.3.4, we establish exponential convergence in the H^1-norm. Next, in Theorem 4.3.8, we show that if one makes weaker assumptions on the data interpolation function, and if feedback control is applied only to horizontal variables, then exponential convergence in the H^1 norm holds as well. Finally, in Section 4.3.3, we establish a rigorous connection between data assimilation and the concept of determining quantities, first introduced in [63], and further studied in [37, 67, 86–88].

4.1.1 Background on Data Assimilation

We now describe the general idea of the data assimilation scheme we use for the 2D MHD equations, based on the idea of feedback control, that was developed by Azouni, Olson and Titi in [6,7] in the context of the 2D Navier-Stokes equations. In the study of a dynamical system in the form,

$$\frac{d}{dt}Y = F(Y), \tag{4.1}$$

subject to certain boundary conditions, one normally tries to show that unique solutions will arise given any initial value

$$Y(0) = Y_0,$$

in a certain space, and that the solution will change in a continuous way with respect to a change in the initial value.

The problem arises in practice that the initial value may not be known exactly, but it may approximate the true initial value of a given observable, for example the temperature, which we would like to predict the value of in the future. The continuous dependence on initial data addresses this issue, in that if the initial approximation is close enough to the true value, then the solution we obtain will accurately approximate the true value of the observable for some period of time. However, usually the length of time the approximation is guaranteed to be good is short, in that the error may grow exponentially in time. Also, the initial measurement may need to give a very close approximation to the true initial value, but in practice measurements may only be available on a coarse grid, limiting the accuracy of the initial approximation and thus limiting both the accuracy the solution can be guaranteed to have, as well as the duration for which this accuracy can be guaranteed.

Data assimilation is the method where, to compensate for this limit to the accuracy of the measured initial condition, measurements are taken of the observable as time goes on (over the same possibly coarse grid on which the initial value is approximated) and fed back into the differential equation (giving a different equation, called the data assimilation equation) in such a way that the solution will become a better approximation as time goes on. This gives us the accuracy we need to apply the continuous dependence on initial data and say the prediction will be accurate for some duration from that time onwards.

The data assimilation algorithm (the way measurements are introduced to the differential equation) can take different forms, but the one we consider here was first introduced by Azouani, Olson, and Titi in [6, 7]. Given that the true value of the observable at time t is $Y(t)$, then the data assimilation equation will be:

$$\begin{aligned} \frac{d}{dt}\tilde{Y} &= F(\tilde{Y}) + \mu(\mathrm{I}_h(Y) - \mathrm{I}_h(\tilde{Y})) \\ &= F(\tilde{Y}) + \mu\,\mathrm{I}_h(Y - \tilde{Y}), \end{aligned} \tag{4.2}$$

where the second equality in the above equation follows because we assume the interpolant operator, I_h, is linear. Here, μ will be an adequately chosen tuning parameter. In addition, we will assume that for all $u \in H^1$, I_h satisfies one of the following:

$$\|u - \mathrm{I}_h(u)\|_{L^2} \leq c_1 h \|\nabla u\|_{L^2}, \tag{4.3}$$

or

$$\|u - \mathrm{I}_h(u)\|_{L^2} \leq c_2 h \|\nabla u\|_{L^2} + c_3 h^2 \|\Delta u\|_{L^2}. \tag{4.4}$$

Many relevant examples of operators satisfy one of these two conditions, including the projection onto the low modes, finite volume element operators, and nodal interpolant operators. For more information, see, e.g. [6, 70, 104].

4.2 Data assimilation algorithms for the MHD

Now, we describe the data assimilation algorithms we will study. Following the ideas of [6, 7] we incorporate measurements obtained from a fixed reference solution (of which we want to predict future values) through a damping term. This will "steer" the data assimilation solutions to the reference solution exponentially in time. In what sense we will have convergence depends on the type of interpolant I_h with which we take measurements.

The results are separated by the type of interpolant considered and by which measurements are recorded. We assume that for a given reference solution, $(\boldsymbol{u}, \boldsymbol{b})$, of (3.2), we have data being collected on some subset of the fields $\{u_1, u_2, b_1, b_2\}$. A feedback control term could be introduced into the evolution equation of any variable on which we are collecting data, and so we can consider a different algorithm for each combination of the variables we assume to be measuring. We consider algorithms which require measurements taken only on the first components, u_1 and b_1 (which is the same as measuring v_1 and w_1), by measuring all the components of $\boldsymbol{u}$ and $\boldsymbol{b}$, or by measuring either the sum $\boldsymbol{u} + \boldsymbol{b}$ or the difference $\boldsymbol{u} - \boldsymbol{b}$ only. We frame our results in terms of the Elsässer variables, not in terms of $\boldsymbol{u}$ and $\boldsymbol{b}$.

Remark 4.2.1 *As the pressure field, p, does not have an evolution equation, we cannot directly make use of any data collected on p with an equation like (4.2). Therefore, we do not consider taking pressure measurements.*

In the following, let $(\boldsymbol{v}, \boldsymbol{w})$ be a fixed solution of (3.3), and denote the data assimilation variables by $\boldsymbol{V}$ and $\boldsymbol{W}$, which will approximate $\boldsymbol{v}$ and $\boldsymbol{w}$ respectively. I_h may satisfy either (4.3) or (4.4); we will analyze each case separately. Because we are introducing the feedback term into the equations, the magnetic field will no longer be divergence free (in general). Therefore, to explicitly enforce the divergence free conditions on the data assimilation variables without making the systems overdetermined, we also introduce a potential field, ∇q.

As for the original system, in the following systems we consider periodic boundary conditions, and assume that $\tilde{\mathcal{P}}$ and q have zero space average. First, we have the following algorithm which utilizes measurements taken on all components (so measuring $\boldsymbol{u}$ and $\boldsymbol{b}$):

Algorithm 4.2.2 *Solve*

$$\partial_t \boldsymbol{V} - \alpha\Delta\boldsymbol{V} - \beta\Delta\boldsymbol{W} + (\boldsymbol{W}\cdot\nabla)\,\boldsymbol{V} = -\nabla\tilde{\mathcal{P}} - \nabla q + \boldsymbol{f} + \mu\,\mathrm{I}_h(\boldsymbol{v} - \boldsymbol{V}) \tag{4.5a}$$

$$\partial_t \boldsymbol{W} - \alpha\Delta\boldsymbol{W} - \beta\Delta\boldsymbol{V} + (\boldsymbol{V}\cdot\nabla)\,\boldsymbol{W} = -\nabla\tilde{\mathcal{P}} + \nabla q + \boldsymbol{g} + \mu\,\mathrm{I}_h(\boldsymbol{w} - \boldsymbol{W}) \tag{4.5b}$$

$$\nabla\cdot\boldsymbol{V} = 0, \quad \nabla\cdot\boldsymbol{W} = 0, \tag{4.5c}$$

for $(\boldsymbol{V},\boldsymbol{W})$ *with the initial conditions* $\boldsymbol{V}(0)\equiv\boldsymbol{W}(0)\equiv 0$.

We would also like to consider cases where we do not need to collect data on all of the variables. In the next algorithm, we only require data to be collected on the horizontal components of $\boldsymbol{v}$ and $\boldsymbol{w}$ (which is equivalent to measuring u_1 and b_1):

Algorithm 4.2.3 *Solve*

$$\partial_t \boldsymbol{V} - \alpha\Delta \boldsymbol{V} - \beta\Delta \boldsymbol{W} + (\boldsymbol{W}\cdot\nabla)\,\boldsymbol{V} = -\nabla\tilde{\mathcal{P}} - \nabla q + \boldsymbol{f} + \mu\,\mathrm{I}_h(v_1 - V_1)\boldsymbol{e}_1 \tag{4.6a}$$

$$\partial_t \boldsymbol{W} - \alpha\Delta \boldsymbol{W} - \beta\Delta \boldsymbol{V} + (\boldsymbol{V}\cdot\nabla)\,\boldsymbol{W} = -\nabla\tilde{\mathcal{P}} + \nabla q + \boldsymbol{g} + \mu\,\mathrm{I}_h(w_1 - W_1)\boldsymbol{e}_1 \tag{4.6b}$$

$$\nabla\cdot\boldsymbol{V} = 0, \quad \nabla\cdot\boldsymbol{W} = 0, \tag{4.6c}$$

for $(\boldsymbol{V},\boldsymbol{W})$ *with the initial condition* $\boldsymbol{V}(0)\equiv\boldsymbol{W}(0)\equiv 0$.

Finally, taking measurements on only $\boldsymbol{v}$:

Algorithm 4.2.4 *Solve*

$$\partial_t \boldsymbol{V} - \alpha\Delta \boldsymbol{V} - \beta\Delta \boldsymbol{W} + (\boldsymbol{W}\cdot\nabla)\,\boldsymbol{V} = -\nabla\tilde{\mathcal{P}} - \nabla q + \boldsymbol{f} + \mu\,\mathrm{I}_h(\boldsymbol{v} - \boldsymbol{V}) \tag{4.7a}$$

$$\partial_t \boldsymbol{W} - \alpha\Delta \boldsymbol{W} - \beta\Delta \boldsymbol{V} + (\boldsymbol{V}\cdot\nabla)\,\boldsymbol{W} = -\nabla\tilde{\mathcal{P}} + \nabla q + \boldsymbol{g} \tag{4.7b}$$

$$\nabla\cdot\boldsymbol{V} = 0, \quad \nabla\cdot\boldsymbol{W} = 0, \tag{4.7c}$$

for $(\boldsymbol{V},\boldsymbol{W})$ *with the initial condition* $\boldsymbol{V}(0)\equiv\boldsymbol{W}(0)\equiv 0$.

Remark 4.2.5 *Although we chose to consider taking measurements on the first components of* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *in Algorithm 4.2.3, we could instead use the second components with no substantial differences. Likewise, in Algorithm 4.2.4 we could also consider taking measurements on* $\boldsymbol{w}$ *and we would obtain similar results.*

Remark 4.2.6 *In the above we chose to make the initial conditions* 0*, but in fact the initial conditions may be chosen essentially arbitrarily, albeit in accordance with the existence theorems. Theorem 4.3.8 additionally requires that the initial conditions satisfy an upper bound of the form* (4.9).

Remark 4.2.7 *Here we first constructed the Elsässer variables from the original variables* $\boldsymbol{u}$ *and* $\boldsymbol{b}$ *after nondimensionalizing, and then proceeded to define the various data assimilation algorithms and variables. However, since the transformations were linear, if we were to define each data assimilation algorithm using the original variables, in the process defining data assimilation variables* $\boldsymbol{U}$ *and* $\boldsymbol{B}$*, and then nondimensionalize and change to the Elsässer variables, we would arrive at the same systems above. So, all our results apply to the corresponding algorithms formulated in terms of the original variables (we give these explicitly in Chapter 5).*

We define weak solutions for all the systems mentioned in the distributional sense in the usual way. See [119] for a precise definition in the case of (3.2) (the other systems are similar). In addition to being a weak solution, we say $(\boldsymbol{v}, \boldsymbol{w})$ (or $(\boldsymbol{V}, \boldsymbol{W})$) is a global strong solution of (3.3) (or (4.5), (4.6), or (4.7)) if

$$\boldsymbol{v}, \boldsymbol{w} \in L^2(0, T; H^2) \cap L^\infty(0, T; H^1), \quad \forall T > 0.$$

In [119], it was shown that if $\operatorname{ess\,sup}_{[0,\infty)} \|\boldsymbol{f}\|_{L^2} < \infty$ and $\boldsymbol{u}_0, \boldsymbol{b}_0 \in H^1$, then there exists a unique global strong solution to (3.2) (which can be transformed to a solution of (3.3)). Therefore, we will be assuming that, in addition to being space periodic and divergence free,

$$\operatorname{ess\,sup}_{[0,\infty)} \max\{\|\boldsymbol{f}\|_{L^2}, \|\boldsymbol{g}\|_{L^2}\} < \infty \quad \text{and} \quad \|\nabla \boldsymbol{u}_0\|_{L^2}, \|\nabla \boldsymbol{b}_0\|_{L^2} < \infty.$$

The proofs of the corresponding existence and uniqueness results for solutions of the systems in Algorithms 4.2.2-4.2.4 are similar, and are omitted. We only state and prove the corresponding convergence results.

Our analyses will have to take into account the amount of energy being added to the system by the forcing functions, so to this end we define the Grashof number, G, to be

$$G := \frac{8}{\lambda_1} \max\{\frac{1}{\nu^2}, \frac{1}{\lambda^2}\} \limsup_{t\to\infty} \left(\max\left\{\|\boldsymbol{f}(t)\|_{L^2([0,L]^2)}, \quad \frac{1}{\sqrt{\rho_0\mu_0}}\|\boldsymbol{g}(t)\|_{L^2([0,L]^2)}\right\}\right).$$

where $\lambda_1 := \frac{4\pi^2}{L^2}$ is the smallest eigenvalue of the Stokes operator on the space of functions with space average zero on $[0, L]^2$ under periodic boundary conditions [64].

We note here that G can be expressed in terms of the forcing functions and parameters of the Elsässer variable formulation:

$$G = \frac{\max\{Re^2, Rm^2\}}{\pi^2} \limsup_{t\to\infty} \left(\max\{\|\boldsymbol{f}(t) + \boldsymbol{g}(t)\|_{L^2}, \|\boldsymbol{f}(t) - \boldsymbol{g}(t)\|_{L^2}\}\right),$$

hence,

$$G \geq \frac{1}{\pi^2(\alpha-\beta)^2} \limsup_{t\to\infty} \left(\max\{\|\boldsymbol{f}(t)\|_{L^2}, \|\boldsymbol{g}(t)\|_{L^2}\}\right),$$

Before we get to the main theorems, we first state the following bounds for the reference solution to the MHD system. Moreover, we prove (4.8), which follows standard arguments from the Navier-Stokes theory (see, e.g., [40, 123]). The proofs of (4.9) and (4.10) can be obtained by modifying the corresponding proofs from the Navier-Stokes theory in a similar way (see, e.g. [51, 119] for more details on (4.9) and the appendix of [56] for (4.10)).

Proposition 4.2.8 (Upper Bounds on Solutions of the MHD) *Let $(\boldsymbol{v}, \boldsymbol{w})$ be a solution of (3.3). Then there is a $t_0 > 0$ and constants $c_M > 0$ and $C = \frac{81}{4}c_L^8$ such that for all $t \geq t_0$ and any $T > 0$,*

$$\int_t^{t+T} \left(\|\nabla \boldsymbol{v}(s)\|_{L^2}^2 + \|\nabla \boldsymbol{w}(s)\|_{L^2}^2\right) ds \leq (1 + T\pi^2(\alpha - \beta))(\alpha - \beta)G^2, \tag{4.8}$$

$$\|\nabla \boldsymbol{v}(t)\|_{L^2}^2 + \|\nabla \boldsymbol{w}(t)\|_{L^2}^2 \leq 10\pi^2(\alpha-\beta)^2 G^2 e^{CG^4}, \tag{4.9}$$

$$\|\Delta \boldsymbol{v}(t)\|_{L^2}^2 + \|\Delta \boldsymbol{w}(t)\|_{L^2}^2 \leq$$
$$c_M(\alpha-\beta)^2 G^2 \left(1 + \left(1 + G^2 e^{CG^4}\right)\left(1 + e^{CG^4} + G^4 e^{CG^4}\right)\right). \tag{4.10}$$

Proof of (4.8) See the appendix. □

4.3 Statements of the Results

4.3.1 Results for Type 1 Interpolants

Theorem 4.3.1 *Let* $(\boldsymbol{v}, \boldsymbol{w})$ *be a strong solution of* (3.3) *which at time* $t = 0$ *has evolved enough so that Proposition 4.2.8 holds with* $t_0 = 0$*. Let* I_h *satisfy* (4.3)*, where*

$$h \leq c_1^{-1}(\alpha-\beta)^{\frac{1}{2}}\mu^{-\frac{1}{2}}, \text{ and } \quad \mu > \tfrac{\pi^2(c_L^4+(\alpha-\beta)^4)}{\alpha-\beta} G^2$$

(so $h \sim G^{-1}$*). Then there is a unique strong solution,* $(\boldsymbol{V}, \boldsymbol{W})$*, of* (4.5) *corresponding to* $(\boldsymbol{v}, \boldsymbol{w})$ *which exists globally in time, and furthermore* $\|\boldsymbol{v}(t) - \boldsymbol{V}(t)\|_{L^2} + \|\boldsymbol{w}(t) - \boldsymbol{W}(t)\|_{L^2} \to 0$ *exponentially as* $t \to \infty$*.*

Theorem 4.3.2 *Let* $(\boldsymbol{v}, \boldsymbol{w})$ *be a strong solution of* (3.3) *which at time* $t = 0$ *has evolved enough so that Proposition 4.2.8 holds with* $t_0 = 0$*. Let* I_h *satisfy* (4.3)*, where*

$$h \leq c_1^{-1}(\alpha-\beta)^{\frac{1}{2}}\mu^{-\frac{1}{2}}, \text{ and } \quad \mu > 32\pi^2 c^2(\alpha-\beta)\left(\tilde{c} + 2\ln G + CG^4\right) G^2$$

(so $h \sim G^{-3}$*). Then there is a unique strong solution,* $(\boldsymbol{V}, \boldsymbol{W})$*, of* (4.6) *corresponding to* $(\boldsymbol{v}, \boldsymbol{w})$ *which exists globally in time, and furthermore* $\|\boldsymbol{v}(t) - \boldsymbol{V}(t)\|_{L^2} + \|\boldsymbol{w}(t) - \boldsymbol{W}(t)\|_{L^2} \to 0$ *exponentially as* $t \to \infty$*.*

Theorem 4.3.3 *Let $(\boldsymbol{v}, \boldsymbol{w})$ be a strong solution of (3.3) which at time $t = 0$ has evolved enough so that Proposition 4.2.8 holds with $t_0 = 0$. Let I_h satisfy (4.3), where*

$$h \leq c_1^{-1}(\alpha - \beta)^{\frac{1}{2}}\mu^{-\frac{1}{2}}, \text{ and } \quad \mu > \frac{\pi^2 c_L^4 G^2 (4 + (\alpha-\beta)^2 G^2)^2}{16(\alpha - \beta)}$$

(so $h \sim G^{-3}$). Then there is a unique strong solution, $(\boldsymbol{V}, \boldsymbol{W})$, of (4.7) corresponding to $(\boldsymbol{v}, \boldsymbol{w})$ which exists globally in time, and furthermore $\|\boldsymbol{v}(t) - \boldsymbol{V}(t)\|_{L^2} + \|\boldsymbol{w}(t) - \boldsymbol{W}(t)\|_{L^2} \to 0$ exponentially as $t \to \infty$.

In the next three theorems, by using the L^2 convergence results we just established, we show that solutions of (4.5), (4.6), and (4.7) will converge exponentially in time to the reference solution in the stronger topology of the H^1-norm.

Theorem 4.3.4 *Let $(\boldsymbol{v}, \boldsymbol{w})$ be a strong solution of (3.3) which at time $t = 0$ has evolved enough so that Proposition 4.2.8 holds with $t_0 = 0$. Let I_h satisfy (4.3), where*

$$h < (2\sqrt{2}c_1)^{-1}(\alpha - \beta)^{\frac{1}{2}}\mu^{-\frac{1}{2}}, \text{ and } \quad \mu > \frac{\pi^2(c_L^4 + (\alpha-\beta)^4)}{\alpha - \beta}G^2$$

(so $h \sim G^{-1}$). Then there is a unique strong solution, $(\boldsymbol{V}, \boldsymbol{W})$, of (4.5) corresponding to $(\boldsymbol{v}, \boldsymbol{w})$ which exists globally in time, and furthermore $\|\boldsymbol{v}(t) - \boldsymbol{V}(t)\|_{H^1} + \|\boldsymbol{w}(t) - \boldsymbol{W}(t)\|_{H^1} \to 0$ exponentially as $t \to \infty$.

Theorem 4.3.5 *Let $(\boldsymbol{v}, \boldsymbol{w})$ be a strong solution of (3.3) which at time $t = 0$ has evolved enough so that Proposition 4.2.8 holds with $t_0 = 0$. Let I_h satisfy (4.3), where*

$$h < (2\sqrt{2}c_1)^{-1}(\alpha - \beta)^{\frac{1}{2}}\mu^{-\frac{1}{2}}, \text{ and } \quad \mu > 32\pi^2 c^2 (\alpha - \beta)\left(\tilde{c} + 2\ln G + CG^4\right)G^2$$

(so $h \sim G^{-3}$). Then there is a unique strong solution, $(\boldsymbol{V}, \boldsymbol{W})$, of (4.6) corresponding to $(\boldsymbol{v}, \boldsymbol{w})$ which exists globally in time, and furthermore $\|\boldsymbol{v}(t) - \boldsymbol{V}(t)\|_{H^1} + \|\boldsymbol{w}(t) - \boldsymbol{W}(t)\|_{H^1} \to 0$ exponentially as $t \to \infty$.

Theorem 4.3.6 *Let $(\boldsymbol{v}, \boldsymbol{w})$ be a strong solution of* (3.3) *which at time $t = 0$ has evolved enough so that Proposition 4.2.8 holds with $t_0 = 0$. Let I_h satisfy* (4.3)*, where*

$$h < (2\sqrt{2}c_1)^{-1}(\alpha-\beta)^{\frac{1}{2}}\mu^{-\frac{1}{2}}, \text{ and } \quad \mu > \frac{\pi^2 c_L^4 G^2(4+(\alpha-\beta)^2G^2)^2}{16(\alpha-\beta)}$$

(so $h \sim G^{-3}$). Then there is a unique strong solution, $(\boldsymbol{V}, \boldsymbol{W})$, of (4.7) *corresponding to $(\boldsymbol{v}, \boldsymbol{w})$ which exists globally in time, and furthermore $\|\boldsymbol{v}(t) - \boldsymbol{V}(t)\|_{H^1} + \|\boldsymbol{w}(t) - \boldsymbol{W}(t)\|_{H^1} \to 0$ exponentially as $t \to \infty$.*

Remark 4.3.7 *Observing the Poincaré inequality, the results of Theorems 4.3.4-4.3.6 seem to imply those of Theorems 4.3.1-4.3.3, but the spatial resolution is required to be slightly finer for the H^1 results. Also, based on our analysis, there may be a longer period of time that must pass before exponential convergence is observed in the H^1-norm than in the L^2-norm (see the estimates in* (4.37) *and* (4.40)*). However, we point out that in computational results regarding data assimilation in the context of the one-dimensional Kuramoto-Sivasinsky equation, convergence times for both norms are almost identical (see [99] for more details).*

4.3.2 Results for Type 2 Interpolants

Theorem 4.3.8 *Let $(\boldsymbol{v}, \boldsymbol{w})$ be a strong solution of* (3.3)*, which at time $t = 0$ has evolved enough so that Proposition 4.2.8 holds with $t_0 = 0$. Then $h \sim G^{-6}e^{-CG^4}$ and $\mu \sim G^{12}e^{2CG^4}$ can be chosen so that if I_h satisfies* (4.4) *then there is a unique strong solution $(\boldsymbol{V}, \boldsymbol{W})$ of* (4.6) *corresponding to $(\boldsymbol{v}, \boldsymbol{w})$ which exists globally in time, and $\|\boldsymbol{v}(t) - \boldsymbol{V}(t)\|_{H^1} + \|\boldsymbol{w}(t) - \boldsymbol{W}(t)\|_{H^1} \to 0$ exponentially as $t \to \infty$.*

Remark 4.3.9 *Similar theorems hold for the cases of measurements on all variables and one Elsässer variable (although not as direct corollaries, since the dynamical systems involved are slightly different). However, in the case of measuring all variables we do not find much improvement in the restrictions on h and μ.*

4.3.3 Determining Interpolants

In order to prove that there are finitely many (say N) determining modes, one needs to show that if $(\boldsymbol{v}^{(1)}, \boldsymbol{w}^{(1)})$ and $(\boldsymbol{v}^{(2)}, \boldsymbol{w}^{(2)})$ are different solutions of (3.3) with possibly different forcing terms and initial data, then knowing that $\|P_N(\boldsymbol{v}^{(1)}, \boldsymbol{w}^{(1)}) - P_N(\boldsymbol{v}^{(2)}, \boldsymbol{w}^{(2)})\|_{L^2} \to 0$ is sufficient to conclude that $\|(\boldsymbol{v}^{(1)}, \boldsymbol{w}^{(1)}) - (\boldsymbol{v}^{(2)}, \boldsymbol{w}^{(2)})\|_{L^2} \to 0$, where P_N denotes the projection onto the modes with magnitude at most N. In general, we replace P_N by a different operator, say I_h, and ask the question of whether the knowledge inherent in I_h is "determining".

In the following theorems, we show that the data assimilation results we have obtained can be adapted to show that the interpolant operators, I_h, are determining. We do this by first generalizing the convergence results we developed in the previous theorems to allow for the evolution equations of the reference solution and the data assimilation solution to have different forcing terms, which converge in L^2 as $t \to \infty$, at the cost of losing the exponential rate of convergence of the solutions. We also allow for the reference solution to be perturbed by a function which decays in L^2.

We illustrate the ideas for the algorithm studied in Theorem 4.3.1, i.e. with measurements taken on all variables and for I_h satisfying (4.3), but the results can be obtained for all the other cases as well. So, we can show that operators which satisfy (4.3) or (4.4) and use measurements on $(\boldsymbol{v}, \boldsymbol{w})$, (v_1, w_1), or $\boldsymbol{v}$, are determining in the sense of convergence in L^2 and H^1.

Theorem 4.3.10 *Let I_h satisfy (4.3) and let $(\boldsymbol{v}, \boldsymbol{w})$ be a reference solution of (3.3). Then if μ and h satisfy the hypotheses of Theorem 4.3.1, and if $\|\delta^{(1)}(t)\|_{L^2}, \|\delta^{(2)}(t)\|_{L^2} \to 0$ and $\|\mathrm{I}_h(\epsilon^{(1)}(t))\|_{L^2}, \|\mathrm{I}_h(\epsilon^{(2)}(t))\|_{L^2} \to 0$ as $t \to \infty$, there are unique $\boldsymbol{V}, \boldsymbol{W}$, q and $\tilde{\mathcal{p}}$ which satisfy the following modified version of (4.5):*

System 4.3.11

$$\partial_t \boldsymbol{V} - \alpha\Delta\boldsymbol{V} + \beta\Delta\boldsymbol{W} + (\boldsymbol{W} \cdot \nabla)\,\boldsymbol{V} = -\nabla\tilde{\mathcal{p}} - \nabla q + \boldsymbol{f} + \delta^{(1)} + \mu\,\mathrm{I}_h(\boldsymbol{v} + \epsilon^{(1)} - \boldsymbol{V}), \tag{4.11a}$$

$$\partial_t \boldsymbol{W} - \alpha\Delta\boldsymbol{W} + \beta\Delta\boldsymbol{V} + (\boldsymbol{V} \cdot \nabla)\,\boldsymbol{W} = -\nabla\tilde{\mathcal{p}} + \nabla q + \boldsymbol{g} + \delta^{(2)} + \mu\,\mathrm{I}_h(\boldsymbol{w} + \epsilon^{(2)} - \boldsymbol{W}), \tag{4.11b}$$

$$\nabla \cdot \boldsymbol{V} = 0,\ \nabla \cdot \boldsymbol{W} = 0, \tag{4.11c}$$

subject to the initial conditions $\boldsymbol{V}(0) \equiv 0, \boldsymbol{W}(0) \equiv 0$,

and furthermore, $\|\boldsymbol{v} - \boldsymbol{V}\|_{L^2}, \|\boldsymbol{w} - \boldsymbol{W}\|_{L^2} \to 0$ as $t \to \infty$.

In the next theorem we illustrate the result that if an interpolant I_h satisfies the conditions for the generalized data assimilation theorem, then I_h is determining, for the case of the generalized version of Theorem 4.3.1. Note that the projection onto the low modes, P_N, is an example of an interpolant operator I_h for which the theorem applies, provided that $h := \frac{1}{N} \lesssim G^{-1}$. Hence, the following theorem shows that there are finitely many determining modes for instance.

Theorem 4.3.12 *Let* $(\boldsymbol{v}^{(1)}, \boldsymbol{w}^{(1)})$ *and* $(\boldsymbol{v}^{(2)}, \boldsymbol{w}^{(2)})$ *be solutions of* (3.3) *with forcing terms* $\boldsymbol{f}^{(1)}, \boldsymbol{g}^{(1)}$ *and* $\boldsymbol{f}^{(2)}, \boldsymbol{g}^{(2)}$ *respectively, and suppose that* $\|\boldsymbol{f}^{(1)}(t) - \boldsymbol{f}^{(2)}(t)\|_{L^2} \to 0$ *and* $\|\boldsymbol{g}^{(1)}(t) - \boldsymbol{g}^{(2)}(t)\|_{L^2} \to 0$ *as* $t \to \infty$*. Let* I_h *satisfy* (4.3) *where*

$$
\begin{aligned}
h &< \frac{\alpha - \beta}{\pi c_1 \sqrt{c_L^4 + (\alpha-\beta)^4}} G^{-1}, \ \text{and} \\
G &:= \frac{1}{\pi^2 (\alpha-\beta)^2} \limsup_{t\to\infty} \left(\max\{\|\boldsymbol{f}^{(1)}(t)\|_{L^2}, \|\boldsymbol{g}^{(1)}(t)\|_{L^2}\} \right) \\
&= \frac{1}{\pi^2 (\alpha-\beta)^2} \limsup_{t\to\infty} \left(\max\{\|\boldsymbol{f}^{(2)}(t)\|_{L^2}, \|\boldsymbol{g}^{(2)}(t)\|_{L^2}\} \right),
\end{aligned}
$$

and suppose that $\|\mathrm{I}_h(\boldsymbol{v}^{(1)}(t) - \boldsymbol{v}^{(2)}(t))\|_{L^2}, \|\mathrm{I}_h(\boldsymbol{w}^{(1)}(t) - \boldsymbol{w}^{(2)}(t))\|_{L^2} \to 0$ *as* $t \to \infty$*.*

Then $\|\boldsymbol{v}^{(1)}(t) - \boldsymbol{v}^{(2)}(t)\|_{L^2}, \|\boldsymbol{w}^{(1)}(t) - \boldsymbol{w}^{(2)}(t)\|_{L^2} \to 0$ *as well.*

4.4 Proofs of the Results

4.4.1 Proofs of L^2 Convergence Results with Type 1 Interpolants

Before we get to the proofs of the main theorems, we first collect the various estimates needed for the bilinear term in the following lemma.

Lemma 4.4.1 *Let* $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in H^1$ *be divergence free. Then the following inequalities hold for any* $\epsilon, \delta > 0$:

$$
\begin{aligned}
(a) &\left| \int_\Omega (\boldsymbol{u} \cdot \nabla) \boldsymbol{v} \cdot \boldsymbol{w} \, dxdy \right| \\
&\le \frac{c_L \delta}{4} \|\nabla \boldsymbol{u}\|_{L^2}^2 + \frac{\epsilon}{2} \|\nabla \boldsymbol{w}\|_{L^2}^2 + \frac{c_L \delta}{4} \|\nabla \boldsymbol{v}\|_{L^2}^2 \|\boldsymbol{u}\|_{L^2}^2 + \frac{c_L^2}{8\epsilon\delta^2} \|\nabla \boldsymbol{v}\|_{L^2}^2 \|\boldsymbol{w}\|_{L^2}^2, \quad (4.12)
\end{aligned}
$$

or

$$\left|\int_\Omega (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{v}\cdot\boldsymbol{w}\,dxdy\right|$$
$$\leq \frac{c_L\delta}{4}\|\nabla\boldsymbol{w}\|_{L^2}^2 + \frac{\epsilon}{2}\|\nabla\boldsymbol{u}\|_{L^2}^2 + \frac{c_L\delta}{4}\|\nabla\boldsymbol{v}\|_{L^2}^2\|\boldsymbol{w}\|_{L^2}^2 + \frac{c_L^2}{8\epsilon\delta^2}\|\nabla\boldsymbol{v}\|_{L^2}^2\|\boldsymbol{u}\|_{L^2}^2, \quad (4.13)$$

(b) $\left|\int_\Omega (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{v}\cdot\boldsymbol{w}\,dxdy\right|$

$$\leq c\delta\|\nabla\boldsymbol{u}\|_{L^2}^2 + c\delta\|\nabla\boldsymbol{w}\|_{L^2}^2 + \frac{c}{\delta}\|\nabla\boldsymbol{v}\|_{L^2}^2\left(\|u_1\|_{L^2}^2 + \|w_1\|_{L^2}^2\right)$$
$$+ \frac{c}{\delta}\|\nabla\boldsymbol{v}\|_{L^2}^2\left(1+\ln\frac{\|\nabla u_1\|_{L^2}}{2\pi\|u_1\|_{L^2}}\right)\|u_1\|_{L^2}^2 + \frac{c}{\delta}\|\nabla\boldsymbol{v}\|_{L^2}^2\left(1+\ln\frac{\|\nabla w_1\|_{L^2}}{2\pi\|w_1\|_{L^2}}\right)\|w_1\|_{L^2}^2. \quad (4.14)$$

Proof. See the appendix. □

The following lemma will be used in our analyses of the algorithms using measurements on only the first components of the reference solutions, where we will need to make use of (1.13), (1.14), or (4.14). The proof is given in [61] (see page 371).

Lemma 4.4.2 *Let* $\varphi(r) = r - \gamma(1+\ln(r))$, *for some* $\gamma > 0$. *Then* $\forall r \geq 1$,

$$\varphi(r) \geq -\gamma\ln(\gamma).$$

Proof of Theorem 4.3.1 Let $\boldsymbol{\eta} = \boldsymbol{v} - \boldsymbol{V}$ and $\boldsymbol{\zeta} = \boldsymbol{w} - \boldsymbol{W}$. Then $\boldsymbol{\eta}$ satisfies:

$$\partial_t\boldsymbol{\eta} - \alpha\Delta\boldsymbol{\eta} - \beta\Delta\boldsymbol{\zeta} + (\boldsymbol{w}\cdot\nabla)\,\boldsymbol{v} - (\boldsymbol{W}\cdot\nabla)\,\boldsymbol{V} = -\nabla(\mathcal{P} - \tilde{\mathcal{P}} - q) - \mu\,\mathrm{I}_h(\boldsymbol{\eta}).$$

Using the fact that $(\boldsymbol{w}\cdot\nabla)\,\boldsymbol{v} - (\boldsymbol{W}\cdot\nabla)\,\boldsymbol{V} = (\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v} + (\boldsymbol{W}\cdot\nabla)\,\boldsymbol{\eta}$ we write:

$$\partial_t\boldsymbol{\eta} - \alpha\Delta\boldsymbol{\eta} - \beta\Delta\boldsymbol{\zeta} + (\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v} + (\boldsymbol{W}\cdot\nabla)\,\boldsymbol{\eta} = -\nabla(\mathcal{P} - \tilde{\mathcal{P}} - q) - \mu\,\mathrm{I}_h(\boldsymbol{\eta}).$$

Taking the inner product with $\boldsymbol{\eta}$ we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\eta}\|_{L^2}^2+\alpha\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\beta\left\langle\nabla\boldsymbol{\zeta},\nabla\boldsymbol{\eta}\right\rangle+\left\langle\left(\boldsymbol{\zeta}\cdot\nabla\right)\boldsymbol{v},\boldsymbol{\eta}\right\rangle=-\left\langle\nabla(\mathcal{p}-\tilde{\mathcal{p}}-q),\boldsymbol{\eta}\right\rangle-\mu\left\langle \mathrm{I}_h(\boldsymbol{\eta}),\boldsymbol{\eta}\right\rangle.$$

Now, by the divergence free condition,

$$-\left\langle\nabla(\mathcal{p}-\tilde{\mathcal{p}}-q),\boldsymbol{\eta}\right\rangle:=-\int_\Omega\nabla(\mathcal{p}-\tilde{\mathcal{p}}-q)\cdot\boldsymbol{\eta}\,dxdy=\int_\Omega(\mathcal{p}-\tilde{\mathcal{p}}-q)\cdot(\nabla\cdot\boldsymbol{\eta})\,dxdy=0.$$

By applying Cauchy-Schwarz inequality and (1.3),

$$|\beta\left\langle\nabla\boldsymbol{\zeta},\nabla\boldsymbol{\eta}\right\rangle|\leq\frac{\beta}{2}\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\frac{\beta}{2}\|\nabla\boldsymbol{\zeta}\|_{L^2}^2,$$

and by rewriting $\left\langle \mathrm{I}_h(\boldsymbol{\eta}),\boldsymbol{\eta}\right\rangle=\left\langle \mathrm{I}_h(\boldsymbol{\eta})-\boldsymbol{\eta},\boldsymbol{\eta}\right\rangle+\left\langle\boldsymbol{\eta},\boldsymbol{\eta}\right\rangle$, we have:

$$-\mu\left\langle \mathrm{I}_h(\boldsymbol{\eta}),\boldsymbol{\eta}\right\rangle=-\mu\left\langle \mathrm{I}_h(\boldsymbol{\eta})-\boldsymbol{\eta},\boldsymbol{\eta}\right\rangle-\mu\|\boldsymbol{\eta}\|_{L^2}^2.$$

Thus, we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\eta}\|_{L^2}^2+\left(\alpha-\frac{\beta}{2}\right)\|\nabla\boldsymbol{\eta}\|_{L^2}^2-\frac{\beta}{2}\|\nabla\boldsymbol{\zeta}\|_{L^2}^2+\mu\|\boldsymbol{\eta}\|_{L^2}^2\\
&\leq-\left\langle\left(\boldsymbol{\zeta}\cdot\nabla\right)\boldsymbol{v},\boldsymbol{\eta}\right\rangle-\mu\left\langle \mathrm{I}_h(\boldsymbol{\eta})-\boldsymbol{\eta},\boldsymbol{\eta}\right\rangle\\
&\leq\left|\left\langle\left(\boldsymbol{\zeta}\cdot\nabla\right)\boldsymbol{v},\boldsymbol{\eta}\right\rangle\right|+\mu\left|\left\langle \mathrm{I}_h(\boldsymbol{\eta})-\boldsymbol{\eta},\boldsymbol{\eta}\right\rangle\right|\\
&\leq\left|\left\langle\left(\boldsymbol{\zeta}\cdot\nabla\right)\boldsymbol{v},\boldsymbol{\eta}\right\rangle\right|+\mu\|\,\mathrm{I}_h(\boldsymbol{\eta})-\boldsymbol{\eta}\|_{L^2}\|\boldsymbol{\eta}\|_{L^2}\\
&\leq\left|\left\langle\left(\boldsymbol{\zeta}\cdot\nabla\right)\boldsymbol{v},\boldsymbol{\eta}\right\rangle\right|+\mu c_1h\|\nabla\boldsymbol{\eta}\|_{L^2}\|\boldsymbol{\eta}\|_{L^2}\\
&\leq\left|\left\langle\left(\boldsymbol{\zeta}\cdot\nabla\right)\boldsymbol{v},\boldsymbol{\eta}\right\rangle\right|+\frac{\mu c_1^2h^2}{2}\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\frac{\mu}{2}\|\boldsymbol{\eta}\|_{L^2}^2,
\end{aligned}$$

where in the last three lines we used Cauchy-Schwarz inequality, the definition of I_h, and Young's inequality. This leaves us with:

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\eta}\|_{L^2}^2+\left(\alpha-\frac{\beta}{2}\right)\|\nabla\boldsymbol{\eta}\|_{L^2}^2-\frac{\beta}{2}\|\nabla\boldsymbol{\zeta}\|_{L^2}^2+\frac{\mu}{2}\|\boldsymbol{\eta}\|_{L^2}^2\leq\left|\left\langle\left(\boldsymbol{\zeta}\cdot\nabla\right)\boldsymbol{v},\boldsymbol{\eta}\right\rangle\right|+\frac{\mu c_1^2h^2}{2}\|\nabla\boldsymbol{\eta}\|_{L^2}^2.$$

Proceeding the same way for $\boldsymbol{\zeta}$, we have the following equations:

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\eta}\|_{L^2}^2+\left(\alpha-\frac{\beta}{2}-\frac{\mu c_1^2h^2}{2}\right)\|\nabla\boldsymbol{\eta}\|_{L^2}^2-\frac{\beta}{2}\|\nabla\boldsymbol{\zeta}\|_{L^2}^2+\frac{\mu}{2}\|\boldsymbol{\eta}\|_{L^2}^2\leq\left|\int_\Omega\left(\boldsymbol{\zeta}\cdot\nabla\right)\boldsymbol{v}\cdot\boldsymbol{\eta}\,dxdy\right|, \tag{4.15}$$

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\zeta}\|_{L^2}^2+\left(\alpha-\frac{\beta}{2}-\frac{\mu c_1^2h^2}{2}\right)\|\nabla\boldsymbol{\zeta}\|_{L^2}^2-\frac{\beta}{2}\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\frac{\mu}{2}\|\boldsymbol{\zeta}\|_{L^2}^2\leq\left|\int_\Omega\left(\boldsymbol{\eta}\cdot\nabla\right)\boldsymbol{w}\cdot\boldsymbol{\zeta}\,dxdy\right|. \tag{4.16}$$

We estimate the integrals in these equations using (4.12), with $\epsilon=\frac{\alpha-\beta}{2}$ and $\delta=\frac{\alpha-\beta}{c_L}$, and obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\eta}\|_{L^2}^2&+\left(\alpha-\frac{\beta}{2}-\frac{\mu c_1^2h^2}{2}-\frac{\alpha-\beta}{4}\right)\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\left(-\frac{\beta}{2}-\frac{\alpha-\beta}{4}\right)\|\nabla\boldsymbol{\zeta}\|_{L^2}^2\\&+\left(\frac{\mu}{2}-\frac{c_L^4}{4(\alpha-\beta)^3}\|\nabla\boldsymbol{v}\|_{L^2}^2\right)\|\boldsymbol{\eta}\|_{L^2}^2+\left(-\frac{\alpha-\beta}{4}\|\nabla\boldsymbol{v}\|_{L^2}^2\right)\|\boldsymbol{\zeta}\|_{L^2}^2\leq 0,\end{aligned} \tag{4.17}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\zeta}\|_{L^2}^2&+\left(\alpha-\frac{\beta}{2}-\frac{\mu c_1^2h^2}{2}-\frac{\alpha-\beta}{4}\right)\|\nabla\boldsymbol{\zeta}\|_{L^2}^2+\left(-\frac{\beta}{2}-\frac{\alpha-\beta}{4}\right)\|\nabla\boldsymbol{\eta}\|_{L^2}^2\\&+\left(\frac{\mu}{2}-\frac{c_L^4}{4(\alpha-\beta)^3}\|\nabla\boldsymbol{w}\|_{L^2}^2\right)\|\boldsymbol{\zeta}\|_{L^2}^2+\left(-\frac{\alpha-\beta}{4}\|\nabla\boldsymbol{w}\|_{L^2}^2\right)\|\boldsymbol{\eta}\|_{L^2}^2\leq 0.\end{aligned} \tag{4.18}$$

Then, adding (4.17) and (4.18), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\eta}\|_{L^2}^2&+\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\zeta}\|_{L^2}^2+\left(\frac{\alpha-\beta}{2}-\frac{\mu c_1^2h^2}{2}\right)\left(\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\|\nabla\boldsymbol{\zeta}\|_{L^2}^2\right)\\&+\left[\frac{\mu}{2}-\left(\frac{c_L^4}{4(\alpha-\beta)^3}+\frac{\alpha-\beta}{4}\right)\left(\|\nabla\boldsymbol{v}\|_{L^2}^2+\|\nabla\boldsymbol{w}\|_{L^2}^2\right)\right]\left(\|\boldsymbol{\eta}\|_{L^2}^2+\|\boldsymbol{\zeta}\|_{L^2}^2\right)\leq 0.\end{aligned} \tag{4.19}$$

Thus, defining $Y(t)=\|\boldsymbol{\eta}(t)\|_{L^2}^2+\|\boldsymbol{\zeta}(t)\|_{L^2}^2$ and $Z(t)=\|\nabla\boldsymbol{v}(t)\|_{L^2}^2+\|\nabla\boldsymbol{w}(t)\|_{L^2}^2$, we have

$$\frac{d}{dt}Y+\psi Y\leq 0, \tag{4.20}$$

where $\psi(t):=\mu-\left(\frac{c_L^4+(\alpha-\beta)^4}{2(\alpha-\beta)^3}\right)Z(t)$, provided that $\mu c_1^2h^2\leq\alpha-\beta$.

By Proposition 4.2.8 with $T = \frac{1}{\pi^2(\alpha-\beta)}$, ψ satisfies (1.16b) and if

$$\mu - \frac{c_L^4 + (\alpha-\beta)^4}{2T(\alpha-\beta)^3}(1+T\pi^2(\alpha-\beta))(\alpha-\beta)G^2 > 0 \iff \mu > \frac{\pi^2(c_L^4+(\alpha-\beta)^4)}{\alpha-\beta}G^2,$$

then ψ also satisfies (1.16a), so we can apply Proposition 1.0.3 to Y and conclude that $(\boldsymbol{V}, \boldsymbol{W})$ converges exponentially in time to $(\boldsymbol{v}, \boldsymbol{w})$.

The requirement on h is

$$h < \left(\frac{\alpha-\beta}{c_1^2\mu}\right)^{1/2} < \frac{\alpha-\beta}{\pi c_1\sqrt{c_L^4+(\alpha-\beta)^4}}\,G^{-1},$$

so $h \sim G^{-1}$. □

Proof of Theorem 4.3.2 Let $\boldsymbol{\eta} = \boldsymbol{v} - \boldsymbol{V}$ and $\boldsymbol{\zeta} = \boldsymbol{w} - \boldsymbol{W}$. Then $\boldsymbol{\eta}$ satisfies:

$$\partial_t\boldsymbol{\eta} - \alpha\Delta\boldsymbol{\eta} - \beta\Delta\boldsymbol{\zeta} + (\boldsymbol{w}\cdot\nabla)\,\boldsymbol{v} - (\boldsymbol{W}\cdot\nabla)\,\boldsymbol{V} = -\nabla(\mathcal{P} - \tilde{\mathcal{P}} - q) - \mu\,\mathrm{I}_h(\eta_1)\boldsymbol{e}_1.$$

Using the fact that $(\boldsymbol{w}\cdot\nabla)\,\boldsymbol{v} - (\boldsymbol{W}\cdot\nabla)\,\boldsymbol{V} = (\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v} + (\boldsymbol{W}\cdot\nabla)\,\boldsymbol{\eta}$ we write:

$$\partial_t\boldsymbol{\eta} - \alpha\Delta\boldsymbol{\eta} - \beta\Delta\boldsymbol{\zeta} + (\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v} + (\boldsymbol{W}\cdot\nabla)\,\boldsymbol{\eta} = -\nabla(\mathcal{P} - \tilde{\mathcal{P}} - q) - \mu\,\mathrm{I}_h(\eta_1)\boldsymbol{e}_1.$$

Taking the inner product with $\boldsymbol{\eta}$ we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\eta}\|_{L^2}^2 + \alpha\|\nabla\boldsymbol{\eta}\|_{L^2}^2 + \beta\left\langle\nabla\boldsymbol{\zeta}, \nabla\boldsymbol{\eta}\right\rangle + \left\langle(\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v}, \boldsymbol{\eta}\right\rangle = -\left\langle\nabla(\mathcal{P} - \tilde{\mathcal{P}} - q), \boldsymbol{\eta}\right\rangle - \mu\left\langle\mathrm{I}_h(\eta_1), \eta_1\right\rangle.$$

Now, by the divergence free condition, we have:

$$-\left\langle\nabla(\mathcal{P} - \tilde{\mathcal{P}} - q), \boldsymbol{\eta}\right\rangle := -\int_\Omega \nabla(\mathcal{P} - \tilde{\mathcal{P}} - q)\cdot\boldsymbol{\eta}\,dxdy = \int_\Omega(\mathcal{P} - \tilde{\mathcal{P}} - q)\cdot(\nabla\cdot\boldsymbol{\eta})\,dxdy = 0.$$

By applying Cauchy-Schwarz inequality and (1.3),

$$\left|\beta\left\langle\nabla\boldsymbol{\zeta}, \nabla\boldsymbol{\eta}\right\rangle\right| \le \frac{\beta}{2}\|\nabla\boldsymbol{\eta}\|_{L^2}^2 + \frac{\beta}{2}\|\nabla\boldsymbol{\zeta}\|_{L^2}^2,$$

and by rewriting $\langle \mathrm{I}_h(\eta_1), \eta_1 \rangle = \langle \mathrm{I}_h(\eta_1) - \eta_1, \eta_1 \rangle + \langle \eta_1, \eta_1 \rangle$, we have:

$$-\mu \langle \mathrm{I}_h(\eta_1), \eta_1 \rangle = -\mu \langle \mathrm{I}_h(\eta_1) - \eta_1, \eta_1 \rangle - \mu \|\eta_1\|_{L^2}^2.$$

Thus, we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\eta}\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2}\right)\|\nabla\boldsymbol{\eta}\|_{L^2}^2 - \frac{\beta}{2}\|\nabla\boldsymbol{\zeta}\|_{L^2}^2 + \mu\|\eta_1\|_{L^2}^2 \\
&\leq -\langle (\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v}, \boldsymbol{\eta}\rangle - \mu \langle \mathrm{I}_h(\eta_1) - \eta_1, \eta_1 \rangle \\
&\leq |\langle (\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v}, \boldsymbol{\eta}\rangle| + \mu\,|\langle \mathrm{I}_h(\eta_1) - \eta_1, \eta_1 \rangle| \\
&\leq |\langle (\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v}, \boldsymbol{\eta}\rangle| + \mu\|\,\mathrm{I}_h(\eta_1) - \eta_1\|_{L^2}\|\eta_1\|_{L^2} \\
&\leq |\langle (\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v}, \boldsymbol{\eta}\rangle| + \mu c_1 h\|\nabla\eta_1\|_{L^2}\|\eta_1\|_{L^2} \\
&\leq |\langle (\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v}, \boldsymbol{\eta}\rangle| + \frac{\mu c_1^2 h^2}{2}\|\nabla\eta_1\|_{L^2}^2 + \frac{\mu}{2}\|\eta_1\|_{L^2}^2,
\end{aligned}$$

where in the last three lines we used the Cauchy-Schwarz inequality, the definition of I_h, and Young's inequality. This leaves us with:

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\eta}\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2}\right)\|\nabla\boldsymbol{\eta}\|_{L^2}^2 - \frac{\beta}{2}\|\nabla\boldsymbol{\zeta}\|_{L^2}^2 + \frac{\mu}{2}\|\eta_1\|_{L^2}^2 \leq |\langle (\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v}, \boldsymbol{\eta}\rangle| + \frac{\mu c_1^2 h^2}{2}\|\nabla\eta_1\|_{L^2}^2,$$

or equivalently,

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\eta}\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} - \frac{\mu c_1^2 h^2}{2}\right)\|\nabla\boldsymbol{\eta}\|_{L^2}^2 - \frac{\beta}{2}\|\nabla\boldsymbol{\zeta}\|_{L^2}^2 + \frac{\mu}{2}\|\eta_1\|_{L^2}^2 \leq \left|\int_\Omega (\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v}\cdot\boldsymbol{\eta}\,dxdy\right|. \tag{4.21}$$

Now we apply Lemma 4.4.1 to estimate the nonlinear term with (4.14), yielding:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\eta}\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} - \frac{\mu c_1^2 h^2}{2} - c\delta\right)\|\nabla\boldsymbol{\eta}\|_{L^2}^2 + \left(-\frac{\beta}{2} - c\delta\right)\|\nabla\boldsymbol{\zeta}\|_{L^2}^2 \\
&+ \left[\frac{\mu}{2} - \frac{c}{\delta}\|\nabla\boldsymbol{v}\|_{L^2}^2 - \frac{c}{\delta}\|\nabla\boldsymbol{v}\|_{L^2}^2\left(1 + \ln\frac{\|\nabla\eta_1\|_{L^2}}{2\pi\|\eta_1\|_{L^2}}\right)\right]\|\eta_1\|_{L^2}^2 \\
&+ \left[-\frac{c}{\delta}\|\nabla\boldsymbol{v}\|_{L^2}^2 - \frac{c}{\delta}\|\nabla\boldsymbol{v}\|_{L^2}^2\left(1 + \ln\frac{\|\nabla\zeta_1\|_{L^2}}{2\pi\|\zeta_1\|_{L^2}}\right)\right]\|\zeta_1\|_{L^2}^2 \leq 0.
\end{aligned} \tag{4.22}$$

Proceeding similarly with $\boldsymbol{\zeta}$ we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\zeta}\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} - \frac{\mu c_1^2 h^2}{2} - c\delta\right)\|\nabla\boldsymbol{\zeta}\|_{L^2}^2 + \left(-\frac{\beta}{2} - c\delta\right)\|\nabla\boldsymbol{\eta}\|_{L^2}^2 \\
&+ \left[\frac{\mu}{2} - \frac{c}{\delta}\|\nabla\boldsymbol{w}\|_{L^2}^2 - \frac{c}{\delta}\|\nabla\boldsymbol{w}\|_{L^2}^2\left(1 + \ln\frac{\|\nabla\zeta_1\|_{L^2}}{2\pi\|\zeta_1\|_{L^2}}\right)\right]\|\zeta_1\|_{L^2}^2 \\
&+ \left[-\frac{c}{\delta}\|\nabla\boldsymbol{w}\|_{L^2}^2 - \frac{c}{\delta}\|\nabla\boldsymbol{w}\|_{L^2}^2\left(1 + \ln\frac{\|\nabla\eta_1\|_{L^2}}{2\pi\|\eta_1\|_{L^2}}\right)\right]\|\eta_1\|_{L^2}^2 \le 0. \qquad (4.23)
\end{aligned}$$

Now, adding (4.22) and (4.23) and defining $Z(t) = \|\nabla\boldsymbol{v}(t)\|_{L^2}^2 + \|\nabla\boldsymbol{w}(t)\|_{L^2}^2$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\eta}\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\boldsymbol{\zeta}\|_{L^2}^2 \\
&+ \left(\alpha - \beta - \frac{\mu c_1^2 h^2}{2} - 2c\delta\right)\|\nabla\boldsymbol{\eta}\|_{L^2}^2 + \left(\alpha - \beta - \frac{\mu c_1^2 h^2}{2} - 2c\delta\right)\|\nabla\boldsymbol{\zeta}\|_{L^2}^2 \\
&+ \left[\frac{\mu}{2} - \frac{c}{\delta}Z - \frac{c}{\delta}Z\left(1 + \ln\frac{\|\nabla\eta_1\|_{L^2}}{2\pi\|\eta_1\|_{L^2}}\right)\right]\|\eta_1\|_{L^2}^2 \\
&+ \left[\frac{\mu}{2} - \frac{c}{\delta}Z - \frac{c}{\delta}Z\left(1 + \ln\frac{\|\nabla\zeta_1\|_{L^2}}{2\pi\|\zeta_1\|_{L^2}}\right)\right]\|\zeta_1\|_{L^2}^2 \le 0. \qquad (4.24)
\end{aligned}$$

Since $\alpha > \beta$,

$$\gamma := (\alpha - \beta) - \frac{\mu c_1^2 h^2}{2} - 2c\delta \ge \frac{(\alpha - \beta)}{4} > 0,$$

provided that $h \le (\alpha - \beta)^{\frac{1}{2}} c_1^{-1} \mu^{-\frac{1}{2}}$ and by choosing $\delta = \frac{(\alpha-\beta)}{8c}$.

We want to apply Lemma 4.4.2 to the logarithmic terms in (4.24). To this end note that by (1.5), $\frac{\|\nabla\eta_1\|_{L^2}}{2\pi\|\eta_1\|_{L^2}} \ge 1$, so $\ln\frac{\|\nabla\eta_1\|_{L^2}^2}{4\pi^2\|\eta_1\|_{L^2}^2} \ge \ln\frac{\|\nabla\eta_1\|_{L^2}}{2\pi\|\eta_1\|_{L^2}}$. Next, we write

$$\gamma\|\nabla\boldsymbol{\eta}\|_{L^2}^2 \ge \frac{\gamma}{2}\|\nabla\boldsymbol{\eta}\|_{L^2}^2 + \frac{4\pi^2\gamma}{2}\frac{\|\nabla\eta_1\|_{L^2}^2}{4\pi^2\|\eta_1\|_{L^2}^2}\|\eta_1\|_{L^2}^2,$$

and consider

$$\begin{aligned}
&2\pi^2\gamma\frac{\|\nabla\eta_1\|_{L^2}^2}{4\pi^2\|\eta_1\|_{L^2}^2}\|\eta_1\|_{L^2}^2 - \frac{c}{\delta}Z\left(1 + \ln\frac{\|\nabla\eta_1\|_{L^2}^2}{4\pi^2\|\eta_1\|_{L^2}^2}\right)\|\eta_1\|_{L^2}^2 \\
&= 2\pi^2\gamma\left(\frac{\|\nabla\eta_1\|_{L^2}^2}{4\pi^2\|\eta_1\|_{L^2}^2} - \frac{c}{2\pi^2\gamma\delta}Z\left(1 + \ln\frac{\|\nabla\eta_1\|_{L^2}^2}{4\pi^2\|\eta_1\|_{L^2}^2}\right)\right)\|\eta_1\|_{L^2}^2.
\end{aligned}$$

By Lemma 4.4.2,

$$\frac{\|\nabla\eta_1\|_{L^2}^2}{4\pi^2\|\eta_1\|_{L^2}^2} - \frac{c}{2\pi^2\gamma\delta}Z\left(1+\ln\frac{\|\nabla\eta_1\|_{L^2}^2}{4\pi^2\|\eta_1\|_{L^2}^2}\right) \geq -\frac{c}{2\pi^2\gamma\delta}Z\left(\ln\frac{c}{2\pi^2\gamma\delta}Z\right). \tag{4.25}$$

Hence, using (4.25) and defining $Y(t) = \|\boldsymbol{\eta}(t)\|_{L^2}^2 + \|\boldsymbol{\zeta}(t)\|_{L^2}^2$, we rewrite (4.24) as

$$\frac{1}{2}\frac{d}{dt}Y+\frac{\gamma}{2}\left(\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\|\nabla\boldsymbol{\zeta}\|_{L^2}^2\right)+\left[\frac{\mu}{2}-\frac{c}{\delta}Z\left(1+\ln\frac{c}{2\pi^2\gamma\delta}Z\right)\right]\left(\|\eta_1\|_{L^2}^2+\|\zeta_1\|_{L^2}^2\right) \leq 0.$$

By (1.5),

$$\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\|\nabla\boldsymbol{\zeta}\|_{L^2}^2 \geq 4\pi^2\left(\|\boldsymbol{\eta}\|_{L^2}^2+\|\boldsymbol{\zeta}\|_{L^2}^2\right) \geq 4\pi^2\left(\|\eta_2\|_{L^2}^2+\|\zeta_2\|_{L^2}^2\right),$$

and so

$$\frac{d}{dt}Y+\min\left\{4\pi^2\gamma\,,\ \mu-\frac{2c}{\delta}Z\left(1+\ln\frac{c}{2\pi^2\gamma\delta}Z\right)\right\}Y \leq 0. \tag{4.26}$$

Let

$$\psi(t) := \min\left\{4\pi^2\gamma\,,\ \mu-\frac{2c}{\delta}Z(t)\left(1+\ln\frac{c}{2\pi^2\gamma\delta}Z(t)\right)\right\},$$

and in order to apply Proposition 1.0.3 we only need to show that ψ satisfies (1.16a) and (1.16b). It is sufficient to show that for some $T, t_0 > 0$,

$$\mu - \limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\frac{2c}{\delta}Z(s)\left(1+\ln\frac{c}{2\pi^2\gamma\delta}Z(s)\right)\,ds > 0, \tag{4.27}$$

and

$$\sup_{s>t_0} Z(s)\left(1+\ln\frac{c}{2\pi^2\gamma\delta}Z(s)\right)\,ds < \infty. \tag{4.28}$$

In fact, (4.28) follows directly from (4.9) with the t_0 given there.

To see (4.27), by Proposition 4.2.8 with $T = \frac{1}{\pi^2(\alpha-\beta)}$, we have:

$$
\begin{aligned}
\lim_{t\to\infty} \frac{1}{T}\int_t^{t+T} & \frac{2c}{\delta} Z(s)\left(1+\ln\frac{c}{2\pi^2\gamma\delta}Z(s)\right)\,ds \\
&\le \frac{2c}{\delta T}\left(1+\ln\frac{c}{2\pi^2\gamma\delta}10\pi^2(\alpha-\beta)^2G^2e^{CG^4}\right)\lim_{t\to\infty}\int_t^{t+T} Z(s)\,ds \\
&\le \frac{2c}{\delta T}\left(\tilde{c}+2\ln G+CG^4\right)(1+T\pi^2(\alpha-\beta))(\alpha-\beta)G^2, \\
&= 32\pi^2c^2(\alpha-\beta)\left(\tilde{c}+2\ln G+CG^4\right)G^2.
\end{aligned}
$$

Therefore, (4.27) holds by choosing $\mu > 32\pi^2c^2(\alpha-\beta)\left(\tilde{c}+2\ln G+CG^4\right)G^2$. In addition, the requirement $h \le \frac{(\alpha-\beta)^{\frac{1}{2}}}{c_1}\mu^{-\frac{1}{2}}$ implies $h \sim G^{-3}$. $\square$

Proof of Theorem 4.3.3 Let $\boldsymbol{\eta} = \boldsymbol{v} - \boldsymbol{V}$ and $\boldsymbol{\zeta} = \boldsymbol{w} - \boldsymbol{W}$. Similarly to how we showed (4.21), the equation we obtain for $\boldsymbol{\eta}$ is

$$
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\eta}\|_{L^2}^2+\left(\alpha-\frac{\beta}{2}-\frac{\mu c_1^2h^2}{2}\right)\|\nabla\boldsymbol{\eta}\|_{L^2}^2-\frac{\beta}{2}\|\nabla\boldsymbol{\zeta}\|_{L^2}^2+\frac{\mu}{2}\|\boldsymbol{\eta}\|_{L^2}^2 \le \left|\int_\Omega (\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v}\cdot\boldsymbol{\eta}\,dxdy\right|, \tag{4.29}
$$

but now the equation for $\boldsymbol{\zeta}$ is

$$
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\zeta}\|_{L^2}^2+\left(\alpha-\frac{\beta}{2}\right)\|\nabla\boldsymbol{\zeta}\|_{L^2}^2-\frac{\beta}{2}\|\nabla\boldsymbol{\eta}\|_{L^2}^2 \le \left|\int_\Omega (\boldsymbol{\eta}\cdot\nabla)\,\boldsymbol{w}\cdot\boldsymbol{\zeta}\,dxdy\right|. \tag{4.30}
$$

We estimate the integral in (4.29) using (4.12), so (4.29) becomes:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\eta}\|_{L^2}^2 &+ \left(\alpha-\frac{\beta}{2}-\frac{\mu c_1^2h^2}{2}-\frac{\epsilon}{2}\right)\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\left(-\frac{\beta}{2}-\frac{c_L\delta}{4}\right)\|\nabla\boldsymbol{\zeta}\|_{L^2}^2 \\
&+\left(\frac{\mu}{2}-\frac{c_L^2}{8\epsilon\delta^2}\|\nabla\boldsymbol{v}\|_{L^2}^2\right)\|\boldsymbol{\eta}\|_{L^2}^2+\left(-\frac{c_L\delta}{4}\|\nabla\boldsymbol{v}\|_{L^2}^2\right)\|\boldsymbol{\zeta}\|_{L^2}^2 \le 0,
\end{aligned} \tag{4.31}
$$

Similarly, we estimate the integral in (4.30) using (4.13), and get:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\zeta}\|_{L^2}^2 &+ \left(\alpha-\frac{\beta}{2}-\frac{c_L\delta}{4}\right)\|\nabla\boldsymbol{\zeta}\|_{L^2}^2+\left(-\frac{\beta}{2}-\frac{\epsilon}{2}\right)\|\nabla\boldsymbol{\eta}\|_{L^2}^2 \\
&+\left(-\frac{c_L\delta}{4}\|\nabla\boldsymbol{w}\|_{L^2}^2\right)\|\boldsymbol{\zeta}\|_{L^2}^2+\left(-\frac{c_L^2}{8\epsilon\delta^2}\|\nabla\boldsymbol{w}\|_{L^2}^2\right)\|\boldsymbol{\eta}\|_{L^2}^2 \le 0.
\end{aligned} \tag{4.32}
$$

Adding (4.31) and (4.32),

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\eta}\|_{L^2}^2+\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\zeta}\|_{L^2}^2+\left(\alpha-\beta-\frac{\mu c_1^2 h^2}{2}-\epsilon\right)\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\left(\alpha-\beta-\frac{c_L\delta}{2}\right)\|\nabla\boldsymbol{\zeta}\|_{L^2}^2$$
$$+\left(\frac{\mu}{2}-\frac{c_L^2}{8\epsilon\delta^2}\left(\|\nabla\boldsymbol{v}\|_{L^2}^2+\|\nabla\boldsymbol{w}\|_{L^2}^2\right)\right)\|\boldsymbol{\eta}\|_{L^2}^2+\left(-\frac{c_L\delta}{4}\left(\|\nabla\boldsymbol{v}\|_{L^2}^2+\|\nabla\boldsymbol{w}\|_{L^2}^2\right)\right)\|\boldsymbol{\zeta}\|_{L^2}^2\le 0.$$

Now, if we choose

$$h\le\frac{(\alpha-\beta)^{1/2}}{c_1}\mu^{-1/2},$$

and $\epsilon=\frac{\alpha-\beta}{2}$, then $\alpha-\beta-\frac{\mu c_1^2h^2}{2}-\epsilon\ge 0$.

Also, by choosing $\delta<\frac{\alpha-\beta}{c_L}$, we have

$$\gamma:=\alpha-\beta-\frac{c_L\delta}{2}>\frac{\alpha-\beta}{2}>0.$$

Then by applying (1.5) we obtain $\gamma\|\nabla\boldsymbol{\zeta}\|_{L^2}^2\ge\gamma 4\pi^2\|\boldsymbol{\zeta}\|_{L^2}^2$. Hence, defining $Y(t)=\|\boldsymbol{\eta}(t)\|_{L^2}^2+\|\boldsymbol{\zeta}(t)\|_{L^2}^2$ and $Z(t)=\|\nabla\boldsymbol{v}(t)\|_{L^2}^2+\|\nabla\boldsymbol{w}(t)\|_{L^2}^2$, we have:

$$\frac{d}{dt}Y+\psi Y\le 0, \tag{4.33}$$

where $\psi(t):=\min\left\{\mu-\frac{c_L^2}{4\epsilon\delta^2}Z(t)\,,\,8\pi^2\gamma-\frac{c_L\delta}{2}Z(t)\right\}$. Using Proposition 4.2.8 similarly as before, with $T=\frac{1}{\pi^2(\alpha-\beta)}$, ψ satisfies (1.16b) as well as (1.16a) provided that

$$\delta<\frac{\alpha-\beta}{c_L}\frac{4}{4+(\alpha-\beta)^2G^2}\implies 8\pi^2\gamma-\frac{c_L\delta}{2T}(1+T\pi^2(\alpha-\beta))(\alpha-\beta)G^2>4(\alpha-\beta)>0,$$

and

$$\mu>\frac{\pi^2c_L^4G^2(4+(\alpha-\beta)^2G^2)^2}{16(\alpha-\beta)}\implies\mu-\frac{c_L^2}{4\epsilon\delta^2T}(1+T\pi^2(\alpha-\beta))(\alpha-\beta)G^2>0.$$

By choosing such a μ and δ, we can apply Proposition 1.0.3 to conclude that $(\boldsymbol{V},\boldsymbol{W})$ converges exponentially in time to $(\boldsymbol{v},\boldsymbol{w})$.

Now the requirement we needed on h implies

$$h < \frac{4(\alpha-\beta)}{\pi c_1 c_L^2 G(4+(\alpha-\beta)^2 G^2)},$$

so $h \sim G^{-3}$. $\square$

4.4.2 Proof of H^1 Convergence Results with Type 1 Interpolants

Proof of Theorem 4.3.4. By denoting $\boldsymbol{\eta} = \boldsymbol{v} - \boldsymbol{V}$ and $\boldsymbol{\zeta} = \boldsymbol{w} - \boldsymbol{W}$ and subtracting the equations for $(\boldsymbol{v}, \boldsymbol{w})$ and $(\boldsymbol{V}, \boldsymbol{W})$, we obtain the following equations for $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$

$$\partial_t \boldsymbol{\eta} - \alpha\Delta\boldsymbol{\eta} - \beta\Delta\boldsymbol{\zeta} + (\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v} + (\boldsymbol{W}\cdot\nabla)\,\boldsymbol{\eta} = -\nabla(\mathcal{P} - \tilde{\mathcal{P}} - q) - \mu\,\mathrm{I}_h(\boldsymbol{\eta}),$$

$$\partial_t \boldsymbol{\zeta} - \alpha\Delta\boldsymbol{\zeta} - \beta\Delta\boldsymbol{\eta} + (\boldsymbol{\eta}\cdot\nabla)\,\boldsymbol{w} + (\boldsymbol{V}\cdot\nabla)\,\boldsymbol{\zeta} = -\nabla(\mathcal{P} - \tilde{\mathcal{P}} - q) - \mu\,\mathrm{I}_h(\boldsymbol{\zeta}).$$

Taking the inner product with $-\Delta\boldsymbol{\eta}$ and $-\Delta\boldsymbol{\zeta}$, respectively, we obtain:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{\eta}\|_{L^2}^2 + \alpha\|\Delta\boldsymbol{\eta}\|_{L^2}^2 &= -\beta\left\langle\Delta\boldsymbol{\zeta}, \Delta\boldsymbol{\eta}\right\rangle + \left\langle(\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v}, \Delta\boldsymbol{\eta}\right\rangle + \left\langle(\boldsymbol{W}\cdot\nabla)\,\boldsymbol{\eta}, \Delta\boldsymbol{\eta}\right\rangle \\
&\quad + \left\langle\nabla(\mathcal{P} - \tilde{\mathcal{P}} - q), \Delta\boldsymbol{\eta}\right\rangle + \mu\left\langle\mathrm{I}_h(\boldsymbol{\eta}), \Delta\boldsymbol{\eta}\right\rangle, \\
\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{\zeta}\|_{L^2}^2 + \alpha\|\Delta\boldsymbol{\zeta}\|_{L^2}^2 &= -\beta\left\langle\Delta\boldsymbol{\eta}, \Delta\boldsymbol{\zeta}\right\rangle + \left\langle(\boldsymbol{\eta}\cdot\nabla)\,\boldsymbol{w}, \Delta\boldsymbol{\zeta}\right\rangle + \left\langle(\boldsymbol{V}\cdot\nabla)\,\boldsymbol{\zeta}, \Delta\boldsymbol{\zeta}\right\rangle \\
&\quad + \left\langle\nabla(\mathcal{P} - \tilde{\mathcal{P}} - q), \Delta\boldsymbol{\zeta}\right\rangle + \mu\left\langle\mathrm{I}_h(\boldsymbol{\zeta}), \Delta\boldsymbol{\zeta}\right\rangle.
\end{aligned}$$

Then, by the divergence-free condition,

$$\left\langle\nabla(\mathcal{P} - \tilde{\mathcal{P}} - q), \Delta\boldsymbol{\eta}\right\rangle = -\int_\Omega (\mathcal{P} - \tilde{\mathcal{P}} - q)\cdot\Delta(\nabla\cdot\boldsymbol{\eta})\,dxdy = 0,$$

and similarly

$$\left\langle\nabla(\mathcal{P} - \tilde{\mathcal{P}} - q), \Delta\boldsymbol{\zeta}\right\rangle = 0.$$

Also, by applying Cauchy-Schwarz inequality and (1.3), we have

$$-\beta \langle \Delta\boldsymbol{\zeta}, \Delta\boldsymbol{\eta}\rangle \leq \frac{\beta}{2}\|\Delta\boldsymbol{\eta}\|_{L^2}^2 + \frac{\beta}{2}\|\Delta\boldsymbol{\zeta}\|_{L^2}^2.$$

Rewriting $\langle \mathrm{I}_h(\boldsymbol{\eta}), -\Delta\boldsymbol{\eta}\rangle = \langle \mathrm{I}_h(\boldsymbol{\eta}) - \boldsymbol{\eta}, -\Delta\boldsymbol{\eta}\rangle + \langle \boldsymbol{\eta}, \Delta\boldsymbol{\eta}\rangle$, we have,

$$-\mu \langle \mathrm{I}_h(\boldsymbol{\eta}), -\Delta\boldsymbol{\eta}\rangle = -\mu \langle \mathrm{I}_h(\boldsymbol{\eta}) - \boldsymbol{\eta}, -\Delta\boldsymbol{\eta}\rangle - \mu\|\nabla\boldsymbol{\eta}\|_{L^2}^2,$$

and similarly,

$$-\mu \langle \mathrm{I}_h(\boldsymbol{\zeta}), -\Delta\boldsymbol{\zeta}\rangle = -\mu \langle \mathrm{I}_h(\boldsymbol{\zeta}) - \boldsymbol{\zeta}, -\Delta\boldsymbol{\zeta}\rangle - \mu\|\nabla\boldsymbol{\zeta}\|_{L^2}^2.$$

Adding up the equations for $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla\boldsymbol{\eta}\|_{L^2}^2 + \|\nabla\boldsymbol{\zeta}\|_{L^2}^2\right) + (\alpha-\beta)\left(\|\Delta\boldsymbol{\eta}\|_{L^2}^2 + \|\Delta\boldsymbol{\zeta}\|_{L^2}^2\right)\\
&\leq |\langle(\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v}, \Delta\boldsymbol{\eta}\rangle| + |\langle(\boldsymbol{\eta}\cdot\nabla)\,\boldsymbol{w}, \Delta\boldsymbol{\zeta}\rangle| + |\langle(\boldsymbol{W}\cdot\nabla)\,\boldsymbol{\eta}, \Delta\boldsymbol{\eta}\rangle| + |\langle(\boldsymbol{V}\cdot\nabla)\,\boldsymbol{\zeta}, \Delta\boldsymbol{\zeta}\rangle|\\
&\qquad + \mu\,|\langle \mathrm{I}_h(\boldsymbol{\eta}) - \boldsymbol{\eta}, \Delta\boldsymbol{\eta}\rangle| + \mu\,|\langle \mathrm{I}_h(\boldsymbol{\zeta}) - \boldsymbol{\zeta}, \Delta\boldsymbol{\zeta}\rangle| - \mu\left(\|\nabla\boldsymbol{\eta}\|_{L^2}^2 + \|\nabla\boldsymbol{\zeta}\|_{L^2}^2\right).
\end{aligned}$$

Due to the properties of I_h, we have

$$\begin{aligned}
\mu\,|\langle \mathrm{I}_h(\boldsymbol{\eta}) - \boldsymbol{\eta}, \Delta\boldsymbol{\eta}\rangle| &\leq \mu\|\,\mathrm{I}_h(\boldsymbol{\eta}) - \boldsymbol{\eta}\|_{L^2}\|\Delta\boldsymbol{\eta}\|_{L^2} \leq \mu c_1 h\|\nabla\boldsymbol{\eta}\|_{L^2}\|\Delta\boldsymbol{\eta}\|_{L^2}\\
&\leq \frac{4\mu^2 c_1^2 h^2}{\alpha-\beta}\|\nabla\boldsymbol{\eta}\|_{L^2}^2 + \frac{\alpha-\beta}{16}\|\Delta\boldsymbol{\eta}\|_{L^2}^2,
\end{aligned}$$

and similarly, we obtain

$$\mu\,|\langle \mathrm{I}_h(\boldsymbol{\zeta}) - \boldsymbol{\zeta}, \Delta\boldsymbol{\zeta}\rangle| \leq \frac{4\mu^2 c_1^2 h^2}{\alpha-\beta}\|\nabla\boldsymbol{\zeta}\|_{L^2}^2 + \frac{\alpha-\beta}{16}\|\Delta\boldsymbol{\zeta}\|_{L^2}^2.$$

Next, we estimate the nonlinear terms. First, by Hölder's and Sobolev inequalities, we obtain

$$\begin{aligned}
|\langle(\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v},\Delta\boldsymbol{\eta}\rangle| &\leq \int_\Omega |\boldsymbol{\zeta}||\nabla\boldsymbol{v}||\Delta\boldsymbol{\eta}|\,dxdy \leq \|\boldsymbol{\zeta}\|_{L^4}\|\nabla\boldsymbol{v}\|_{L^4}\|\Delta\boldsymbol{\eta}\|_{L^2}\\
&\leq \|\boldsymbol{\zeta}\|_{L^2}^{1/2}\|\nabla\boldsymbol{\zeta}\|_{L^2}^{1/2}\|\nabla\boldsymbol{v}\|_{L^2}^{1/2}\|\Delta\boldsymbol{v}\|_{L^2}^{1/2}\|\Delta\boldsymbol{\eta}\|_{L^2}\\
&\leq \frac{4}{\alpha-\beta}\|\nabla\boldsymbol{v}\|_{L^2}\|\Delta\boldsymbol{v}\|_{L^2}\|\boldsymbol{\zeta}\|_{L^2}\|\nabla\boldsymbol{\zeta}\|_{L^2}+\frac{\alpha-\beta}{16}\|\Delta\boldsymbol{\eta}\|_{L^2}^2\\
&\leq \frac{4}{2\pi(\alpha-\beta)}\|\nabla\boldsymbol{v}\|_{L^2}\|\Delta\boldsymbol{v}\|_{L^2}\|\boldsymbol{\zeta}\|_{L^2}\|\Delta\boldsymbol{\zeta}\|_{L^2}+\frac{\alpha-\beta}{16}\|\Delta\boldsymbol{\eta}\|_{L^2}^2\\
&\leq \frac{1}{4\pi^2}\left(\frac{4}{\alpha-\beta}\right)^3\|\nabla\boldsymbol{v}\|_{L^2}^2\|\Delta\boldsymbol{v}\|_{L^2}^2\|\boldsymbol{\zeta}\|_{L^2}^2+\frac{\alpha-\beta}{16}\left(\|\Delta\boldsymbol{\eta}\|_{L^2}^2+\|\Delta\boldsymbol{\zeta}\|_{L^2}^2\right),
\end{aligned}$$

where we used Poincaré's and Young's inequalities. The estimate for $\langle(\boldsymbol{\eta}\cdot\nabla)\,\boldsymbol{w},\Delta\boldsymbol{\zeta}\rangle$ is similar, i.e., we have

$$|\langle(\boldsymbol{\eta}\cdot\nabla)\,\boldsymbol{w},\Delta\boldsymbol{\zeta}\rangle| \leq \frac{1}{4\pi^2}\left(\frac{4}{\alpha-\beta}\right)^3\|\nabla\boldsymbol{w}\|_{L^2}^2\|\Delta\boldsymbol{w}\|_{L^2}^2\|\boldsymbol{\eta}\|_{L^2}^2+\frac{\alpha-\beta}{16}\left(\|\Delta\boldsymbol{\eta}\|_{L^2}^2+\|\Delta\boldsymbol{\zeta}\|_{L^2}^2\right).$$

Regarding $\langle(\boldsymbol{W}\cdot\nabla)\,\boldsymbol{\eta},\Delta\boldsymbol{\eta}\rangle$, we first rewrite it as

$$\langle(\boldsymbol{W}\cdot\nabla)\,\boldsymbol{\eta},\Delta\boldsymbol{\eta}\rangle = \langle(\boldsymbol{w}\cdot\nabla)\,\boldsymbol{\eta},\Delta\boldsymbol{\eta}\rangle - \langle(\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{\eta},\Delta\boldsymbol{\eta}\rangle = I + II.$$

In order to estimate I, we first observe that by the periodic boundary conditions, we have

$$\|\nabla\boldsymbol{\eta}\|_{L^2}^2 = \int_\Omega \nabla\boldsymbol{\eta}\cdot\nabla\boldsymbol{\eta}\,dxdy = -\int_\Omega \boldsymbol{\eta}\Delta\boldsymbol{\eta}\,dxdy \leq \|\boldsymbol{\eta}\|_{L^2}\|\Delta\boldsymbol{\eta}\|_{L^2}. \tag{4.34}$$

Thus, we integrate by parts and proceed to estimate I as

$$\begin{aligned}
\langle (\boldsymbol{w}\cdot\nabla)\,\boldsymbol{\eta},\Delta\boldsymbol{\eta}\rangle &= \sum_{i,j,k=1}^{2}\int_\Omega \boldsymbol{w}_i\partial_i\boldsymbol{\eta}_k\partial_{jj}^2\boldsymbol{\eta}_k\,dxdy = -\sum_{i,j,k=1}^{2}\int_\Omega \partial_j\boldsymbol{w}_i\partial_i\boldsymbol{\eta}_k\partial_j\boldsymbol{\eta}_k\,dxdy\\
&\le \int_\Omega |\nabla\boldsymbol{w}||\nabla\boldsymbol{\eta}|^2\,dxdy \le \|\nabla\boldsymbol{w}\|_{L^2}\|\nabla\boldsymbol{\eta}\|_{L^2}\|\Delta\boldsymbol{\eta}\|_{L^2}\\
&\le \frac{4}{\alpha-\beta}\|\nabla\boldsymbol{w}\|_{L^2}^2\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\frac{\alpha-\beta}{16}\|\Delta\boldsymbol{\eta}\|_{L^2}^2\\
&\le \frac{4}{\alpha-\beta}\|\nabla\boldsymbol{w}\|_{L^2}^2\|\boldsymbol{\eta}\|_{L^2}\|\Delta\boldsymbol{\eta}\|_{L^2}+\frac{\alpha-\beta}{16}\|\Delta\boldsymbol{\eta}\|_{L^2}^2\\
&\le \left(\frac{4}{\alpha-\beta}\right)^3\|\nabla\boldsymbol{w}\|_{L^2}^4\|\boldsymbol{\eta}\|_{L^2}^2+\frac{\alpha-\beta}{8}\|\Delta\boldsymbol{\eta}\|_{L^2}^2.
\end{aligned}$$

By similar estimates and the analogy of (4.34) for $\boldsymbol{\zeta}$, i.e.,

$$\|\nabla\boldsymbol{\zeta}\|_{L^2}^2\le\|\boldsymbol{\zeta}\|_{L^2}\|\Delta\boldsymbol{\zeta}\|_{L^2},$$

we estimate II as

$$\begin{aligned}
-\langle (\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{\eta},\Delta\boldsymbol{\eta}\rangle &\le \int_\Omega |\nabla\boldsymbol{\zeta}||\nabla\boldsymbol{\eta}|^2\,dxdy \le \|\nabla\boldsymbol{\zeta}\|_{L^2}\|\nabla\boldsymbol{\eta}\|_{L^2}\|\Delta\boldsymbol{\eta}\|_{L^2}\\
&\le \frac{4}{\alpha-\beta}\|\nabla\boldsymbol{\eta}\|_{L^2}^2\|\nabla\boldsymbol{\zeta}\|_{L^2}^2+\frac{\alpha-\beta}{16}\|\Delta\boldsymbol{\eta}\|_{L^2}^2\\
&\le \frac{4}{\alpha-\beta}\|\boldsymbol{\eta}\|_{L^2}\|\boldsymbol{\zeta}\|_{L^2}\|\Delta\boldsymbol{\eta}\|_{L^2}\|\Delta\boldsymbol{\zeta}\|_{L^2}+\frac{\alpha-\beta}{16}\|\Delta\boldsymbol{\eta}\|_{L^2}^2\\
&\le \frac{2}{\alpha-\beta}\|\boldsymbol{\eta}\|_{L^2}\|\boldsymbol{\zeta}\|_{L^2}\left(\|\Delta\boldsymbol{\eta}\|_{L^2}^2+\|\Delta\boldsymbol{\zeta}\|_{L^2}^2\right)+\frac{\alpha-\beta}{16}\|\Delta\boldsymbol{\eta}\|_{L^2}^2.
\end{aligned}$$

By a similar approach, we have

$$\langle (\boldsymbol{V}\cdot\nabla)\,\boldsymbol{\zeta},\Delta\boldsymbol{\zeta}\rangle = \langle (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{\zeta},\Delta\boldsymbol{\zeta}\rangle - \langle (\boldsymbol{\eta}\cdot\nabla)\,\boldsymbol{\zeta},\Delta\boldsymbol{\zeta}\rangle = III+IV,$$

and III is bounded by

$$|\langle (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{\zeta},\Delta\boldsymbol{\zeta}\rangle| \le \left(\frac{4}{\alpha-\beta}\right)^3\|\nabla\boldsymbol{v}\|_{L^2}^4\|\boldsymbol{\zeta}\|_{L^2}^2+\frac{\alpha-\beta}{8}\|\Delta\boldsymbol{\zeta}\|_{L^2}^2,$$

while we estimate IV as

$$-\langle(\boldsymbol{\eta}\cdot\nabla)\,\boldsymbol{\zeta},\Delta\boldsymbol{\zeta}\rangle \le \frac{2}{\alpha-\beta}\|\boldsymbol{\eta}\|_{L^2}\|\boldsymbol{\zeta}\|_{L^2}\left(\|\Delta\boldsymbol{\zeta}\|_{L^2}^2+\|\Delta\boldsymbol{\eta}\|_{L^2}^2\right)+\frac{\alpha-\beta}{16}\|\Delta\boldsymbol{\zeta}\|_{L^2}^2.$$

Combining all the above estimates, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\|\nabla\boldsymbol{\zeta}\|_{L^2}^2\right)+\underbrace{\left(\frac{\alpha-\beta}{2}-\frac{4}{\alpha-\beta}\|\boldsymbol{\eta}\|_{L^2}\|\boldsymbol{\zeta}\|_{L^2}\right)}_{V}\left(\|\Delta\boldsymbol{\eta}\|_{L^2}^2+\|\Delta\boldsymbol{\zeta}\|_{L^2}^2\right)\\
&\le\Bigg[\underbrace{\frac{1}{4\pi^2}\left(\frac{4}{\alpha-\beta}\right)^3\left(\|\nabla\boldsymbol{v}\|_{L^2}^2\|\Delta\boldsymbol{v}\|_{L^2}^2+\|\nabla\boldsymbol{w}\|_{L^2}^2\|\Delta\boldsymbol{w}\|_{L^2}^2\right)}_{VI}\\
&\qquad+\underbrace{\left(\frac{4}{\alpha-\beta}\right)^3\left(\|\nabla\boldsymbol{v}\|_{L^2}^4+\|\nabla\boldsymbol{w}\|_{L^2}^4\right)}_{VII}\Bigg]\left(\|\boldsymbol{\eta}\|_{L^2}^2+\|\boldsymbol{\zeta}\|_{L^2}^2\right)\\
&\qquad+\Bigg(\underbrace{\frac{4\mu^2c_1^2h^2}{\alpha-\beta}}_{VIII}-\mu\Bigg)\left(\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\|\nabla\boldsymbol{\zeta}\|_{L^2}^2\right).
\end{aligned}\tag{4.35}$$

Now choose h such that

$$VIII=\frac{4\mu^2c_1^2h^2}{\alpha-\beta}<\frac{\mu}{2}.$$

Thus, we have

$$h^2<\frac{\alpha-\beta}{8\mu c_1^2}.\tag{4.36}$$

Moreover, by Theorem 4.3.1, we know that after a sufficiently large time T_1, $\|\boldsymbol{\eta}\|_{L^2}$ and $\|\boldsymbol{\zeta}\|_{L^2}$ are small enough so that we have

$$\|\boldsymbol{\eta}\|_{L^2}\|\boldsymbol{\zeta}\|_{L^2}\le\frac{(\alpha-\beta)^2}{16},\tag{4.37}$$

which implies that $V\ge 0$, so we have:

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\|\nabla\boldsymbol{\zeta}\|_{L^2}^2\right)+\frac{\mu}{2}\left(\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\|\nabla\boldsymbol{\zeta}\|_{L^2}^2\right)\le(VI+VII)\left(\|\boldsymbol{\eta}\|_{L^2}^2+\|\boldsymbol{\zeta}\|_{L^2}^2\right).$$

Define $Y(t) = \|\nabla \boldsymbol{\eta}(t)\|_{L^2}^2 + \|\nabla \boldsymbol{\zeta}(t)\|_{L^2}^2$, and by appealing to Proposition 4.2.8, we see that $VI + VII$ is bounded by some number $\frac{M_G}{2}$. Also, by Theorem 4.3.1 we know that there exists constants $K, a > 0$ such that $\|\boldsymbol{\eta}(t)\|_{L^2}^2 + \|\boldsymbol{\zeta}(t)\|_{L^2}^2 \leq K e^{-at}$, $\forall t \geq T_1$. Putting all of this together, we have the following for all $t > T_1$:

$$\begin{aligned}
&\frac{d}{dt} Y(t) + \mu Y(t) \leq M_G K e^{-at}, \\
&\Rightarrow \frac{d}{dt}\left(e^{\mu t} Y(t)\right) \leq M_G K e^{(\mu - a)t}, \\
\Rightarrow e^{\mu t} Y(t) - e^{\mu T_1} Y(T_1) &\leq \frac{M_G K}{\mu - a} e^{(\mu - a)t} - \frac{M_G K}{\mu - a} e^{(\mu - a)T_1}, \\
&\Rightarrow Y(t) \leq Y(T_1) e^{-\mu(t - T_1)} + \frac{M_G K}{\mu - a}\left(e^{-at} - e^{-\mu(t - T_1) - a T_1}\right).
\end{aligned}$$

Therefore, $Y(t) = \|\nabla \boldsymbol{\eta}(t)\|_{L^2}^2 + \|\nabla \boldsymbol{\zeta}(t)\|_{L^2}^2 \to 0$ exponentially as $t \to \infty$ as long as μ and h satisfy the conditions of Theorem 4.3.1, as well as the new requirement (4.36). So, choosing

$$\mu > \frac{\pi^2 (c_L^4 + (\alpha - \beta)^4)}{\alpha - \beta} G^2, \quad \text{and} \quad h < \frac{\alpha - \beta}{2\sqrt{2}\pi c_1 \sqrt{c_L^4 + (\alpha - \beta)^4}} G^{-1},$$

we have exponential convergence. $\square$

Next, we prove the H^1 decay estimates for the data assimilation scenario where measurement is only on v_1 and w_1.

Proof of Theorem 4.3.5. We still denote the difference of solutions to (3.3) and (4.6) by $\boldsymbol{\eta} = \boldsymbol{v} - \boldsymbol{V}$ and $\boldsymbol{\zeta} = \boldsymbol{w} - \boldsymbol{W}$. Similarly to the beginning of the proof of

Theorem 4.3.4, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\|\nabla\boldsymbol{\zeta}\|_{L^2}^2\right)+(\alpha-\beta)\left(\|\Delta\boldsymbol{\eta}\|_{L^2}^2+\|\Delta\boldsymbol{\zeta}\|_{L^2}^2\right)$$
$$\leq |\langle(\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v},\Delta\boldsymbol{\eta}\rangle|+|\langle(\boldsymbol{\eta}\cdot\nabla)\,\boldsymbol{w},\Delta\boldsymbol{\zeta}\rangle|+|\langle(\boldsymbol{W}\cdot\nabla)\,\boldsymbol{\eta},\Delta\boldsymbol{\eta}\rangle|+|\langle(\boldsymbol{V}\cdot\nabla)\,\boldsymbol{\zeta},\Delta\boldsymbol{\zeta}\rangle|$$
$$+\mu\,|\langle \mathrm{I}_h(\eta_1)-\eta_1,\Delta\eta_1\rangle|+\mu\,|\langle \mathrm{I}_h(\zeta_1)-\zeta_1,\Delta\zeta_1\rangle|-\mu\|\nabla\eta_1\|_{L^2}^2-\mu\|\nabla\zeta_1\|_{L^2}^2,$$

as well as

$$\mu\,|\langle \mathrm{I}_h(\eta_1)-\eta_1,\Delta\eta_1\rangle|\leq\frac{4\mu^2c_1^2h^2}{\alpha-\beta}\|\nabla\eta_1\|_{L^2}^2+\frac{\alpha-\beta}{16}\|\Delta\eta_1\|_{L^2}^2,$$

and

$$\mu\,|\langle \mathrm{I}_h(\zeta_1)-\zeta_1,\Delta\zeta_1\rangle|\leq\frac{4\mu^2c_1^2h^2}{\alpha-\beta}\|\nabla\zeta_1\|_{L^2}^2+\frac{\alpha-\beta}{16}\|\Delta\zeta_1\|_{L^2}^2.$$

The estimates for the nonlinear terms are also similar. Namely, we have

$$|\langle(\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v},\Delta\boldsymbol{\eta}\rangle|\leq\frac{1}{4\pi^2}\left(\frac{4}{\alpha-\beta}\right)^3\|\nabla\boldsymbol{v}\|_{L^2}^2\|\Delta\boldsymbol{v}\|_{L^2}^2\|\boldsymbol{\zeta}\|_{L^2}^2+\frac{\alpha-\beta}{16}\left(\|\Delta\boldsymbol{\eta}\|_{L^2}^2+\|\Delta\boldsymbol{\zeta}\|_{L^2}^2\right),$$

and

$$|\langle(\boldsymbol{\eta}\cdot\nabla)\,\boldsymbol{w},\Delta\boldsymbol{\zeta}\rangle|\leq\frac{1}{4\pi^2}\left(\frac{4}{\alpha-\beta}\right)^3\|\nabla\boldsymbol{w}\|_{L^2}^2\|\Delta\boldsymbol{w}\|_{L^2}^2\|\boldsymbol{\eta}\|_{L^2}^2+\frac{\alpha-\beta}{16}\left(\|\Delta\boldsymbol{\eta}\|_{L^2}^2+\|\Delta\boldsymbol{\zeta}\|_{L^2}^2\right).$$

Also, by rewriting

$$\langle(\boldsymbol{W}\cdot\nabla)\,\boldsymbol{\eta},\Delta\boldsymbol{\eta}\rangle=\langle(\boldsymbol{w}\cdot\nabla)\,\boldsymbol{\eta},\Delta\boldsymbol{\eta}\rangle-\langle(\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{\eta},\Delta\boldsymbol{\eta}\rangle$$

we obtain

$$\langle(\boldsymbol{w}\cdot\nabla)\,\boldsymbol{\eta},\Delta\boldsymbol{\eta}\rangle\leq\left(\frac{4}{\alpha-\beta}\right)^3\|\nabla\boldsymbol{w}\|_{L^2}^4\|\boldsymbol{\eta}\|_{L^2}^2+\frac{\alpha-\beta}{8}\|\Delta\boldsymbol{\eta}\|_{L^2}^2,$$

and

$$-\langle(\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{\eta},\Delta\boldsymbol{\eta}\rangle\leq\frac{2}{\alpha-\beta}\|\boldsymbol{\eta}\|_{L^2}\|\boldsymbol{\zeta}\|_{L^2}\left(\|\Delta\boldsymbol{\eta}\|_{L^2}^2+\|\Delta\boldsymbol{\zeta}\|_{L^2}^2\right)+\frac{\alpha-\beta}{16}\|\Delta\boldsymbol{\eta}\|_{L^2}^2.$$

Estimates for

$$\langle (\boldsymbol{V}\cdot\nabla)\,\boldsymbol{\zeta},\Delta\boldsymbol{\zeta}\rangle = \langle (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{\zeta},\Delta\boldsymbol{\zeta}\rangle - \langle (\boldsymbol{\eta}\cdot\nabla)\,\boldsymbol{\zeta},\Delta\boldsymbol{\zeta}\rangle$$

also follow similarly, and we obtain

$$\langle (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{\zeta},\Delta\boldsymbol{\zeta}\rangle \leq \left(\frac{4}{\alpha-\beta}\right)^3 \|\nabla\boldsymbol{v}\|_{L^2}^4\|\boldsymbol{\zeta}\|_{L^2}^2 + \frac{\alpha-\beta}{8}\|\Delta\boldsymbol{\zeta}\|_{L^2}^2,$$

and

$$-\langle (\boldsymbol{\eta}\cdot\nabla)\,\boldsymbol{\zeta},\Delta\boldsymbol{\zeta}\rangle \leq \frac{2}{\alpha-\beta}\|\boldsymbol{\eta}\|_{L^2}\|\boldsymbol{\zeta}\|_{L^2}\left(\|\Delta\boldsymbol{\zeta}\|_{L^2}^2+\|\Delta\boldsymbol{\eta}\|_{L^2}^2\right) + \frac{\alpha-\beta}{16}\|\Delta\boldsymbol{\zeta}\|_{L^2}^2.$$

Combining all the above estimates, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\|\nabla\boldsymbol{\zeta}\|_{L^2}^2\right) + \underbrace{\left(\frac{\alpha-\beta}{2} - \frac{4}{\alpha-\beta}\|\boldsymbol{\eta}\|_{L^2}\|\boldsymbol{\zeta}\|_{L^2}\right)}_{V}\left(\|\Delta\boldsymbol{\eta}\|_{L^2}^2+\|\Delta\boldsymbol{\zeta}\|_{L^2}^2\right)\\
&\quad\leq \Bigg[\underbrace{\left(\frac{1}{4\pi^2}\left(\frac{4}{\alpha-\beta}\right)^3\left(\|\nabla\boldsymbol{v}\|_{L^2}^2\|\Delta\boldsymbol{v}\|_{L^2}^2+\|\nabla\boldsymbol{w}\|_{L^2}^2\|\Delta\boldsymbol{w}\|_{L^2}^2\right)\right)}_{VI}\\
&\qquad + \underbrace{\left(\frac{4}{\alpha-\beta}\right)^3\left(\|\nabla\boldsymbol{v}\|_{L^2}^4+\|\nabla\boldsymbol{w}\|_{L^2}^4\right)}_{VII}\Bigg]\left(\|\boldsymbol{\eta}\|_{L^2}^2+\|\boldsymbol{\zeta}\|_{L^2}^2\right)\\
&\qquad + \Bigg(\underbrace{\frac{4\mu^2c_1^2h^2}{\alpha-\beta}}_{VIII} - \mu\Bigg)\left(\|\nabla\eta_1\|_{L^2}^2+\|\nabla\zeta_1\|_{L^2}^2\right).
\end{aligned} \tag{4.38}$$

We choose h such that

$$VIII = \frac{4\mu^2c_1^2h^2}{\alpha-\beta} < \frac{\mu}{2}. \tag{4.39}$$

In view of Theorem 4.3.2, after sufficiently large time $T_2>0$, $\|\boldsymbol{\eta}\|_{L^2}$ and $\|\boldsymbol{\zeta}\|_{L^2}$ are small enough so that

$$\|\boldsymbol{\eta}\|_{L^2}\|\boldsymbol{\zeta}\|_{L^2} < \frac{(\alpha-\beta)^2}{16}. \tag{4.40}$$

Thus, $V > \frac{1}{4}(\alpha-\beta) > 0$. Let us denote $Y(t) = \|\nabla\boldsymbol{\eta}\|_{L^2}^2 + \|\nabla\boldsymbol{\zeta}\|_{L^2}^2$. Then, for all $t > T_2$, by applying Poincaré's inequality to the second term on the left-hand side of (4.38), it follows, due to (4.39), that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}Y(t) + \pi^2(\alpha-\beta)Y(t) &\leq M_G\left(\|\boldsymbol{\eta}\|_{L^2}^2 + \|\boldsymbol{\zeta}\|_{L^2}^2\right) + (VIII-\mu)\left(\|\nabla\eta_1\|_{L^2}^2 + \|\nabla\zeta_1\|_{L^2}^2\right) \\
&\leq M_G\left(\|\boldsymbol{\eta}\|_{L^2}^2 + \|\boldsymbol{\zeta}\|_{L^2}^2\right) \\
&\leq K' M_G e^{-a't},
\end{aligned}$$

where $K' > 0$ and $a' > 0$ chosen so that is such that $\|\boldsymbol{\eta}\|_{L^2}^2 + \|\boldsymbol{\zeta}\|_{L^2}^2 \leq K' M_G e^{-a't}$ for all $t > T_2$ (this is permitted due to Theorem 4.3.2). This implies

$$\frac{d}{dt}\left(Y(t)e^{2\pi^2(\alpha-\beta)t}\right) \leq K' M_G e^{2\pi^2(\alpha-\beta)t}e^{-a't}.$$

Integrating, we arrive at

$$Y(t) \leq Y(T_2)e^{-2\pi^2(\alpha-\beta)(t-T_2)} + \frac{K' M_G}{2\pi^2(\alpha-\beta) - a'}\left(e^{-ta'} - e^{-2\pi^2(\alpha-\beta)(t-T_2) - a'T_2}\right).$$

(Note that, if necessary, one may choose a' slightly smaller so that $2\pi^2(\alpha-\beta) \neq a'$.) In particular, $Y(t) = \|\nabla\boldsymbol{\eta}\|_{L^2}^2 + \|\nabla\boldsymbol{\zeta}\|_{L^2}^2$ decays exponentially in time for all $t > T_2$, with h and μ chosen so that

$$\mu > 32\pi^2 c^2(\alpha-\beta)\left(\tilde{c} + 2\ln G + CG^4\right)G^2$$

and

$$h < (2\sqrt{2}c_1)^{-1}(\alpha-\beta)^{\frac{1}{2}}\mu^{-\frac{1}{2}} < (8\sqrt{2}\pi c_1 c)^{-1}\left(\tilde{c} + 2\ln G + CG^4\right)^{-\frac{1}{2}}G^{-1}.$$

Thus, the proof of Theorem 4.3.5 is complete. □

Proof of Theorem 4.3.6. The proof goes similarly as that of Theorem 4.3.5. For the sake of simplicity, we omit the details here. □

4.4.3 Proofs of the Results for Type 2 Interpolants

Lemma 4.4.3 *Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in H^2$ be divergence free. Then the following inequalities hold:*

$$
\begin{aligned}
(a)\quad \left|\int_\Omega (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{v}\cdot\Delta\boldsymbol{w}\,dxdy\right| \leq\ & 3c_T\|\nabla u_1\|_{L^2}\|\nabla\boldsymbol{v}\|_{L^2}\|\Delta\boldsymbol{w}\|_{L^2}\left(1+\ln\frac{\|\Delta u_1\|_{L^2}}{2\pi\|\nabla u_1\|_{L^2}}\right)^{1/2} \\
&+(c_T+4c_B)\|\Delta\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{v}\|_{L^2}\|\nabla w_1\|_{L^2}\left(1+\ln\frac{\|\Delta w_1\|_{L^2}}{2\pi\|\nabla w_1\|_{L^2}}\right)^{1/2} \\
&+2c_T\|\nabla\boldsymbol{u}\|_{L^2}\|\Delta\boldsymbol{v}\|_{L^2}\|\nabla w_1\|_{L^2}\left(1+\ln\frac{\|\Delta w_1\|_{L^2}}{2\pi\|\nabla w_1\|_{L^2}}\right)^{1/2},
\end{aligned}
\tag{4.41}
$$

$$
(b)\quad \left|\int_\Omega (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{v}\cdot\Delta\boldsymbol{v}\,dxdy\right| \leq (2c_B+5c_T)\|\nabla\boldsymbol{u}\|_{L^2}\|\nabla v_1\|_{L^2}\|\Delta\boldsymbol{v}\|_{L^2}\left(1+\ln\frac{\|\Delta v_1\|_{L^2}}{2\pi\|\nabla v_1\|_{L^2}}\right)^{1/2}. \tag{4.42}
$$

Proof. See the appendix. □

In the following proof of Theorem 4.3.8, we simultaneously establish a bound like (4.9) for the data assimilation solution, because the proof requires such an estimate.

Proof of Theorem 4.3.8 Since $(\boldsymbol{V}, \boldsymbol{W})$ is a strong solution and $\boldsymbol{V}_0 \equiv \boldsymbol{W}_0 \equiv 0$, there is a largest time $T_0 \in (0, \infty]$ such that

$$\sup_{t\in[0,T_0)} (\|\nabla\boldsymbol{V}(t)\|_{L^2}^2 + \|\nabla\boldsymbol{W}(t)\|_{L^2}^2) \leq 50\pi^2(\alpha-\beta)^2G^2e^{CG^4}.$$

Suppose that $T_0 < \infty$. Then we know that

$$\limsup_{t\to T_0^-}(\|\nabla \boldsymbol{V}(t)\|_{L^2}^2+\|\nabla \boldsymbol{W}(t)\|_{L^2}^2) = \sup_{t\in[0,T_0)}(\|\nabla \boldsymbol{V}(t)\|_{L^2}^2+\|\nabla \boldsymbol{W}(t)\|_{L^2}^2) = 50\pi^2(\alpha-\beta)^2 G^2 e^{CG^4}. \tag{4.43}$$

Let $\boldsymbol{\eta} = \boldsymbol{v} - \boldsymbol{V}$ and $\boldsymbol{\zeta} = \boldsymbol{w} - \boldsymbol{W}$. Then we have the following equation for $\boldsymbol{\eta}$:

$$\partial_t \boldsymbol{\eta} - \alpha\Delta\boldsymbol{\eta} - \beta\Delta\boldsymbol{\zeta} + (\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v} + (\boldsymbol{W}\cdot\nabla)\,\boldsymbol{\eta} = -\nabla(p - \tilde{p} - q) - \mu\,\mathrm{I}_h(\eta_1)\boldsymbol{e}_1.$$

Taking the inner product with $-\Delta\boldsymbol{\eta}$, we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{\eta}\|_{L^2}^2 + \alpha\|\Delta\boldsymbol{\eta}\|_{L^2}^2 + \beta\left\langle\Delta\boldsymbol{\zeta},\Delta\boldsymbol{\eta}\right\rangle - \left\langle(\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v},\Delta\boldsymbol{\eta}\right\rangle - \left\langle(\boldsymbol{W}\cdot\nabla)\,\boldsymbol{\eta},\Delta\boldsymbol{\eta}\right\rangle \\
&= \left\langle\nabla(p - \tilde{p} - q),\Delta\boldsymbol{\eta}\right\rangle - \mu\left\langle\mathrm{I}_h(\eta_1),-\Delta\eta_1\right\rangle
\end{aligned}$$

Now, by the divergence free condition, we have:

$$\left\langle\nabla(p - \tilde{p} - q),\Delta\boldsymbol{\eta}\right\rangle = -\int_\Omega (p - \tilde{p} - q)\cdot\Delta(\nabla\cdot\boldsymbol{\eta})\,dxdy = 0,$$

and by applying Cauchy-Schwarz inequality and (1.3),

$$|\beta\left\langle\Delta\boldsymbol{\zeta},\Delta\boldsymbol{\eta}\right\rangle| \le \frac{\beta}{2}\|\Delta\boldsymbol{\eta}\|_{L^2}^2 + \frac{\beta}{2}\|\Delta\boldsymbol{\zeta}\|_{L^2}^2.$$

Rewriting $\left\langle\mathrm{I}_h(\eta_1),-\Delta\eta_1\right\rangle = \left\langle\mathrm{I}_h(\eta_1)-\eta_1,-\Delta\eta_1\right\rangle + \left\langle\eta_1,\Delta\eta_1\right\rangle$, we have,

$$-\mu\left\langle\mathrm{I}_h(\eta_1),-\Delta\eta_1\right\rangle = -\mu\left\langle\mathrm{I}_h(\eta_1)-\eta_1,-\Delta\eta_1\right\rangle - \mu\|\nabla\eta_1\|_{L^2}^2,$$

so we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{\eta}\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2}\right)\|\Delta\boldsymbol{\eta}\|_{L^2}^2 - \frac{\beta}{2}\|\Delta\boldsymbol{\zeta}\|_{L^2}^2 + \mu\|\nabla\eta_1\|_{L^2}^2 \\
&\le \left|\left\langle(\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v},\Delta\boldsymbol{\eta}\right\rangle\right| + \left|\left\langle(\boldsymbol{W}\cdot\nabla)\,\boldsymbol{\eta},\Delta\boldsymbol{\eta}\right\rangle\right| + \mu\left|\left\langle\mathrm{I}_h(\eta_1)-\eta_1,\Delta\eta_1\right\rangle\right|.
\end{aligned}$$

By the properties of I_h, we have

$$
\begin{aligned}
\mu\left|\langle \mathrm{I}_h(\eta_1)-\eta_1,\Delta\eta_1\rangle\right| &\le \mu\|\mathrm{I}_h(\eta_1)-\eta_1\|_{L^2}\|\Delta\eta_1\|_{L^2}\\
&\le \mu\left(c_2h\|\nabla\eta_1\|_{L^2}+c_3h^2\|\Delta\eta_1\|_{L^2}\right)\|\Delta\eta_1\|_{L^2}\\
&\le \frac{\mu^2}{2(\alpha-\beta)}(c_2h\|\nabla\eta_1\|_{L^2}+c_3h^2\|\Delta\eta_1\|_{L^2})^2+\frac{\alpha-\beta}{2}\|\Delta\eta_1\|_{L^2}^2\\
&\le \frac{\mu^2c_2^2h^2}{\alpha-\beta}\|\nabla\eta_1\|_{L^2}^2+\frac{\mu^2c_3^2h^4}{\alpha-\beta}\|\Delta\eta_1\|_{L^2}^2+\frac{\alpha-\beta}{2}\|\Delta\eta_1\|_{L^2}^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\left(\frac{\alpha}{2}-\frac{\mu^2c_3^2h^4}{\alpha-\beta}\right)\|\Delta\boldsymbol{\eta}\|_{L^2}^2-\frac{\beta}{2}\|\Delta\boldsymbol{\zeta}\|_{L^2}^2+\mu\left(1-\frac{\mu c_2^2h^2}{\alpha-\beta}\right)\|\nabla\eta_1\|_{L^2}^2\\
&\le\left|\langle(\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v},\Delta\boldsymbol{\eta}\rangle\right|+\left|\langle(\boldsymbol{W}\cdot\nabla)\,\boldsymbol{\eta},\Delta\boldsymbol{\eta}\rangle\right|. \qquad (4.44)
\end{aligned}
$$

Note that $1-\frac{\mu c_2^2h^2}{\alpha-\beta}>\frac{1}{2}$, and $\frac{\mu^2c_3^2h^4}{\alpha-\beta}<\frac{\alpha-\beta}{4}$ as long as

$$
h^2<\frac{\alpha-\beta}{2\mu\max\{c_2^2,c_3\}}. \qquad (4.45)
$$

Now we estimate the nonlinear terms using Lemma 4.4.3. By (4.41), we obtain

$$
\begin{aligned}
\left|\langle(\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v},\Delta\boldsymbol{\eta}\rangle\right| &\le 3c_T\|\nabla\zeta_1\|_{L^2}\|\nabla\boldsymbol{v}\|_{L^2}\|\Delta\boldsymbol{\eta}\|_{L^2}\left(1+\ln\frac{\|\Delta\zeta_1\|_{L^2}}{2\pi\|\nabla\zeta_1\|_{L^2}}\right)^{1/2}\\
&\quad+(c_T+4c_B)\|\Delta\boldsymbol{\zeta}\|_{L^2}\|\nabla\boldsymbol{v}\|_{L^2}\|\nabla\eta_1\|_{L^2}\left(1+\ln\frac{\|\Delta\eta_1\|_{L^2}}{2\pi\|\nabla\eta_1\|_{L^2}}\right)^{1/2}\\
&\quad+2c_T\|\nabla\boldsymbol{\zeta}\|_{L^2}\|\Delta\boldsymbol{v}\|_{L^2}\|\nabla\eta_1\|_{L^2}\left(1+\ln\frac{\|\Delta\eta_1\|_{L^2}}{2\pi\|\nabla\eta_1\|_{L^2}}\right)^{1/2},
\end{aligned}
$$

so by applying (1.3), we obtain

$$
\begin{aligned}
\left|\langle(\boldsymbol{\zeta}\cdot\nabla)\,\boldsymbol{v},\Delta\boldsymbol{\eta}\rangle\right| &\le \frac{\alpha-\beta}{32}\left(\|\Delta\boldsymbol{\eta}\|_{L^2}^2+\|\Delta\boldsymbol{\zeta}\|_{L^2}^2+4\pi^2\|\nabla\boldsymbol{\zeta}\|_{L^2}^2\right)\\
&\quad+\frac{72c_T^2}{(\alpha-\beta)}\|\nabla\zeta_1\|_{L^2}^2\|\nabla\boldsymbol{v}\|_{L^2}^2\left(1+\ln\frac{\|\Delta\zeta_1\|_{L^2}}{2\pi\|\nabla\zeta_1\|_{L^2}}\right)\\
&\quad+\frac{64(1+4\pi^2)(c_T^2+c_B^2)}{4\pi^2(\alpha-\beta)}\left(\|\nabla\boldsymbol{v}\|_{L^2}^2+\|\Delta\boldsymbol{v}\|_{L^2}^2\right)\|\nabla\eta_1\|_{L^2}^2\left(1+\ln\frac{\|\Delta\eta_1\|_{L^2}}{2\pi\|\nabla\eta_1\|_{L^2}}\right).
\end{aligned}
$$

Also, we use (1.5) to write $4\pi^2\|\nabla\boldsymbol{\zeta}\|_{L^2}^2 \le \|\Delta\boldsymbol{\zeta}\|_{L^2}$.

For the other term, we first apply (4.42), and obtain

$$|\langle(\boldsymbol{W}\cdot\nabla)\,\boldsymbol{\eta},\Delta\boldsymbol{\eta}\rangle| \le (2c_B+5c_T)\|\nabla\boldsymbol{W}\|_{L^2}\|\nabla\eta_1\|_{L^2}\|\Delta\boldsymbol{\eta}\|_{L^2}\left(1+\ln\frac{\|\Delta\eta_1\|_{L^2}}{2\pi\|\nabla\eta_1\|_{L^2}}\right)^{1/2}.$$

Then, by (1.3), we have

$$|\langle(\boldsymbol{W}\cdot\nabla)\,\boldsymbol{\eta},\Delta\boldsymbol{\eta}\rangle| \le \frac{\alpha-\beta}{32}\|\Delta\boldsymbol{\eta}\|_{L^2}^2+\frac{200(c_B+c_T)^2}{\alpha-\beta}\|\nabla\boldsymbol{W}\|_{L^2}^2\|\nabla\eta_1\|_{L^2}^2\left(1+\ln\frac{\|\Delta\eta_1\|_{L^2}}{2\pi\|\nabla\eta_1\|_{L^2}}\right).$$

Combining these estimates with (4.44), we have:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\left(\frac{\alpha}{2}-\frac{5(\alpha-\beta)}{16}\right)\|\Delta\boldsymbol{\eta}\|_{L^2}^2-\left(\frac{\beta}{2}+\frac{\alpha-\beta}{16}\right)\|\Delta\boldsymbol{\zeta}\|_{L^2}^2\\
&+\left[\frac{\mu}{2}-\gamma_0\left(\|\nabla\boldsymbol{W}\|_{L^2}^2+\|\nabla\boldsymbol{v}\|_{L^2}^2+\|\Delta\boldsymbol{v}\|_{L^2}^2\right)\left(1+\ln\frac{\|\Delta\eta_1\|_{L^2}}{2\pi\|\nabla\eta_1\|_{L^2}}\right)\right]\|\nabla\eta_1\|_{L^2}^2\\
&-\gamma_0\|\nabla\boldsymbol{v}\|_{L^2}^2\left(1+\ln\frac{\|\Delta\zeta_1\|_{L^2}}{2\pi\|\nabla\zeta_1\|_{L^2}}\right)\|\nabla\zeta_1\|_{L^2}^2\le 0,
\end{aligned}\tag{4.46}$$

where

$$\gamma_0 := \frac{200(c_B+c_T)^2}{\alpha-\beta} = \max\left\{\frac{72c_T^2}{(\alpha-\beta)},\frac{64(1+4\pi^2)(c_T^2+c_B^2)}{4\pi^2(\alpha-\beta)},\frac{200(c_B+c_T)^2}{\alpha-\beta}\right\}.$$

Adding (4.46) with the corresponding inequality for $\frac{d}{dt}\|\nabla\boldsymbol{\zeta}\|_{L^2}^2$, we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{\eta}\|_{L^2}^2+\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{\zeta}\|_{L^2}^2+\frac{\alpha-\beta}{8}\left(\|\Delta\boldsymbol{\eta}\|_{L^2}^2+\|\Delta\boldsymbol{\zeta}\|_{L^2}^2\right)\\
&\quad+\left[\frac{\mu}{2}-\gamma_0\left(\|\nabla\boldsymbol{W}\|_{L^2}^2+\|\nabla\boldsymbol{v}\|_{L^2}^2+\|\nabla\boldsymbol{w}\|_{L^2}^2+\|\Delta\boldsymbol{v}\|_{L^2}^2\right)\left(1+\ln\frac{\|\Delta\eta_1\|_{L^2}}{2\pi\|\nabla\eta_1\|_{L^2}}\right)\right]\|\nabla\eta_1\|_{L^2}^2\\
&\quad+\left[\frac{\mu}{2}-\gamma_0\left(\|\nabla\boldsymbol{V}\|_{L^2}^2+\|\nabla\boldsymbol{v}\|_{L^2}^2+\|\nabla\boldsymbol{w}\|_{L^2}^2+\|\Delta\boldsymbol{w}\|_{L^2}^2\right)\left(1+\ln\frac{\|\Delta\zeta_1\|_{L^2}}{2\pi\|\nabla\zeta_1\|_{L^2}}\right)\right]\|\nabla\zeta_1\|_{L^2}^2\\
&\le 0.
\end{aligned}\tag{4.47}$$

Next, we write

$$\frac{\alpha-\beta}{8}\|\Delta\boldsymbol{\eta}\|_{L^2}^2 \ge \frac{\alpha-\beta}{16}\|\Delta\boldsymbol{\eta}\|_{L^2}^2+\frac{\alpha-\beta}{16}\frac{\|\Delta\eta_1\|_{L^2}^2}{4\pi^2\|\nabla\eta_1\|_{L^2}^2}4\pi^2\|\nabla\eta_1\|_{L^2}^2$$

and

$$\frac{\alpha-\beta}{8}\|\Delta\boldsymbol{\zeta}\|_{L^2}^2 \geq \frac{\alpha-\beta}{16}\|\Delta\boldsymbol{\zeta}\|_{L^2}^2 + \frac{\alpha-\beta}{16}\frac{\|\Delta\zeta_1\|_{L^2}^2}{4\pi^2\|\nabla\zeta_1\|_{L^2}^2}4\pi^2\|\nabla\zeta_1\|_{L^2}^2.$$

Then, by defining

$$r(\boldsymbol{u}) = \frac{\|\Delta u_1\|_{L^2}^2}{4\pi^2\|\nabla u_1\|_{L^2}^2}$$

and

$$\gamma = \frac{4}{\pi^2(\alpha-\beta)}\gamma_0\left(\|\nabla\boldsymbol{V}\|_{L^2}^2 + \|\nabla\boldsymbol{W}\|_{L^2}^2 + \|\nabla\boldsymbol{v}\|_{L^2}^2 + \|\nabla\boldsymbol{w}\|_{L^2}^2 + \|\Delta\boldsymbol{v}\|_{L^2}^2 + \|\Delta\boldsymbol{w}\|_{L^2}^2\right),$$

by (1.5) we can rewrite (4.47) as:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla\boldsymbol{\eta}\|_{L^2}^2 + \|\nabla\boldsymbol{\zeta}\|_{L^2}^2) + \frac{\alpha-\beta}{16}\left(\|\Delta\boldsymbol{\eta}\|_{L^2}^2 + \|\Delta\boldsymbol{\zeta}\|_{L^2}^2\right)\\
&\quad + \left[\frac{\mu}{2} + \frac{\pi^2(\alpha-\beta)}{4}(r(\boldsymbol{\eta}) - \gamma\,(1+\ln r(\boldsymbol{\eta})))\right]\|\nabla\eta_1\|_{L^2}^2\\
&\quad + \left[\frac{\mu}{2} + \frac{\pi^2(\alpha-\beta)}{4}(r(\boldsymbol{\zeta}) - \gamma\,(1+\ln r(\boldsymbol{\zeta})))\right]\|\nabla\zeta_1\|_{L^2}^2 \leq 0.
\end{aligned}$$

Now we apply Lemma 4.4.2 and conclude that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla\boldsymbol{\eta}\|_{L^2}^2 + \|\nabla\boldsymbol{\zeta}\|_{L^2}^2) + \frac{\alpha-\beta}{16}\left(\|\Delta\boldsymbol{\eta}\|_{L^2}^2 + \|\Delta\boldsymbol{\zeta}\|_{L^2}^2\right)\\
&\quad + \left[\frac{\mu}{2} - \frac{\pi^2(\alpha-\beta)}{4}\gamma\ln(\gamma)\right]\|\nabla\eta_1\|_{L^2}^2 + \left[\frac{\mu}{2} - \frac{\pi^2(\alpha-\beta)}{4}\gamma\ln(\gamma)\right]\|\nabla\zeta_1\|_{L^2}^2 \leq 0.
\end{aligned}$$

Using (1.5) again, we have

$$\|\Delta\boldsymbol{\eta}\|_{L^2}^2 + \|\Delta\boldsymbol{\zeta}\|_{L^2}^2 \geq 4\pi^2(\|\nabla\boldsymbol{\eta}\|_{L^2}^2 + \|\nabla\boldsymbol{\zeta}\|_{L^2}^2),$$

so by defining

$$Y = \|\nabla\boldsymbol{\eta}\|_{L^2}^2 + \|\nabla\boldsymbol{\zeta}\|_{L^2}^2,$$

and

$$\psi = \min\left\{\frac{\pi^2(\alpha-\beta)}{2},\ \mu - \frac{\pi^2(\alpha-\beta)}{2}\gamma\ln(\gamma)\right\}$$

we obtain:

$$\frac{d}{dt}Y + \psi Y \leq 0. \tag{4.48}$$

Thus, as long as we choose $\mu > \frac{\pi^2(\alpha-\beta)}{2}(1+\gamma\ln(\gamma))$, we conclude by Gronwall's inequality that

$$Y(t) \leq Y(0)e^{-\pi^2(\alpha-\beta)t/2}, \quad \forall t \in [0, T_0).$$

By (4.43), (4.9), and (4.10),

$$\begin{aligned}\gamma \leq &\frac{4}{\pi^2(\alpha-\beta)}\gamma_0 \Big(60\pi^2(\alpha-\beta)^2G^2e^{CG^4} + c_M(\alpha-\beta)^2 \\ &\times G^2\left[1 + \left(1+G^2e^{CG^4}\right)\left(1+e^{CG^4}+G^4e^{CG^4}\right)\right]\Big) \\ < &\infty,\end{aligned}$$

so on the time interval $[0, T_0)$, such a μ is available. Specifically, it is sufficient to choose

$$\mu \geq 2000(c_B+c_T)^2(20\pi^2+c_M)G^2(1+G^2)^3e^{2CG^4}\left(\tilde{c} + \ln(1+G) + CG^4\right), \tag{4.49}$$

where $\tilde{c} := \ln(250(c_B+c_T)^2(20\pi^2+c_M))/8$, so

$$\mu \sim G^{12}e^{2CG^4}. \tag{4.50}$$

Therefore, for all $t \in [0, T_0)$, we obtain

$$Y(t) \leq Y(0) \leq 2\|\nabla \boldsymbol{v}_0\|_{L^2}^2 + 2\|\nabla \boldsymbol{V}_0\|_{L^2}^2 + 2\|\nabla \boldsymbol{w}_0\|_{L^2}^2 + 2\|\nabla \boldsymbol{W}_0\|_{L^2}^2 \leq 20\pi^2(\alpha-\beta)^2G^2e^{CG^4}.$$

This implies that, in fact,

$$\sup_{t\in[0,T_0)} (\|\nabla \boldsymbol{V}(t)\|_{L^2}^2 + \|\nabla \boldsymbol{W}(t)\|_{L^2}^2) \leq 40\pi^2(\alpha-\beta)^2G^2e^{CG^4},$$

which is a contradiction to (4.43).

Hence we have $T_0 = \infty$, and $(\boldsymbol{V}(t), \boldsymbol{W}(t))$ converges exponentially in time to $(\boldsymbol{v}(t), \boldsymbol{w}(t))$ in the H^1 norm, and we have established the estimate:

$$\sup_{t\in[0,\infty)} (\|\nabla \boldsymbol{V}(t)\|_{L^2}^2 + \|\nabla \boldsymbol{W}(t)\|_{L^2}^2) \le 50\pi^2(\alpha-\beta)^2 G^2 e^{CG^4}.$$

Also, our restriction on μ (4.50) is in fact sufficient to guarantee convergence on $[0,\infty)$, with our restriction (4.45) on h, which we see now means we can choose

$$h \sim G^{-6} e^{-CG^4}.$$

$\square$

4.4.4 Determining Interpolants

Proof of Theorem 4.3.10 The proof proceeds exactly as that of Theorem 4.3.1, where $\delta^{(1)} \equiv \delta^{(2)} \equiv \epsilon^{(1)} \equiv \epsilon^{(2)} \equiv 0$, with a few differences. As before, we let $\boldsymbol{\eta} = \boldsymbol{v} - \boldsymbol{V}$ and then we obtain a differential inequality for $\|\boldsymbol{\eta}\|_{L^2}$. We get the same inequality as before but with two extra terms.

After subtracting the equations for $\boldsymbol{v}$ and $\boldsymbol{V}$, we have $\boldsymbol{f} - (\boldsymbol{f} + \delta^{(1)}) = -\delta^{(1)}$ for the forcing term, and after taking the inner product with $\boldsymbol{\eta}$ we have

$$\left|\int_\Omega \delta^{(1)} \cdot \boldsymbol{\eta}\, dxdy\right| \le \|\delta^{(1)}\|_{L^2}\|\boldsymbol{\eta}\|_{L^2} \le \frac{1}{\mu}\|\delta^{(1)}\|_{L^2}^2 + \frac{\mu}{4}\|\boldsymbol{\eta}\|_{L^2}^2.$$

Also, we have $\mu\, \mathrm{I}_h\left(\boldsymbol{v} + \epsilon^{(1)} - \boldsymbol{V}\right) = \mu\, \mathrm{I}_h\left(\boldsymbol{v} - \boldsymbol{V}\right) + \mu\, \mathrm{I}_h\left(\epsilon^{(1)}\right)$, and after taking the inner product with $\boldsymbol{\eta}$, we obtain

$$\left|\mu \int_\Omega \mathrm{I}_h(\epsilon^{(1)}) \cdot \boldsymbol{\eta}\, dxdy\right| \le \mu\|\, \mathrm{I}_h(\epsilon^{(1)})\|_{L^2}\|\boldsymbol{\eta}\|_{L^2} \le \mu\|\, \mathrm{I}_h(\epsilon^{(1)})\|_{L^2}^2 + \frac{\mu}{4}\|\boldsymbol{\eta}\|_{L^2}^2.$$

We have similar additions for the inequality we derive for $\boldsymbol{\zeta} := \boldsymbol{w} - \boldsymbol{W}$.

Thus, letting $Y(t) = \|\boldsymbol{\eta}(t)\|_{L^2}^2 + \|\boldsymbol{\zeta}(t)\|_{L^2}^2$ and proceeding as before, we eventually get:

$$\frac{d}{dt}Y + \psi Y \leq \varphi,$$

where

$$\psi(t) := \frac{\mu}{2} - \left(\frac{c_L^4 + (\alpha-\beta)^4}{2(\alpha-\beta)^3}\right)\left(\|\nabla \boldsymbol{v}\|_{L^2}^2 + \|\nabla \boldsymbol{w}\|_{L^2}^2\right),$$

and

$$\varphi(t) := \frac{1}{\mu}\left(\|\delta^{(1)}\|_{L^2}^2 + \|\delta^{(2)}\|_{L^2}^2\right) + \mu\left(\|\operatorname{I}_h(\epsilon^{(1)})\|_{L^2}^2 + \|\operatorname{I}_h(\epsilon^{(2)})\|_{L^2}^2\right).$$

Since $\|\delta(1)\|_{L^2}, \|\delta(2)\|_{L^2} \to 0$ and $\|\operatorname{I}_h(\epsilon^{(1)})\|_{L^2}, \|\operatorname{I}_h(\epsilon^{(2)})\|_{L^2} \to 0$, we have $\|\varphi\|_{L^2} \to 0$. Therefore, by Proposition 1.0.3, $\|\boldsymbol{v} - \boldsymbol{V}\|_{L^2}, \|\boldsymbol{w} - \boldsymbol{W}\|_{L^2} \to 0$ as $t \to \infty$. $\square$

Proof of Theorem 4.3.12 Let $\mu = \frac{(\alpha-\beta)}{c_1^2 h^2}$. Then h, I_h, and μ satisfy Theorem 4.3.1 with $(\boldsymbol{v}^{(1)}, \boldsymbol{w}^{(1)})$ as the reference solution. Let $(\boldsymbol{V}, \boldsymbol{W})$ be the corresponding solution.

Then $\|\boldsymbol{v}^{(1)}(t) - \boldsymbol{V}(t)\|_{L^2} \to 0$ and $\|\boldsymbol{w}^{(1)}(t) - \boldsymbol{W}(t)\|_{L^2} \to 0$, and for some q and $\mathcal{P}$, $\boldsymbol{V}$ and $\boldsymbol{W}$ satisfy the following equations:

$$\begin{aligned}\partial_t \boldsymbol{V} - \alpha\Delta \boldsymbol{V} + \beta\Delta \boldsymbol{W} + (\boldsymbol{W}\cdot\nabla)\boldsymbol{V} + \nabla\mathcal{P} + \nabla q &= \boldsymbol{f}^{(1)} + \mu\operatorname{I}_h\left(\boldsymbol{v}^{(1)} - \boldsymbol{V}\right)\\ &= \boldsymbol{f}^{(2)} + (\boldsymbol{f}^{(1)} - \boldsymbol{f}^{(2)}) + \mu\operatorname{I}_h\left(\boldsymbol{v}^{(2)} + (\boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)}) - \boldsymbol{V}\right),\end{aligned}$$

$$\begin{aligned}\partial_t \boldsymbol{W} - \alpha\Delta \boldsymbol{W} + \beta\Delta \boldsymbol{V} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{W} + \nabla\mathcal{P} - \nabla q &= \boldsymbol{g}^{(1)} + \mu\operatorname{I}_h\left(\boldsymbol{w}^{(1)} - \boldsymbol{W}\right)\\ &= \boldsymbol{g}^{(2)} + (\boldsymbol{g}^{(1)} - \boldsymbol{g}^{(2)}) + \mu\operatorname{I}_h\left(\boldsymbol{w}^{(2)} + (\boldsymbol{w}^{(1)} - \boldsymbol{w}^{(2)}) - \boldsymbol{W}\right).\end{aligned}$$

Therefore, setting $\delta^{(1)} := \boldsymbol{f}^{(1)} - \boldsymbol{f}^{(2)}$ and $\delta^{(2)} := \boldsymbol{g}^{(1)} - \boldsymbol{g}^{(2)}$, and $\epsilon^{(1)} := \boldsymbol{v}^{(1)} - \boldsymbol{v}^{(2)}$ and $\epsilon^{(2)} := \boldsymbol{w}^{(1)} - \boldsymbol{w}^{(2)}$, we see that $(\boldsymbol{V}, \boldsymbol{W})$ must be the unique solution guaranteed by Theorem 4.3.10, with $(\boldsymbol{v}^{(2)}, \boldsymbol{w}^{(2)})$ as the reference solution. Therefore $\|\boldsymbol{v}^{(2)}(t) - \boldsymbol{V}(t)\|_{L^2} \to 0$ and $\|\boldsymbol{w}^{(2)}(t) - \boldsymbol{W}(t)\|_{L^2} \to 0$.

Thus,

$$\|\boldsymbol{v}^{(1)}(t) - \boldsymbol{v}^{(2)}(t)\|_{L^2} \leq \|\boldsymbol{v}^{(1)}(t) - \boldsymbol{V}(t)\|_{L^2} + \|\boldsymbol{V}(t) - \boldsymbol{v}^{(2)}(t)\|_{L^2} \to 0,$$

and

$$\|\boldsymbol{w}^{(1)}(t) - \boldsymbol{w}^{(2)}(t)\|_{L^2} \leq \|\boldsymbol{w}^{(1)}(t) - \boldsymbol{W}(t)\|_{L^2} + \|\boldsymbol{W}(t) - \boldsymbol{w}^{(2)}(t)\|_{L^2} \to 0.$$

□

4.5 Concluding Remarks

We have shown that, in the language of the reformulated equations, solutions $(\boldsymbol{V}, \boldsymbol{W})$ of the data assimilation equations will converge to the corresponding reference solution $(\boldsymbol{v}, \boldsymbol{w})$ in L^2, even if measurements are only taken for only one of $\boldsymbol{v}$ and $\boldsymbol{w}$. This equates to having to take measurements on either $\boldsymbol{u} + \boldsymbol{b}$ or $\boldsymbol{u} - \boldsymbol{b}$. Could one prove that it is sufficient to collect data on just $\boldsymbol{u}$ or just $\boldsymbol{b}$ and still get convergence, similar to the result for the reformulated variables?

If one were to consider collecting data only on the magnetic field, $\boldsymbol{b}$, then the problem is evident when we take $\boldsymbol{b}(t) \equiv \boldsymbol{B}(t) \equiv \boldsymbol{g} \equiv 0$ for all $t \geq 0$, because we then have $\boldsymbol{u}$ and $\boldsymbol{U}$ satisfying the Navier-Stokes equations with different initial conditions and no data assimilation. Hence, there is an asymmetry between the original system and the reformulated system.

The answer to the question for collecting data on the velocity field, $\boldsymbol{u}$, is open. However, as we demonstrated that the algorithm works with knowledge of only the sum of measurements on $\boldsymbol{u}$ and $\boldsymbol{b}$, it may be that the knowledge of the velocity field is what makes this work, and so a $\boldsymbol{u}$-measurement only algorithm is hopeful. However, besause it seems we should not be able to prove the convergence of a $\boldsymbol{b}$-measurement only algorithm, and the Elsässer variable formulation does not distinguish between $\boldsymbol{u}$ and $\boldsymbol{b}$, a proof of a $\boldsymbol{u}$-measurement only algorithm would have to be in terms of the original variables.

Chapter 5: Numerical efficacy study of data assimilation for the 2D magnetohydrodynamic equations

5.1 Introduction and Theory

It is our current goal to test the algorithms we considered in Chapter 4 to see how well they perform in practice. We are interested in approximating a reference solution of the MHD equations for which we have measurement data, so, for the sake of transparency, we will consider the MHD equations in terms of the original variables and before non-dimensionalizing, i.e.,

System 5.1.1 (MHD)

$$\partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + \tfrac{1}{\rho}\nabla\left(p + \tfrac{1}{2\mu_0}\left|\boldsymbol{b}\right|^2\right) = \tfrac{1}{\rho\mu_0}(\boldsymbol{b}\cdot\nabla)\,\boldsymbol{b} + \boldsymbol{f}, \tag{5.1a}$$

$$\partial_t \boldsymbol{b} - \lambda\Delta\boldsymbol{b} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{b} + \nabla q = (\boldsymbol{b}\cdot\nabla)\,\boldsymbol{u} + \boldsymbol{g}, \tag{5.1b}$$

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{b} = 0, \tag{5.1c}$$

on the domain $\Omega = [0, 2\pi]^2$, equipped with periodic boundary conditions.

Previously we considered a general interpolant operator which satisfied either (4.3) or (4.4). We now restrict our attention to the case that I_h is the projection onto the low modes, P_N, defined as follows: if $\boldsymbol{u}$ has the Fourier expansion

$$\boldsymbol{u}(t,x) = \sum_{k\in\mathbb{Z}^2} \hat{\boldsymbol{u}}(t,k)e^{ik\cdot x},$$

then

$$\mathrm{P}_N\,\boldsymbol{u}(t,x) = \sum_{|k|\le N} \hat{\boldsymbol{u}}(t,k)e^{ik\cdot x}.$$

So in this case, $h = \frac{1}{N}$, and we have that for $\boldsymbol{u} \in H^1$,

$$\|\,\mathrm{P}_N\,\boldsymbol{u} - \boldsymbol{u}\|_{L^2}^2 = \sum_{|k|>N} |\hat{\boldsymbol{u}}(t,k)|^2 \le \tfrac{1}{N^2} \sum_{|k|>N} |k|^2\,|\hat{\boldsymbol{u}}(t,k)|^2 \le \tfrac{1}{N^2}\|\nabla u\|_{L^2}^2.$$

Therefore $\|\,\mathrm{P}_N\,\boldsymbol{u} - \boldsymbol{u}\|_{L^2} \to 0$ as $N \to \infty$, and we have that P_N satisfies (4.3):

$$\|\,\mathrm{P}_N\,\boldsymbol{u} - \boldsymbol{u}\|_{L^2} \lesssim h\|\nabla u\|_{L^2}. \tag{5.2}$$

We will also consider Algorithms 4.2.2, 4.2.3, and 4.2.4 in terms of the original variables:

Algorithm 5.1.2 *Solve*

$$\partial_t\boldsymbol{U} - \nu\Delta\boldsymbol{U} + (\boldsymbol{U}\cdot\nabla)\,\boldsymbol{U} - \tfrac{1}{\rho\mu_0}\,(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B} = -\tfrac{1}{\rho}\nabla\tilde{p} + \boldsymbol{f} + \mu\,\mathrm{P}_N(\boldsymbol{u}-\boldsymbol{U}),$$

$$\partial_t\boldsymbol{B} - \lambda\Delta\boldsymbol{B} + (\boldsymbol{U}\cdot\nabla)\,\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{U} = -\nabla q + \boldsymbol{g} + \mu\,\mathrm{P}_N(\boldsymbol{b}-\boldsymbol{B}),$$

$$\nabla\cdot\boldsymbol{U} = 0, \nabla\cdot\boldsymbol{B} = 0,$$

for $(\boldsymbol{U},\boldsymbol{B})$ *with the initial condition* $\boldsymbol{U}(0), \boldsymbol{B}(0) \equiv 0$*.*

Algorithm 5.1.3 *Solve*

$$\partial_t U_1 - \nu \Delta U_1 + (\boldsymbol{U}\cdot\nabla) U_1 - \tfrac{1}{\rho\mu_0}(\boldsymbol{B}\cdot\nabla) B_1 = -\tfrac{1}{\rho}\partial_x \tilde{p} + f_1 + \mu\, \mathrm{P}_N(u_1 - U_1),$$

$$\partial_t U_2 - \nu \Delta U_2 + (\boldsymbol{U}\cdot\nabla) U_2 - \tfrac{1}{\rho\mu_0}(\boldsymbol{B}\cdot\nabla) B_2 = -\tfrac{1}{\rho}\partial_y \tilde{p} + f_2,$$

$$\partial_t B_1 - \lambda \Delta B_1 + (\boldsymbol{U}\cdot\nabla) B_1 - (\boldsymbol{B}\cdot\nabla) U_1 = -\partial_x q + g_1 + \mu\, \mathrm{P}_N(b_1 - B_1),$$

$$\partial_t B_2 - \lambda \Delta B_2 + (\boldsymbol{U}\cdot\nabla) B_2 - (\boldsymbol{B}\cdot\nabla) U_2 = -\partial_y q + g_2,$$

$$\nabla\cdot\boldsymbol{U} = 0, \nabla\cdot\boldsymbol{B} = 0,$$

for $(\boldsymbol{U}, \boldsymbol{B})$ *with the initial condition* $\boldsymbol{U}(0), \boldsymbol{B}(0) \equiv 0$.

Algorithm 5.1.4 *Solve*

$$\partial_t \boldsymbol{U} - \nu \Delta \boldsymbol{U} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{U} - \tfrac{1}{\rho\mu_0}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} = -\tfrac{1}{\rho}\nabla\tilde{p} + \boldsymbol{f} + \tfrac{1}{2}\mu\, \mathrm{P}_N(\boldsymbol{u} + \tfrac{1}{\sqrt{\rho\mu_0}}\boldsymbol{b} - \boldsymbol{U} - \tfrac{1}{\sqrt{\rho\mu_0}}\boldsymbol{B}),$$

$$\partial_t \boldsymbol{B} - \lambda \Delta \boldsymbol{B} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{U} = -\nabla q + \boldsymbol{g} + \tfrac{1}{2}\mu\, \mathrm{P}_N(\boldsymbol{u} + \tfrac{1}{\sqrt{\rho\mu_0}}\boldsymbol{b} - \boldsymbol{U} - \tfrac{1}{\sqrt{\rho\mu_0}}\boldsymbol{B}),$$

$$\nabla\cdot\boldsymbol{U} = 0, \nabla\cdot\boldsymbol{B} = 0,$$

for $(\boldsymbol{U}, \boldsymbol{B})$ *with the initial condition* $\boldsymbol{U}(0), \boldsymbol{B}(0) \equiv 0$.

In Chapter 4, we showed that if

$$h^2\mu \lesssim \min\{\nu, \lambda\}, \tag{5.3}$$

and

$$\mu \gtrsim \tfrac{1+\min\{\nu,\lambda\}^4}{\min\{\nu,\lambda\}} G^2, \tag{5.4}$$

then the solutions of Algorithm 5.1.2 will converge to the reference solution $(\boldsymbol{u}, \boldsymbol{b})$ at an exponential rate (like $e^{-\mu t}$). We also showed that Algorithm 5.1.3 and Algorithm 5.1.4 will succeed, provided the following greater restriction on μ holds:

$$\mu = \mathcal{O}(G^6). \tag{5.5}$$

The theoretical results above ensure that the algorithms will work provided that the appropriate conditions on μ and h are satisfied, and that the data and solutions to the systems in the algorithms are exact. These rigorous estimates, if sharp, would require a prohibitive amount of data: $N = \frac{1}{h} \sim G$ ((5.3) and (5.4)) or worse $N \sim G^3$ ((5.3) and (5.5)). We want to demonstrate that the algorithms are effective using data that is much more coarse than the estimates require.

5.2 The computational setting

All of our computations were performed on the supercomputer Karst at Indian University, using dedalus, an open source pseudo spectral package (see `http://dedalus-project.org/`). An implicit/explicit Runga Kutta 222 timestepping scheme was used, where the linear terms were solved explicitly and the nonlinear terms implicitly.

5.2.1 Reference Solution

To test the data assimilation algorithms, we first compute a reference solution which we then try to recover using only coarse projections. The desired reference solution should exhibit some nontrivial time-dependent behavior so as to adequately test the performance of the algorithm. In addition, for practical matters, we would like to have a relatively simple force, which can be accurately represented in low resolution simulations. For this reason, we choose a two mode force for each equation, which generates some interesting dynamics.

Specifically, we define the forces, $\boldsymbol{f}, \boldsymbol{g} : \Omega \to \mathbb{R}^2$, by

$$\boldsymbol{f}(x,y) = 2Re\left(\begin{bmatrix} 2+2i \\ \\ (2+2i)/2 \end{bmatrix} e^{i(x-2y)} + \begin{bmatrix} -6 \\ \\ 0 \end{bmatrix} e^{i(3y)}\right)/M_f, \tag{5.6}$$

$$\boldsymbol{g}(x,y) = 2Re\left(\begin{bmatrix} 4-3i \\ \\ -2(4-3i)/3 \end{bmatrix} e^{i(2x+3y)} + \begin{bmatrix} -3+7i \\ \\ (-3+7i)/5 \end{bmatrix} e^{i(x-5y)}\right)/M_g, \tag{5.7}$$

where M_f and M_g are constants chosen so that $\|f\|_{L^2} = \|g\|_{L^2} = 10$. The modes were chosen essentially arbitrarily, as were the first components of the coefficients (the second components were then chosen to make the forces divergence free).

We also need to choose appropriate values for ν and λ to produce an interesting reference solution. We expect the flow to become more turbulent the smaller we take ν and λ. However, decreasing $\min\{\nu, \lambda\}$ may necessitate increasing the resolution of the computational grid, as well as taking a smaller time step. As a compromise we take

$$\nu = \lambda = .01.$$

With the forces in (5.6) and (5.7), this yields the Grashof number

$$G = 10^5.$$

For simplicity, we set

$$\rho = \mu_0 = 1.$$

With the parameters and forces given, we find the solution is sufficiently resolved when computing on the Fourier grid $[-128, 128] \times [-128, 128]$ and using a $2/3$ dealiasing factor, and taking the timestep $dt = .0001$. Figure 5.1 shows some properties of the computed reference solution for $t \in [0, 729.92]$, including the spectrum, where for a given time interval $[t_0, t_0 + T]$, we define the spectrum, $S : [0, \infty) \to [0, \infty)$, by

$$S(r) = \frac{1}{T} \int_{t_0}^{t_0+T} \sum_{r-\frac{1}{2} \leq |k| < r+\frac{1}{2}} |\hat{\boldsymbol{u}}(t,k)|^2 + \left|\hat{\boldsymbol{b}}(t,k)\right|^2.$$

Note that by Parseval's identity, the sum in the definition of the spectrum is less than $\|(\boldsymbol{u}(t), \boldsymbol{b}(t))\|_{L^2}^2$, where we define $\|(\boldsymbol{u}, \boldsymbol{b})\|$ for any norm $\|\cdot\|$ as

$$\|(\boldsymbol{u}, \boldsymbol{b})\| := \sqrt{\|\boldsymbol{u}\|^2 + \|\boldsymbol{b}\|^2}. \tag{5.8}$$

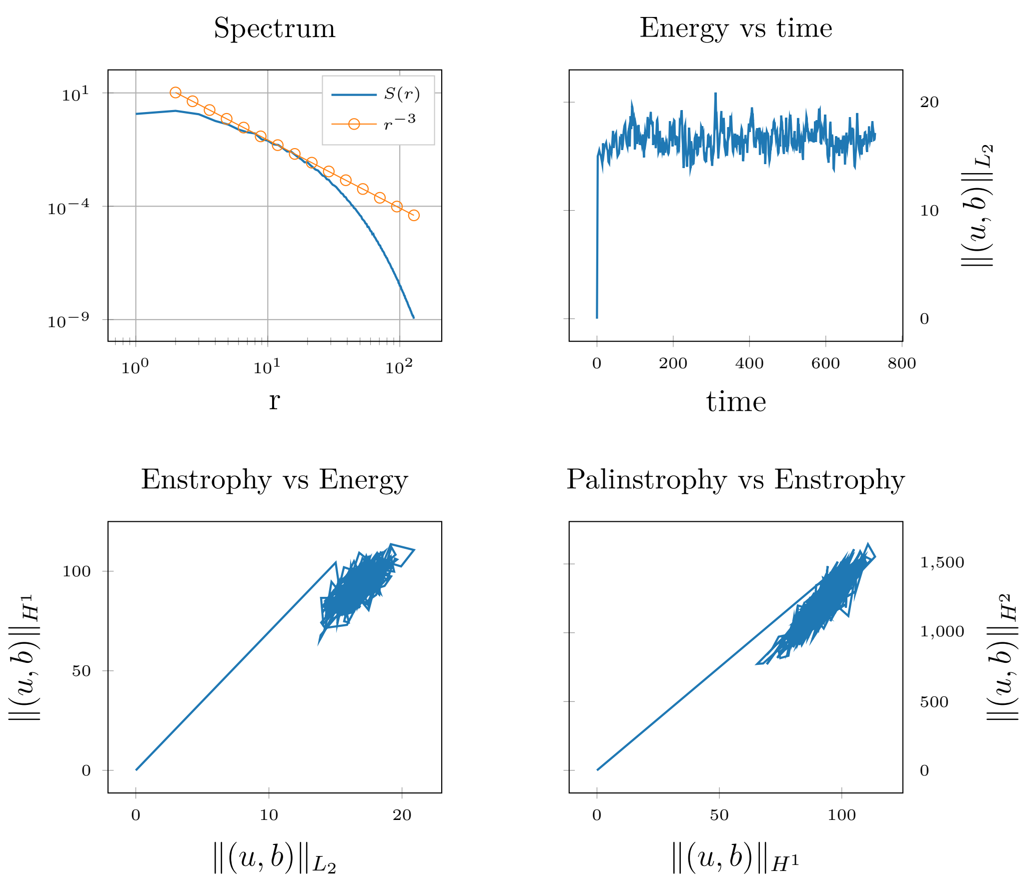

Figure 5.1: Properties of the reference solution on the time interval $[0, 729.92]$

Judging from the tail of the spectrum, we see that the solution seems to be resolved, and that the inertial range is within our computational resolution. Looking at the chaotic but cyclical behavior shown in the Energy vs time graphs, as well as the Enstrophy vs Energy and Palinstrophy vs Enstrophy graphs, we are satisfied that by the time $t = 729.92$ the solution is well past any transient period and is approximating a physical flow.

Figure 5.2 shows the curl of the velocity and magnetic fields of the computed reference solution at time $t = 729.92$, when the data assimilation starts. We see that

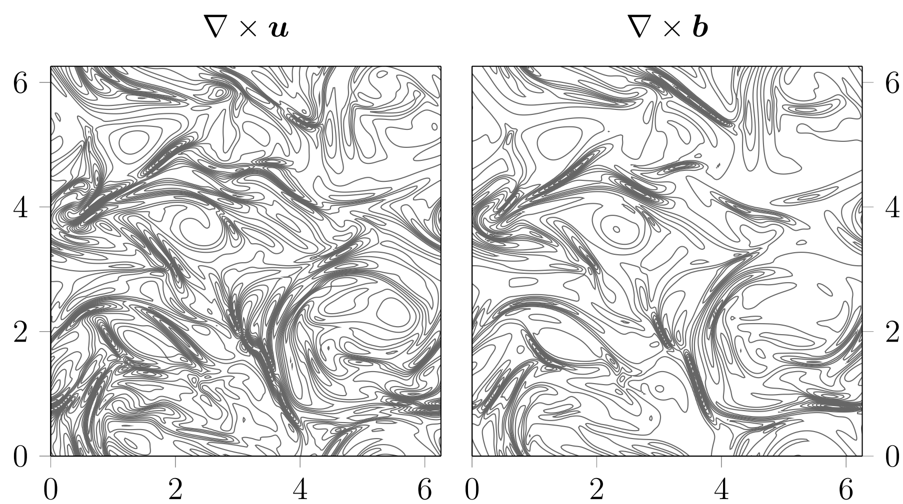

Figure 5.2: Contour lines of the curl of the computed reference solution at time $t = 729.92$.

there are several small eddies and complicated structures for the data assimilation algorithms to attempt to capture.

5.3 Results

All of the following data assimilation simulations were performed starting from $t_0 = 729.92$, with initial data for the algorithm equal to 0, obtaining $(\boldsymbol{U}, \boldsymbol{B})$ and simultaneously computing $(\boldsymbol{u}(t), \boldsymbol{b}(t))$ for $t > t_0$. We compare the resulting evolutions to see how the algorithm performed in terms of the relative error, where we define the relative error as

$$\left(\|\boldsymbol{U}(t) - \boldsymbol{u}(t)\|_{L^2}^2 / \|\boldsymbol{u}(t)\|_{L^2}^2 + \|\boldsymbol{B}(t) - \boldsymbol{b}(t)\|_{L^2}^2 / \|\boldsymbol{b}(t)\|_{L^2}^2\right)^{\frac{1}{2}}. \tag{5.9}$$

For each simulation, we choose an algorithm to use to generate the approximation, (U, B), and set the projection radius N (and so the number of modes we are assuming to have data on will be $(2N+1)^2$), as well as μ, which amplifies the feedback.

We would like to take μ large, because the error decreases like $e^{-c\mu t}$, hence a larger μ increases the convergence rate and therefore reduces the amount of simulation time required. However, if μ becomes too large, it destabilizes the solution over small scales (small scales meaning finer resolutions than the larger, coarser scales captured by I_h) by mixing in the feedback, and the analysis suggests that larger values for μ require smaller values for h. This feedback is compensated by the dissipation, provided condition (5.3) holds.

Hence, given a value for N (and thus h), we have an upper bound for μ in the sense that, with $\nu = \lambda = .01$, (5.3) gives $\mu \lesssim N^2/100$. However, it is unclear what the appropriate value for μ is because of the constants involved. Figure 5.3 shows the minimum error attained during several short simulations of Algorithms 5.1.2,5.1.4, and 5.1.3, for different values of μ and N.

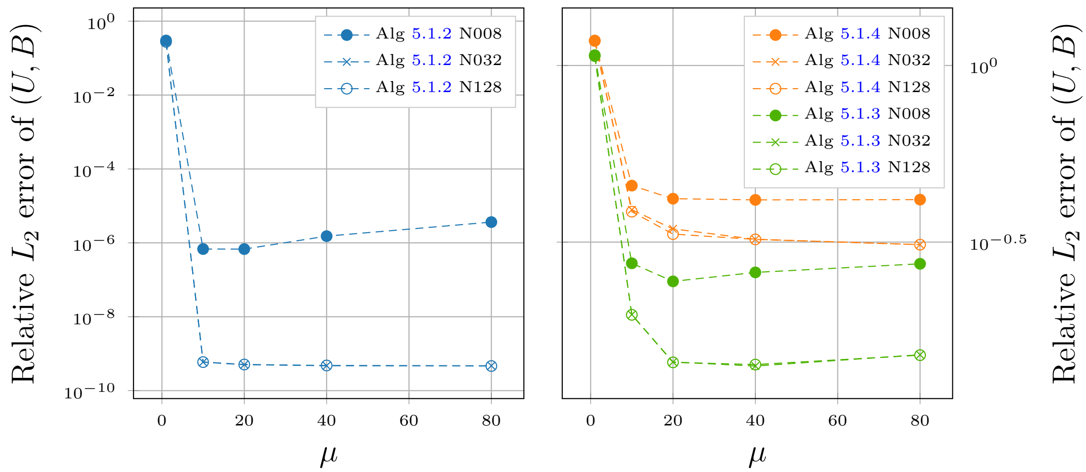

Figure 5.3: Numerical dependence of the data assimilation error on μ and N. The error shown is the minimum error acheived on the time interval $[t_0, t_0 + 5]$, where $t_0 = 729.9$.

We see that the benefit of increasing μ (which is initially great, as the error is exponentially decreasing with μ) quickly diminishes in all cases. Furthermore, in each case increasing N from 32 to 128 does not seem to enable us to increase μ enough to provide a substantial increase in the convergence rate. Based on these results, we decide that $\mu = 20$ is a reasonable value to use in our longer simulations.

Remark 5.3.1 *Reading more into this last observation, one possible explanation is that because our computational grid only supports Fourier modes with magnitude less than 129, it is possible that we cannot take advantage of larger values of μ (even when N is also large) because we are not computing scales small enough for the dissipation to have an effect on the added instabilities introduced by the feedback control.*

After deciding what value to choose for μ, we perform long-time simulations. Figure 5.4 shows the convergence results we obtain for Algorithms 5.1.2, 5.1.3, and 5.1.4, when $N = 128$. As expected, Algorithm 5.1.2 performs much better than Al-

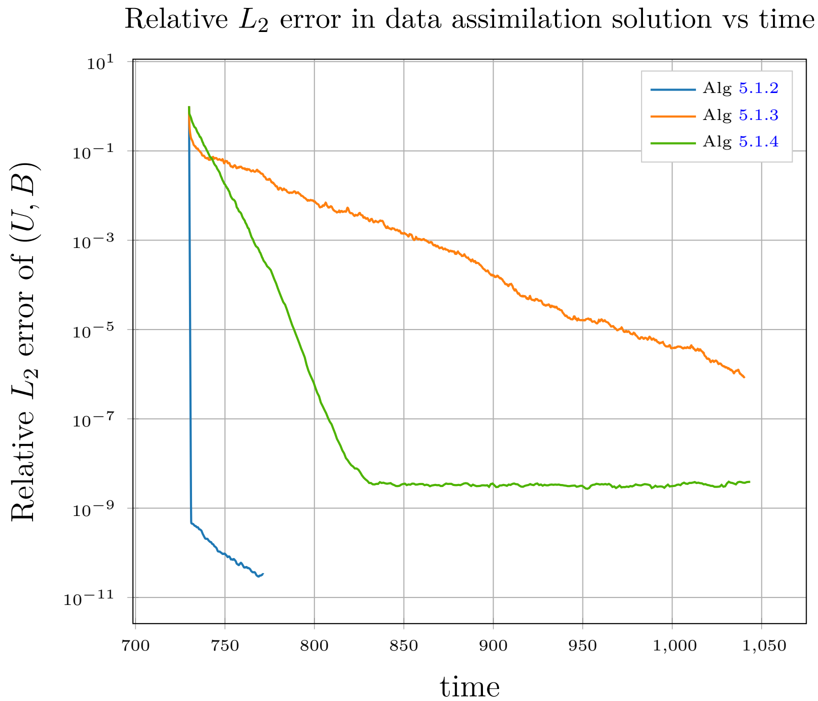

Figure 5.4: Convergence results with projection radius $N = 128$ and damping $\mu = 20$.

gorithm 5.1.3 and Algorithm 5.1.4. Perhaps surprisingly, Algorithm 5.1.4 performs noticeably better than Algorithm 5.1.3. This suggests that the Elssässer variable transformation is more than an algebraic simplification, as it seems to capture some important aspects of magnetohydrodynamics. Algorithm 5.1.3, though it does not perform as well, still shows near monotonic convergence, and it is reasonable to think that if the simulation time were extended, the error would reach 10^{-9}, as do the other algorithms.

5.4 Outside of Theory

In addition to the algorithms we considered in Section 5.3 and Chapter 4, we consider the following algorithms and test their performance numerically, using the same reference solution. Unlike the previous algorithms we considered, the next algorithms have no supporting theory with which to compare. However, the results we obtain here can serve to inform future work.

The most interesting situations of having measurements on only some of the fields are when data is only collected on either the velocity field or the magnetic field alone. We will consider both situations.

In the following algorithm, we collect data on only the velocity field.

Algorithm 5.4.1 *Solve*

$$\partial_t \boldsymbol{U} - \nu \Delta \boldsymbol{U} + (\boldsymbol{U}\cdot\nabla)\,\boldsymbol{U} - \tfrac{1}{\rho\mu_0}\,(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B} = -\tfrac{1}{\rho}\nabla\tilde{p} + \boldsymbol{f} + \mu\,\mathrm{P}_N(\boldsymbol{u}-\boldsymbol{U}),$$

$$\partial_t \boldsymbol{B} - \lambda \Delta \boldsymbol{B} + (\boldsymbol{U}\cdot\nabla)\,\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{U} = -\nabla q + \boldsymbol{g},$$

$$\nabla\cdot\boldsymbol{U} = 0, \nabla\cdot\boldsymbol{B} = 0,$$

for $(\boldsymbol{U}, \boldsymbol{B})$ *with the initial condition* $\boldsymbol{U}(0), \boldsymbol{B}(0) \equiv 0$*.*

Similarly, the next algorithm requires collecting data on only the magnetic field.

Algorithm 5.4.2 *Solve*

$$\partial_t \boldsymbol{U} - \nu \Delta \boldsymbol{U} + (\boldsymbol{U}\cdot\nabla)\,\boldsymbol{U} - \tfrac{1}{\rho\mu_0}\,(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B} = -\tfrac{1}{\rho}\nabla\tilde{p} + \boldsymbol{f},$$

$$\partial_t \boldsymbol{B} - \lambda \Delta \boldsymbol{B} + (\boldsymbol{U}\cdot\nabla)\,\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{U} = -\nabla q + \boldsymbol{g} + \mu\,\mathrm{P}_N(\boldsymbol{b}-\boldsymbol{B}),$$

$$\nabla\cdot\boldsymbol{U} = 0, \nabla\cdot\boldsymbol{B} = 0,$$

for $(\boldsymbol{U}, \boldsymbol{B})$ *with the initial condition* $\boldsymbol{U}(0), \boldsymbol{B}(0) \equiv 0$.

In addition, we can consider how much of an improvement using data from 3 out of the 4 fields might yield compared to only using data from 2 fields. With this in mind, we define the following two algorithms.

Algorithm 5.4.3 *Solve*

$$\partial_t U_1 - \nu \Delta U_1 + (\boldsymbol{U} \cdot \nabla) U_1 - \tfrac{1}{\rho \mu_0} (\boldsymbol{B} \cdot \nabla) B_1 = -\tfrac{1}{\rho} \partial_x \tilde{p} + f_1 + \mu \, \mathrm{P}_N(u_1 - U_1),$$

$$\partial_t U_2 - \nu \Delta U_2 + (\boldsymbol{U} \cdot \nabla) U_2 - \tfrac{1}{\rho \mu_0} (\boldsymbol{B} \cdot \nabla) B_2 = -\tfrac{1}{\rho} \partial_y \tilde{p} + f_2 + \mu \, \mathrm{P}_N(u_2 - U_2),$$

$$\partial_t B_1 - \lambda \Delta B_1 + (\boldsymbol{U} \cdot \nabla) B_1 - (\boldsymbol{B} \cdot \nabla) U_1 = -\partial_x q + g_1 + \mu \, \mathrm{P}_N(b_1 - B_1),$$

$$\partial_t B_2 - \lambda \Delta B_2 + (\boldsymbol{U} \cdot \nabla) B_2 - (\boldsymbol{B} \cdot \nabla) U_2 = -\partial_y q + g_2,$$

$$\nabla \cdot \boldsymbol{U} = 0, \nabla \cdot \boldsymbol{B} = 0,$$

for $(\boldsymbol{U}, \boldsymbol{B})$ *with the initial condition* $\boldsymbol{U}(0), \boldsymbol{B}(0) \equiv 0$.

Algorithm 5.4.4 *Solve*

$$\partial_t U_1 - \nu \Delta U_1 + (\boldsymbol{U} \cdot \nabla) U_1 - \tfrac{1}{\rho \mu_0} (\boldsymbol{B} \cdot \nabla) B_1 = -\tfrac{1}{\rho} \partial_x \tilde{p} + f_1 + \mu \, \mathrm{P}_N(u_1 - U_1),$$

$$\partial_t U_2 - \nu \Delta U_2 + (\boldsymbol{U} \cdot \nabla) U_2 - \tfrac{1}{\rho \mu_0} (\boldsymbol{B} \cdot \nabla) B_2 = -\tfrac{1}{\rho} \partial_y \tilde{p} + f_2,$$

$$\partial_t B_1 - \lambda \Delta B_1 + (\boldsymbol{U} \cdot \nabla) B_1 - (\boldsymbol{B} \cdot \nabla) U_1 = -\partial_x q + g_1 + \mu \, \mathrm{P}_N(b_1 - B_1),$$

$$\partial_t B_2 - \lambda \Delta B_2 + (\boldsymbol{U} \cdot \nabla) B_2 - (\boldsymbol{B} \cdot \nabla) U_2 = -\partial_y q + g_2 + \mu \, \mathrm{P}_N(b_2 - B_2),$$

$$\nabla \cdot \boldsymbol{U} = 0, \nabla \cdot \boldsymbol{B} = 0,$$

for $(\boldsymbol{U}, \boldsymbol{B})$ *with the initial condition* $\boldsymbol{U}(0), \boldsymbol{B}(0) \equiv 0$.

Figure 5.5 shows the convergence results we obtain for the above four algorithms, using the same reference solution as before and $N = 128$, $\mu = 20$ (as

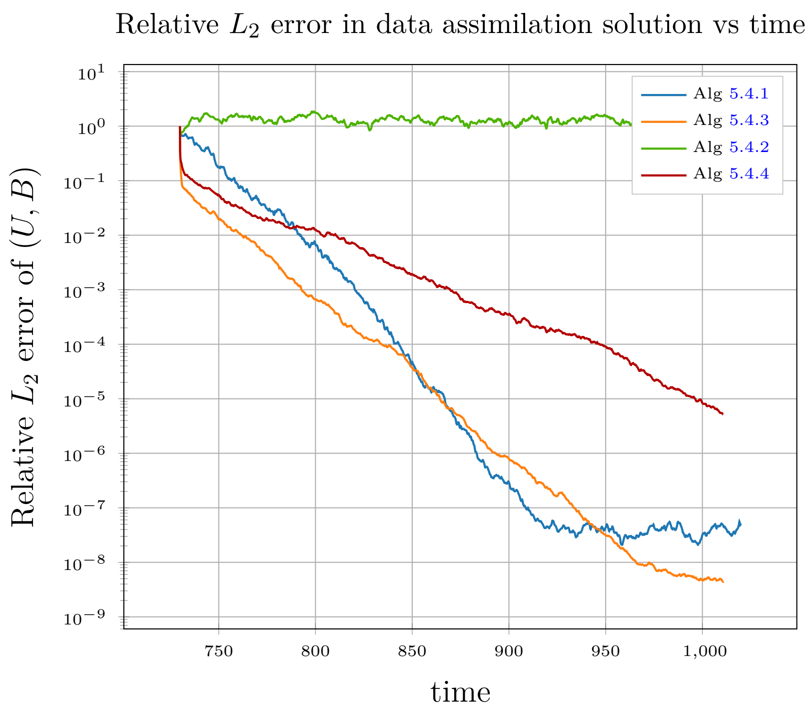

Figure 5.5: Convergence results with projection radius $N = 128$ and damping $\mu = 20$.

before). We see that Algorithm 5.4.2 shows no sign of converging. Also, while Algorithm 5.4.4 shows little improvement over Algorithm 5.1.3, Algorithm 5.4.3 performs much better, although still worse than Algorithm 5.1.4. This is somewhat surprising, as Algorithm 5.1.4 performs better while requiring less measurements.

We also note that Algorithm 5.4.1 seems to be converging.

5.5 Conclusions and Interpretations

We have seen that data assimilation by nudging for the MHD is effective, and works extremely well when measurement data is available on all the fields.

Our results indicate that the Elsässer variable formulation is the most efficient way to use measurement data (without measuring every field), which leads us to suspect it may have deeper meaning.

All of the abridged algorithms seem to work in our study, with the exception of the one which did not incorporate any measurements of the velocity field. This seems to indicate that an algorithm which uses only measurements on the magnetic field is impossible, as was conjectured in Chapter 4.

Our results also indicate that an algorithm which only incorporates velocity field measurements is hopeful.

Appendix A: Technical Proofs

A.1 Chapter 2 Technical Proofs

Proof of Proposition 2.2.3 Observe that for $\mathbf{h}+\mathbf{j}=\mathbf{k},\ \mathbf{h},\mathbf{j},\mathbf{k}\in\widetilde{\mathbb{Z}}^d$, we have

$$|\mathbf{k}|^r \le 2^{r-1}(|\mathbf{h}|^r + |\mathbf{j}|^r).$$

Thus,

$$\begin{aligned}
&|\langle B(\boldsymbol{u},\boldsymbol{v}), A^r e^{2\beta A^{1/2}}\boldsymbol{w}\rangle| \\
&\quad \le \sum_{\mathbf{h}+\mathbf{j}-\mathbf{k}=0} |\widehat{\boldsymbol{u}}(\mathbf{h})||\mathbf{j}||\widehat{\boldsymbol{v}}(\mathbf{j})||\mathbf{k}|^{2r}|\widehat{\boldsymbol{w}}(\mathbf{k})|e^{2\beta|\mathbf{k}|} \\
&\quad \le 2^{r-1}\sum_{\mathbf{h}+\mathbf{j}-\mathbf{k}=0} |\mathbf{h}|^r|\widehat{\boldsymbol{u}}(\mathbf{h})||\mathbf{j}||\widehat{\boldsymbol{v}}(\mathbf{j})||\mathbf{k}|^{r}|\widehat{\boldsymbol{w}}(\mathbf{k})|e^{2\beta|\mathbf{k}|} \\
&\qquad + 2^{r-1}\sum_{\mathbf{h}+\mathbf{j}-\mathbf{k}=0} |\widehat{\boldsymbol{u}}(\mathbf{h})||\mathbf{j}||\mathbf{j}|^r|\widehat{\boldsymbol{v}}(\mathbf{j})||\mathbf{k}|^{r}|\widehat{\boldsymbol{w}}(\mathbf{k})|e^{2\beta|\mathbf{k}|}. \qquad \text{(A.1)}
\end{aligned}$$

Because $\mathbf{j},\mathbf{h},\mathbf{k}\neq 0$, we have $\min\{|\mathbf{j}|,|\mathbf{h}|,|\mathbf{k}|\}\ge 1$ and therefore,

$$|\mathbf{j}| \le |\mathbf{h}|+|\mathbf{k}| \le 2|\mathbf{h}||\mathbf{k}| \text{ which implies } |\mathbf{j}|^{\frac{1}{2}} \lesssim |\mathbf{h}|^{\frac{1}{2}}|\mathbf{k}|^{\frac{1}{2}}. \qquad \text{(A.2)}$$

From (A.1) and (A.2), we have

$$
\begin{aligned}
&|\langle B(\boldsymbol{u},\boldsymbol{v}), A^r e^{2\beta A^{1/2}}\boldsymbol{w}\rangle| \\
&\qquad \le\ 2^{r-1/2}\sum_{\mathbf{h}+\mathbf{j}-\mathbf{k}=0} e^{\beta|\mathbf{h}|}|\mathbf{h}|^{r+\frac{1}{2}}|\widehat{\boldsymbol{u}}(\mathbf{h})|e^{\beta|\mathbf{j}|}|\mathbf{j}|^{\frac{1}{2}}|\widehat{\boldsymbol{v}}(\mathbf{j})||k|^{r+\frac{1}{2}}|\widehat{\boldsymbol{w}}(\mathbf{k})|e^{\beta|\mathbf{k}|} \\
&\qquad +\quad 2^{r-1/2}\sum_{\mathbf{h}+\mathbf{j}-\mathbf{k}=0} e^{\beta|\mathbf{h}|}|\mathbf{h}|^{\frac{1}{2}}|\widehat{\boldsymbol{u}}(\mathbf{h})|e^{\beta|\mathbf{j}|}|\mathbf{j}|^{r+\frac{1}{2}}|\widehat{\boldsymbol{v}}(\mathbf{j})||\mathbf{k}|^{r+\frac{1}{2}}|\widehat{\boldsymbol{w}}(\mathbf{k})|e^{\beta|\mathbf{k}|}, \\
&\qquad \lesssim 2^r\big(\|A^{\frac{1}{4}}e^{\beta A^{1/2}}\boldsymbol{v}\|_{\mathcal{W}}\|A^{\frac{1}{4}}\boldsymbol{u}\|_{G(\beta,r,2)}\|A^{\frac{1}{4}}\boldsymbol{w}\|_{G(\beta,r,2)} \\
&\qquad + \|A^{\frac{1}{4}}e^{\beta A^{1/2}}\boldsymbol{u}\|_{\mathcal{W}}\|A^{\frac{1}{4}}\boldsymbol{v}\|_{G(\beta,r,2)}\|A^{\frac{1}{4}}\boldsymbol{w}\|_{G(\beta,r,2)}\big), \\
&\qquad \lesssim 2^r c_r\big(\|\boldsymbol{v}\|_{G(\beta,r,2)}\|A^{\frac{1}{4}}\boldsymbol{u}\|_{G(\beta,r,2)}\|A^{\frac{1}{4}}\boldsymbol{w}\|_{G(\beta,r,2)} \\
&\qquad + \|\boldsymbol{u}\|_{G(\beta,r,2)}\|A^{\frac{1}{4}}\boldsymbol{v}\|_{G(\beta,r,2)}\|A^{\frac{1}{4}}\boldsymbol{w}\|_{G(\beta,r,2)}\big), \qquad (A.3)
\end{aligned}
$$

where to obtain (A.3) we used (1.2) with $s := r - \frac{1}{2} > \frac{d}{2}$. We readily obtain

$$|\langle B(\boldsymbol{u},\boldsymbol{u}), A^r e^{2\beta A^{1/2}}\boldsymbol{u}\rangle| \lesssim 2^r c_r\|\boldsymbol{u}\|_{G(\beta,r,2)}\|A^{1/4}\boldsymbol{u}\|^2_{G(\beta,r,2)}. \qquad (A.4)$$

□

Proof of Proposition 2.6.2 Observe that for $\mathbf{h}_1+\cdots+\mathbf{h}_n+\mathbf{k}=0,\ \mathbf{h}_i,\mathbf{k}\in\widetilde{\mathbb{Z}}^d$, by the triangle inequality and Proposition 2.6.1, we have

$$|\mathbf{k}|^r \le n^r(|\mathbf{h}_1|^r+\cdots+|\mathbf{h}_n|^r). \qquad (A.5)$$

Denote

$$I \subset \widetilde{\mathbb{Z}}^{d+1}, I = \{(\mathbf{h}_1,\cdots,\mathbf{h}_n,\mathbf{k}) : \mathbf{h}_1+\cdots+\mathbf{h}_n+\mathbf{k}=0, \mathbf{h}_i,\mathbf{k}\in\widetilde{\mathbb{Z}}^d\}.$$

Thus,

$$
\begin{aligned}
|\langle Tu^n, A^r e^{2\beta A^{1/2}} u\rangle| &\lesssim \sum_I |u(\mathbf{h}_1)| \cdots |u(\mathbf{h}_n)||u(\mathbf{k})||\mathbf{k}|^{2r+1} e^{2\beta|\mathbf{k}|} \\
&\lesssim n^r \Bigg(\sum_I |\mathbf{h}_1|^r e^{\beta|\mathbf{h}_1|} |u(\mathbf{h}_1)| \cdots e^{\beta|\mathbf{h}_n|} |u(\mathbf{h}_n)||u(\mathbf{k})||\mathbf{k}|^{r+1} e^{\beta|\mathbf{k}|} \\
&\qquad + \cdots + \sum_I e^{\beta|\mathbf{h}_1|} |u(\mathbf{h}_1)| \cdots |\mathbf{h}_n|^r e^{\beta|\mathbf{h}_n|} |u(\mathbf{h}_n)||u(\mathbf{k})||\mathbf{k}|^{r+1} \mathbf{e}^{\beta|\mathbf{k}|} \Bigg),
\end{aligned}
\tag{A.6}
$$

where to obtain (A.6), we used (A.5) as well as the triangle inequality $|\mathbf{k}| \leq \sum_i |\mathbf{h}_i|$. Because $\min\{|\mathbf{h}_1|, \cdots, |\mathbf{h}_n|, |\mathbf{k}|\} \geq 1$, we have

$$
|\mathbf{k}| \leq \sum_i |\mathbf{h}_i| \leq n|\mathbf{h}_1| \cdots |\mathbf{h}_n|, \text{ which implies } |\mathbf{k}|^{\frac{1}{2}} \lesssim n^{1/2} |\mathbf{h}_1|^{\frac{1}{2}} \cdots |\mathbf{h}_n|^{\frac{1}{2}}.
$$

Consequently, from (A.6), we conclude

$$
\begin{aligned}
&|\langle Tu^n, A^r e^{2\beta A^{1/2}} u\rangle| \\
&\quad \lesssim n^{r+\frac{1}{2}} \Bigg(\sum_I |\mathbf{h}_1|^{r+\frac{1}{2}} e^{\beta|\mathbf{h}_1|} |u(\mathbf{h}_1)| \cdots e^{\beta|\mathbf{h}_n|} |\mathbf{h}_n|^{\frac{1}{2}} |u(\mathbf{h}_n)||u(\mathbf{k})||\mathbf{k}|^{r+\frac{1}{2}} e^{\beta|\mathbf{k}|} \\
&\qquad + \cdots + \sum_I e^{\beta|\mathbf{h}_1|} |\mathbf{h}_1|^{\frac{1}{2}} |u(\mathbf{h}_1)| \cdots e^{\beta|\mathbf{h}_n|} |\mathbf{h}_n|^{r+\frac{1}{2}} |u(\mathbf{h}_n)||u(\mathbf{k})||\mathbf{k}|^{r+\frac{1}{2}} e^{\beta|\mathbf{k}|} \Bigg) \\
&\quad \lesssim n^{r+\frac{3}{2}} (c_r)^{n-1} \|A^{\frac{1}{4}} u\|^2_{G(\beta,r,2)} \|u\|^{n-1}_{G(\beta,r,2)},
\end{aligned}
\tag{A.7}
$$

where the last inequality follows exactly as in the proof of (A.3). This immediately yields (2.31). □

A.2 Chapter 3 Technical Proofs

A.2.1 Leray Energy Inequality

Taking the L^2 inner product of (3.3a) with $\boldsymbol{v}$ we get

$$\frac{d}{dt}\int \|\boldsymbol{v}\|^2 - \int \Delta \boldsymbol{v}\cdot\boldsymbol{v} = 0, \tag{A.8}$$

since one can show that $\int (\boldsymbol{w}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{v} = 0$ and $\int \nabla\left(p - \frac{1}{2}|\frac{1}{2}(\boldsymbol{v}-\boldsymbol{w})|^2\right)\cdot\boldsymbol{v} = 0$ using integration by parts and the fact that $\boldsymbol{v}$ is divergence free. Therefore,

$$\frac{d}{dt}\int \|\boldsymbol{v}\|^2 + \int \Lambda\boldsymbol{v}\cdot\Lambda\boldsymbol{v} = 0, \tag{A.9}$$

i.e.

$$\frac{d}{dt}\|\boldsymbol{v}\|^2 + \|\boldsymbol{v}\|^2_{\dot{H}^1_2} = 0.$$

So, integrating in time, for any $t \geq 0$ we get

$$\|\boldsymbol{v}(t)\|^2 + \int_0^t \|\boldsymbol{v}\|^2_{\dot{H}^1_2} = \|\boldsymbol{v}_0\|^2 \tag{A.10}$$

Similarly,

$$\|\boldsymbol{w}(t)\|^2 + \int_0^t \|\boldsymbol{w}\|^2_{\dot{H}^1_2} = \|\boldsymbol{w}_0\|^2 \tag{A.11}$$

A.2.2 Persistence of Vorticity in L^1

Proof of Theorem 3.2.5 We will show the result for $\boldsymbol{\tau}$, the proof for $\boldsymbol{\psi}$ being identical. Considering the equation for $\boldsymbol{\tau}$ in the vorticity formulation and proceeding as in Constantin and Fefferman [39], if we let $\boldsymbol{\xi} = \dfrac{\boldsymbol{\tau}}{|\boldsymbol{\tau}|}$ and take the inner product with $\boldsymbol{\xi}$ we get:

$$\partial_t|\boldsymbol{\tau}| - \Delta\boldsymbol{\tau}\cdot\boldsymbol{\xi} + (\boldsymbol{w}\cdot\nabla)|\boldsymbol{\tau}| = (\nabla\boldsymbol{v} * \nabla\boldsymbol{w})\cdot\boldsymbol{\xi}$$

For each $\epsilon > 0$ let $f_\epsilon : \mathbb{R} \to \mathbb{R}$ so that $f_\epsilon \in C^2(\mathbb{R})$ and has the following properties:

$$f_\epsilon''(x) \geq 0 \ \ \forall x > \delta, \ \ f_\epsilon''(x) = 0 \ \ \forall x \leq \delta, \ \ f_\epsilon'(x) \in [0,1] \ \ \forall x, \ \ x f_\epsilon'(x) = 1 \ \ \forall x > \epsilon,$$

where $0 < \delta < \epsilon$. Then, by multiplying by $f_\epsilon'(|\boldsymbol{\tau}|)$ and integrating over $|\boldsymbol{\tau}| > \epsilon$ we get:

$$\frac{d}{dt}\int f_\epsilon(|\boldsymbol{\tau}|) + \int f_\epsilon''(|\boldsymbol{\tau}|)|\nabla|\boldsymbol{\tau}||^2 + \int |\boldsymbol{\tau}| f_\epsilon'(|\boldsymbol{\tau}|)|\nabla\boldsymbol{\xi}|^2 = \int f_\epsilon'(|\boldsymbol{\tau}|)(\nabla\boldsymbol{v} * \nabla\boldsymbol{w}) \cdot \boldsymbol{\xi}$$

where we have used the facts that by integrating by parts and using the divergence free conditions we have

$$\int f_\epsilon'(|\boldsymbol{\tau}|)(\boldsymbol{w}\cdot\nabla)|\boldsymbol{\tau}| = \int (\boldsymbol{w}\cdot\nabla) f_\epsilon(|\boldsymbol{\tau}|) = -\int (\nabla\cdot\boldsymbol{w}) f_\epsilon(|\boldsymbol{\tau}|) = 0$$

and

$$\int |\boldsymbol{\tau}| f_\epsilon'(|\boldsymbol{\tau}|)|\nabla\boldsymbol{\xi}|^2 = -\int f_\epsilon'(|\boldsymbol{\tau}|)\Delta\boldsymbol{\tau}\cdot\boldsymbol{\xi} - \int f_\epsilon''(|\boldsymbol{\tau}|)|\nabla|\boldsymbol{\tau}||^2.$$

Now, integrating in time, we get:

$$\begin{aligned}
\int f_\epsilon(|\boldsymbol{\tau}(t)|) - \int f_\epsilon(|\boldsymbol{\tau}_0|) + \int_0^t \int |\boldsymbol{\tau}| f_\epsilon'(|\boldsymbol{\tau}|)|\nabla\boldsymbol{\xi}|^2 &\leq \int_0^t \int |f_\epsilon'(|\boldsymbol{\tau}|)||\nabla\boldsymbol{v} * \nabla\boldsymbol{w}||\boldsymbol{\xi}| \\
&\leq \int_0^t \int |\nabla\boldsymbol{v}||\nabla\boldsymbol{w}| \\
&\leq \int_0^t \|\nabla\boldsymbol{v}\|\|\nabla\boldsymbol{w}\| \\
&\leq \left(\int_0^t \|\nabla\boldsymbol{v}\|^2\right)^{\frac{1}{2}} \left(\int_0^t \|\nabla\boldsymbol{w}\|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

So, by the definition of f_ϵ, letting $\epsilon \to 0$ we have:

$$\begin{aligned}
\int |\boldsymbol{\tau}(t)| + \int_0^t \int |\boldsymbol{\tau}||\nabla\boldsymbol{\xi}|^2 &\leq \left(\int_0^t \|\nabla\boldsymbol{v}\|^2\right)^{\frac{1}{2}} \left(\int_0^t \|\nabla\boldsymbol{w}\|^2\right)^{\frac{1}{2}} + \int |\boldsymbol{\tau}_0| \\
&\leq 4\left(\|\boldsymbol{v}_0\|\|\boldsymbol{w}_0\|\right)^{\frac{1}{2}} + \int |\boldsymbol{\tau}_0|.
\end{aligned}$$

where the last inequality follows from (A.10) and (A.11).

Since t was arbitrary,

$$\sup_{t\geq 0}\|\boldsymbol{\tau}(t)\|_1 \leq 4\left(\|\boldsymbol{v}_0\|\|\boldsymbol{w}_0\|\right)^{\frac{1}{2}} + \|\boldsymbol{\tau}_0\|_1.$$

$\square$

A.3 Chapter 4 Technical Proofs

Proof of Proposition 4.2.8 We provide only a formal proof of (4.8) here. A rigorous proof can be carried out by, e.g., first proving the bounds at the level of finite-dimensional Galerkin truncation, and then passing to a limit.

Taking a (formal) inner-product of (3.3a) with $\boldsymbol{v}$, and of (3.3b) with $\boldsymbol{w}$, using (3.3c) and adding the results, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\boldsymbol{v}\|_{L^2}^2 + \|\boldsymbol{w}\|_{L^2}^2\right) + (\alpha-\beta)\left(\|\nabla\boldsymbol{v}\|_{L^2}^2 + \|\nabla\boldsymbol{w}\|_{L^2}^2\right)\\
&\leq \langle \boldsymbol{f},\boldsymbol{v}\rangle + \langle \boldsymbol{g},\boldsymbol{w}\rangle \leq \|\boldsymbol{f}\|_{L^2}\|\boldsymbol{v}\|_{L^2} + \|\boldsymbol{g}\|_{L^2}\|\boldsymbol{w}\|_{L^2}\\
&\leq \tfrac{1}{8\pi^2(\alpha-\beta)}\left(\|\boldsymbol{f}\|_{L^2}^2 + \|\boldsymbol{g}\|_{L^2}^2\right) + \tfrac{(\alpha-\beta)}{2}4\pi^2\left(\|\boldsymbol{v}\|_{L^2}^2 + \|\boldsymbol{w}\|_{L^2}^2\right)\\
&\leq \tfrac{1}{8\pi^2(\alpha-\beta)}\left(\|\boldsymbol{f}\|_{L^2}^2 + \|\boldsymbol{g}\|_{L^2}^2\right) + \tfrac{(\alpha-\beta)}{2}\left(\|\nabla\boldsymbol{v}\|_{L^2}^2 + \|\nabla\boldsymbol{w}\|_{L^2}^2\right),
\end{aligned}$$

where we used the Poincaré inequality and Young's inequality. Therefore, after collecting terms,

$$\frac{d}{dt}\left(\|\boldsymbol{v}\|_{L^2}^2 + \|\boldsymbol{w}\|_{L^2}^2\right) + (\alpha-\beta)\left(\|\nabla\boldsymbol{v}\|_{L^2}^2 + \|\nabla\boldsymbol{w}\|_{L^2}^2\right) \leq \tfrac{1}{4\pi^2(\alpha-\beta)}\left(\|\boldsymbol{f}\|_{L^2}^2 + \|\boldsymbol{g}\|_{L^2}^2\right), \tag{A.12}$$

and by using the Poincaré inequality on the left hand side,

$$\frac{d}{dt}\left(\|\boldsymbol{v}\|_{L^2}^2+\|\boldsymbol{w}\|_{L^2}^2\right)+4\pi^2(\alpha-\beta)\left(\|\boldsymbol{v}\|_{L^2}^2+\|\boldsymbol{w}\|_{L^2}^2\right)\leq \tfrac{1}{4\pi^2(\alpha-\beta)}\left(\|\boldsymbol{f}\|_{L^2}^2+\|\boldsymbol{g}\|_{L^2}^2\right). \tag{A.13}$$

Then by Grönwall's inequality,

$$\begin{aligned}
&\|\boldsymbol{v}(t)\|_{L^2}^2+\|\boldsymbol{w}(t)\|_{L^2}^2\\
&\leq (\|\boldsymbol{v}(0)\|_{L^2}^2+\|\boldsymbol{w}(0)\|_{L^2}^2)e^{-4\pi^2(\alpha-\beta)t}\\
&\quad+\tfrac{1}{16\pi^4(\alpha-\beta)^2}\text{ess sup}_{s\in[0,t]}\left(\|\boldsymbol{f}(s)\|_{L^2}^2+\|\boldsymbol{g}(s)\|_{L^2}^2\right).
\end{aligned} \tag{A.14}$$

Let $t_*>0$ be large enough so that

$$\text{ess sup}_{t\geq t_*}\left(\|\boldsymbol{f}(t)\|_{L^2}^2+\|\boldsymbol{g}(t)\|_{L^2}^2\right)\leq 2\limsup_{t\to\infty}\left(\|\boldsymbol{f}(t)\|_{L^2}^2+\|\boldsymbol{g}(t)\|_{L^2}^2\right), \tag{A.15}$$

and choose $t_0>t_*$ so that

$$(\|\boldsymbol{v}(t_*)\|_{L^2}^2+\|\boldsymbol{w}(t_*)\|_{L^2}^2)e^{-4\pi^2(\alpha-\beta)(t_0-t_*)}\leq \tfrac{3}{8\pi^4(\alpha-\beta)^2}\limsup_{t\to\infty}\left(\|\boldsymbol{f}(t)\|_{L^2}^2+\|\boldsymbol{g}(t)\|_{L^2}^2\right).$$

Then by using Grönwall's inequality again on (A.13) with initial time t_*, we see that for all $t\geq t_0$,

$$\begin{aligned}
\|\boldsymbol{v}(t)\|_{L^2}^2+\|\boldsymbol{w}(t)\|_{L^2}^2 &\leq (\|\boldsymbol{v}(t_*)\|_{L^2}^2+\|\boldsymbol{w}(t_*)\|_{L^2}^2)e^{-4\pi^2(\alpha-\beta)(t-t_*)}\\
&\quad+\tfrac{1}{16\pi^4(\alpha-\beta)^2}\text{ess sup}_{s\in[t_*,t]}\left(\|\boldsymbol{f}(s)\|_{L^2}^2+\|\boldsymbol{g}(s)\|_{L^2}^2\right)\\
&\leq \tfrac{1}{2\pi^4(\alpha-\beta)^2}\limsup_{s\to\infty}\left(\|\boldsymbol{f}(s)\|_{L^2}^2+\|\boldsymbol{g}(s)\|_{L^2}^2\right).
\end{aligned} \tag{A.16}$$

Next, integrating (A.12) on $[t,t+T]$, and using (A.15),

$$\begin{aligned}
&\|\boldsymbol{v}(t+T)\|_{L^2}^2+\|\boldsymbol{w}(t+T)\|_{L^2}^2+(\alpha-\beta)\int_t^{t+T}\left(\|\nabla\boldsymbol{v}(s)\|_{L^2}^2+\|\nabla\boldsymbol{w}(s)\|_{L^2}^2\right)\,ds\\
&\leq \|\boldsymbol{v}(t)\|_{L^2}^2+\|\boldsymbol{w}(t)\|_{L^2}^2+\tfrac{T}{2\pi^2(\alpha-\beta)}\limsup_{s\to\infty}\left(\|\boldsymbol{f}(s)\|_{L^2}^2+\|\boldsymbol{g}(s)\|_{L^2}^2\right).
\end{aligned}$$

Thus, using (A.16), for $t \geq t_0$,

$$\int_t^{t+T} \left(\|\nabla \boldsymbol{v}(s)\|_{L^2}^2 + \|\nabla \boldsymbol{w}(s)\|_{L^2}^2\right) \, ds$$
$$\leq (1 + \pi^2(\alpha - \beta)T)(\alpha - \beta) \limsup_{s\to\infty} \frac{\|\boldsymbol{f}(s)\|_{L^2}^2 + \|\boldsymbol{g}(s)\|_{L^2}^2}{2\pi^4(\alpha - \beta)^4}, \tag{A.17}$$

which implies (4.8). □

Proof of Lemma 4.4.1 To show (4.12), we first apply (1.4) and (1.3) then (1.12) and (1.3):

$$\left|\int_\Omega (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{v} \cdot \boldsymbol{w} \, dxdy\right| \leq \int_\Omega |\boldsymbol{u}|\, |\nabla \boldsymbol{v}|\, |\boldsymbol{w}| \; dxdy \leq \|\nabla \boldsymbol{v}\|_{L^2} \|\boldsymbol{u}\|_{L^4} \|\boldsymbol{w}\|_{L^4}$$
$$\leq \frac{\delta}{2} \|\nabla \boldsymbol{v}\|_{L^2} \|\boldsymbol{u}\|_{L^4}^2 + \frac{1}{2\delta} \|\nabla \boldsymbol{v}\|_{L^2} \|\boldsymbol{w}\|_{L^4}^2$$
$$\leq \frac{c_L \delta}{2} \|\nabla \boldsymbol{v}\|_{L^2} \|\boldsymbol{u}\|_{L^2} \|\nabla \boldsymbol{u}\|_{L^2} + \frac{c_L}{2\delta} \|\nabla \boldsymbol{v}\|_{L^2} \|\boldsymbol{w}\|_{L^2} \|\nabla \boldsymbol{w}\|_{L^2}.$$
$$\leq \frac{c_L \delta}{2} \left(\frac{1}{2} \|\nabla \boldsymbol{v}\|_{L^2}^2 \|\boldsymbol{u}\|_{L^2}^2 + \frac{1}{2} \|\nabla \boldsymbol{u}\|_{L^2}^2\right) + \frac{1}{2} \frac{c_L^2}{4\epsilon\delta^2} \|\nabla \boldsymbol{v}\|_{L^2}^2 \|\boldsymbol{w}\|_{L^2}^2 + \frac{\epsilon}{2} \|\nabla \boldsymbol{w}\|_{L^2}^2.$$

We obtain (4.13) by switching the roles of $\boldsymbol{u}$ and $\boldsymbol{w}$ after applying (1.4).

The proof of (4.14) requires us to estimate the components of the product differently. First, write

$$\left|\int_\Omega (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{v} \cdot \boldsymbol{w} \, dxdy\right| = \left|\int_\Omega \sum_{i,j=1}^{2} u_i \partial_i v_j w_j \, dxdy\right| \leq \sum_{i,j=1}^{2} \left|\int_\Omega u_i \partial_i v_j w_j \, dxdy\right|,$$

and then we estimate the terms of the sum separately.

(Case: $i = 1, j = 1$) For this case we proceed similarly as in the proof of (4.12), to obtain:

$$
\begin{aligned}
\left|\int_\Omega u_1 \partial_1 v_1 w_1 \, dxdy\right| &\leq \|\nabla v_1\|_{L^2}\|u_1\|_{L^4}\|w_1\|_{L^4} \\
&\leq \frac{c_L}{2}\|\nabla v_1\|_{L^2}\|u_1\|_{L^2}\|\nabla u_1\|_{L^2} + \frac{c_L}{2}\|\nabla v_1\|_{L^2}\|w_1\|_{L^2}\|\nabla w_1\|_{L^2} \\
&\leq \frac{c_L\delta}{4}\|\nabla u_1\|^2_{L^2} + \frac{c_L}{4\delta}\|\nabla v_1\|^2_{L^2}\|u_1\|^2_{L^2} + \frac{c_L\delta}{4}\|\nabla w_1\|^2_{L^2} + \frac{c_L}{4\delta}\|\nabla v_1\|^2_{L^2}\|w_1\|^2_{L^2}.
\end{aligned}
$$

(Case: $i = 1, j = 2$) For this and the next case, we use (1.13):

$$
\begin{aligned}
\left|\int_\Omega u_1 \partial_1 v_2 w_2 \, dxdy\right| &\leq c_B\|\nabla w_2\|_{L^2}\|\nabla v_2\|_{L^2}\|u_1\|_{L^2}\left(1 + \ln\left(\frac{\|\nabla u_1\|_{L^2}}{2\pi\|u_1\|_{L^2}}\right)\right)^{1/2} \\
&\leq \frac{c_B\delta}{2}\|\nabla w_2\|^2_{L^2} + \frac{c_B}{2\delta}\|\nabla v_2\|^2_{L^2}\|u_1\|^2_{L^2}\left(1 + \ln\left(\frac{\|\nabla u_1\|_{L^2}}{2\pi\|u_1\|_{L^2}}\right)\right) \\
&\leq \frac{c_B\delta}{2}\|\nabla w_2\|^2_{L^2} + \frac{c_B}{2\delta}\|\nabla \boldsymbol{v}\|^2_{L^2}\|u_1\|^2_{L^2}\left(1 + \ln\left(\frac{\|\nabla u_1\|_{L^2}}{2\pi\|u_1\|_{L^2}}\right)\right)
\end{aligned}
$$

(Case: $i = 2, j = 1$) Similarly, we obtain:

$$
\left|\int_\Omega u_2 \partial_2 v_1 w_1 \, dxdy\right| \leq \frac{c_B\delta}{2}\|\nabla u_2\|^2_{L^2} + \frac{c_B}{2\delta}\|\nabla \boldsymbol{v}\|^2_{L^2}\|w_1\|^2_{L^2}\left(1 + \ln\left(\frac{\|\nabla w_1\|_{L^2}}{2\pi\|w_1\|_{L^2}}\right)\right)
$$

(Case: $i = 2, j = 2$) Now we use the divergence free conditions (i.e. $\partial_1 u_1 = -\partial_2 u_2$) and integrate by parts in order to obtain integrals in which the second components of $\boldsymbol{u}$ and $\boldsymbol{w}$ do not appear together:

$$
\begin{aligned}
\int_\Omega u_2 \partial_2 v_2 w_2 \, dxdy &= -\int_\Omega \partial_2 u_2 v_2 w_2 \, dxdy - \int_\Omega u_2 v_2 \partial_2 w_2 \, dxdy \\
&= \int_\Omega \partial_1 u_1 v_2 w_2 \, dxdy + \int_\Omega u_2 v_2 \partial_1 w_1 \, dxdy \\
&= -\int_\Omega u_1 \partial_1 v_2 w_2 \, dxdy - \int_\Omega u_1 v_2 \partial_1 w_2 \, dxdy \\
&\quad - \int_\Omega \partial_1 u_2 v_2 w_1 \, dxdy - \int_\Omega u_2 \partial_1 v_2 w_1 \, dxdy.
\end{aligned}
$$

Now, each of these terms can be estimated similarly to the cases where $i \neq j$:

$$\left|\int_\Omega u_1\partial_1 v_2 w_2\, dxdy\right| \leq \frac{c_B\delta}{2}\|\nabla w_2\|_{L^2}^2 + \frac{c_B}{2\delta}\|\nabla \boldsymbol{v}\|_{L^2}^2\|u_1\|_{L^2}^2\left(1+\ln\left(\frac{\|\nabla u_1\|_{L^2}}{2\pi\|u_1\|_{L^2}}\right)\right)$$
$$\left|\int_\Omega u_1 v_2 \partial_1 w_2\, dxdy\right| \leq \frac{c_B\delta}{2}\|\nabla w_2\|_{L^2}^2 + \frac{c_B}{2\delta}\|\nabla \boldsymbol{v}\|_{L^2}^2\|u_1\|_{L^2}^2\left(1+\ln\left(\frac{\|\nabla u_1\|_{L^2}}{2\pi\|u_1\|_{L^2}}\right)\right)$$
$$\left|\int_\Omega \partial_1 u_2 v_2 w_1\, dxdy\right| \leq \frac{c_B\delta}{2}\|\nabla u_2\|_{L^2}^2 + \frac{c_B}{2\delta}\|\nabla \boldsymbol{v}\|_{L^2}^2\|w_1\|_{L^2}^2\left(1+\ln\left(\frac{\|\nabla w_1\|_{L^2}}{2\pi\|w_1\|_{L^2}}\right)\right)$$
$$\left|\int_\Omega u_2 \partial_1 v_2 w_1\, dxdy\right| \leq \frac{c_B\delta}{2}\|\nabla u_2\|_{L^2}^2 + \frac{c_B}{2\delta}\|\nabla \boldsymbol{v}\|_{L^2}^2\|w_1\|_{L^2}^2\left(1+\ln\left(\frac{\|\nabla w_1\|_{L^2}}{2\pi\|w_1\|_{L^2}}\right)\right)$$

Taking the sum of these 7 inequalities obtained from the 4 cases, we have:

$$\left|\int_\Omega (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{v}\cdot\boldsymbol{w}\, dxdy\right| \leq \frac{c_L\delta}{4}\|\nabla u_1\|_{L^2}^2 + \frac{3c_B\delta}{2}\|\nabla u_2\|_{L^2}^2 + \frac{c_L\delta}{4}\|\nabla w_1\|_{L^2}^2 + \frac{3c_B\delta}{2}\|\nabla w_2\|_{L^2}^2$$
$$+\frac{c_L}{4\delta}\|\nabla \boldsymbol{v}\|_{L^2}^2\|u_1\|_{L^2}^2 + \frac{c_L}{4\delta}\|\nabla \boldsymbol{v}\|_{L^2}^2\|w_1\|_{L^2}^2$$
$$+\frac{3c_B}{2\delta}\|\nabla \boldsymbol{v}\|_{L^2}^2\|u_1\|_{L^2}^2\left(1+\ln\left(\frac{\|\nabla u_1\|_{L^2}}{2\pi\|u_1\|_{L^2}}\right)\right)+\frac{3c_B}{2\delta}\|\nabla \boldsymbol{v}\|_{L^2}^2\|w_1\|_{L^2}^2\left(1+\ln\left(\frac{\|\nabla w_1\|_{L^2}}{2\pi\|w_1\|_{L^2}}\right)\right).$$

Setting $c = \max\{\frac{c_L}{4}, \frac{3c_B}{2}\}$ now yields (4.14). $\square$

Proof of Lemma 4.4.3 We start by writing

$$\int_\Omega (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{v}\cdot\Delta\boldsymbol{w}\, dxdy = \int_\Omega u_1\partial_x v_1 \Delta w_1\, dxdy + \int_\Omega u_2\partial_y v_1\Delta w_1\, dxdy$$
$$+\int_\Omega u_1\partial_x v_2 \Delta w_2\, dxdy + \int_\Omega u_2\partial_y v_2\Delta w_2\, dxdy.$$

Now we estimate each term individually.

By (1.14) we have:

$$\begin{aligned}\left|\int_\Omega u_1\partial_x v_1\Delta w_1\, dxdy\right| &\leq c_T\|\nabla u_1\|_{L^2}\|\nabla v_1\|_{L^2}\|\Delta w_1\|_{L^2}\left(1+\ln\frac{\|\Delta u_1\|_{L^2}}{2\pi\|\nabla u_1\|_{L^2}}\right)^{1/2}\\ &\leq c_T\|\nabla u_1\|_{L^2}\|\nabla \boldsymbol{v}\|_{L^2}\|\Delta \boldsymbol{w}\|_{L^2}\left(1+\ln\frac{\|\Delta u_1\|_{L^2}}{2\pi\|\nabla u_1\|_{L^2}}\right)^{1/2},\end{aligned} \tag{A.18}$$

and

$$\left|\int_\Omega u_1\partial_x v_2\Delta w_2\,dxdy\right| \le c_T\|\nabla u_1\|_{L^2}\|\nabla v_2\|_{L^2}\|\Delta w_2\|_{L^2}\left(1+\ln\frac{\|\Delta u_1\|_{L^2}}{2\pi\|\nabla u_1\|_{L^2}}\right)^{1/2}$$
$$\le c_T\|\nabla u_1\|_{L^2}\|\nabla \boldsymbol{v}\|_{L^2}\|\Delta \boldsymbol{w}\|_{L^2}\left(1+\ln\frac{\|\Delta u_1\|_{L^2}}{2\pi\|\nabla u_1\|_{L^2}}\right)^{1/2}. \tag{A.19}$$

Using integration by parts and the divergence free condition, we have:

$$\int_\Omega u_2\partial_y v_1\Delta w_1\,dxdy = -\int_\Omega \partial_x u_2\partial_y v_1\partial_x w_1\,dxdy - \int_\Omega \partial_y u_2\partial_y v_1\partial_y w_1\,dxdy$$
$$+\int_\Omega u_2\partial_{yy}v_2\partial_x w_1\,dxdy - \int_\Omega u_2\partial_{yy}v_1\partial_y w_1\,dxdy,$$

so applying (1.13) to the first two integrals and (1.14) to the second two, we obtain:

$$\left|\int_\Omega u_2\partial_y v_1\Delta w_1\,dxdy\right| \le c_B\|\Delta \boldsymbol{u}\|_{L^2}\|\nabla \boldsymbol{v}\|_{L^2}\|\nabla w_1\|_{L^2}\left(1+\ln\frac{\|\Delta w_1\|_{L^2}}{2\pi\|\nabla w_1\|_{L^2}}\right)^{1/2} \tag{A.20}$$

$$+\, c_T\|\nabla \boldsymbol{u}\|_{L^2}\|\nabla w_1\|_{L^2}\|\Delta \boldsymbol{v}\|_{L^2}\left(1+\ln\frac{\|\Delta w_1\|_{L^2}}{2\pi\|\nabla w_1\|_{L^2}}\right)^{1/2}. \tag{A.21}$$

Again by integrating by parts and using the divergence free condition, we obtain

$$\int_\Omega u_2\partial_y v_2\Delta w_2\,dxdy = \int_\Omega \partial_x u_1 v_2\Delta w_2\,dxdy$$
$$+\int_\Omega \Delta u_2 v_2\partial_x w_1\,dxdy + \int_\Omega u_2\Delta v_2\partial_x w_1\,dxdy$$
$$+\ 2\int_\Omega \partial_x u_2\partial_x v_2\partial_x w_1\,dxdy\ +\ 2\int_\Omega \partial_y u_2\partial_y v_2\partial_x w_1\,dxdy.$$

Now, estimating with (1.13) and (1.14) we have:

$$\begin{aligned}\left|\int_\Omega u_2\partial_y v_2\Delta w_2\,dxdy\right| &\le c_T\|\nabla u_1\|_{L^2}\|\nabla \boldsymbol{v}\|_{L^2}\|\Delta \boldsymbol{w}\|_{L^2}\left(1+\ln\frac{\|\Delta u_1\|_{L^2}}{2\pi\|\nabla u_1\|_{L^2}}\right)^{1/2}\\ &+ c_T\|\Delta \boldsymbol{u}\|_{L^2}\|\nabla \boldsymbol{v}\|_{L^2}\|\nabla w_1\|_{L^2}\left(1+\ln\frac{\|\Delta w_1\|_{L^2}}{2\pi\|\nabla w_1\|_{L^2}}\right)^{1/2}\\ &+ c_T\|\nabla \boldsymbol{u}\|_{L^2}\|\Delta \boldsymbol{v}\|_{L^2}\|\nabla w_1\|_{L^2}\left(1+\ln\frac{\|\Delta w_1\|_{L^2}}{2\pi\|\nabla w_1\|_{L^2}}\right)^{1/2}\\ &+ 4c_B\|\Delta \boldsymbol{u}\|_{L^2}\|\nabla \boldsymbol{v}\|_{L^2}\|\nabla w_1\|_{L^2}\left(1+\ln\frac{\|\Delta w_1\|_{L^2}}{2\pi\|\nabla w_1\|_{L^2}}\right)^{1/2}.\end{aligned} \tag{A.22}$$

Combining (A.18), (A.19), (A.21), and (A.22), we obtain:

$$\begin{aligned}\left|\int_\Omega (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{v}\cdot\Delta\boldsymbol{w}\,dxdy\right| &\le 3c_T\|\nabla u_1\|_{L^2}\|\nabla \boldsymbol{v}\|_{L^2}\|\Delta \boldsymbol{w}\|_{L^2}\left(1+\ln\frac{\|\Delta u_1\|_{L^2}}{2\pi\|\nabla u_1\|_{L^2}}\right)^{1/2}\\ &+ (c_T+4c_B)\|\Delta \boldsymbol{u}\|_{L^2}\|\nabla \boldsymbol{v}\|_{L^2}\|\nabla w_1\|_{L^2}\left(1+\ln\frac{\|\Delta w_1\|_{L^2}}{2\pi\|\nabla w_1\|_{L^2}}\right)^{1/2}\\ &+ 2c_T\|\nabla \boldsymbol{u}\|_{L^2}\|\Delta \boldsymbol{v}\|_{L^2}\|\nabla w_1\|_{L^2}\left(1+\ln\frac{\|\Delta w_1\|_{L^2}}{2\pi\|\nabla w_1\|_{L^2}}\right)^{1/2},\end{aligned}$$

so (a) is proven.

In order to prove (b), we first write

$$\begin{aligned}\int_\Omega (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{v}\cdot\Delta\boldsymbol{v}\,dxdy &= \int_\Omega u_1\partial_x v_1\Delta v_1\,dxdy + \int_\Omega u_2\partial_y v_1\Delta v_1\,dxdy\\ &+ \int_\Omega u_1\partial_x v_2\Delta v_2\,dxdy + \int_\Omega u_2\partial_y v_2\Delta v_2\,dxdy.\end{aligned}$$

Similar to the proof of (a), we proceed to estimate each term individually by appealing to (1.13) or (1.14), by integrating by parts and using the divergence free conditions.

By applying (1.14), we have:

$$\left|\int_\Omega u_1\partial_x v_1\Delta v_1\,dxdy\right| \le c_T\|\nabla \boldsymbol{u}\|_{L^2}\|\nabla v_1\|_{L^2}\|\Delta \boldsymbol{v}\|_{L^2}\left(1+\ln\frac{\|\Delta v_1\|_{L^2}}{2\pi\|\nabla v_1\|_{L^2}}\right)^{1/2}, \tag{A.23}$$

and

$$\left|\int_\Omega u_2\partial_y v_1\Delta v_1\,dxdy\right| \le c_T\|\nabla \boldsymbol{u}\|_{L^2}\|\nabla v_1\|_{L^2}\|\Delta \boldsymbol{v}\|_{L^2}\left(1+\ln\frac{\|\Delta v_1\|_{L^2}}{2\pi\|\nabla v_1\|_{L^2}}\right)^{1/2}, \tag{A.24}$$

and using the divergence free condition, we obtain

$$\begin{aligned}\left|\int_\Omega u_2\partial_y v_2\Delta v_2\,dxdy\right| &= \left|-\int_\Omega u_2\partial_x v_1\Delta v_2\,dxdy\right| \\ &\le c_T\|\nabla \boldsymbol{u}\|_{L^2}\|\nabla v_1\|_{L^2}\|\Delta \boldsymbol{v}\|_{L^2}\left(1+\ln\frac{\|\Delta v_1\|_{L^2}}{2\pi\|\nabla v_1\|_{L^2}}\right)^{1/2}.\end{aligned} \tag{A.25}$$

To estimate the remaining integral, we write:

$$\int_\Omega u_1\partial_x v_2\Delta v_2\,dxdy = \int_\Omega u_1\partial_x v_2\partial_{xx}v_2\,dxdy + \int_\Omega u_1\partial_x v_2\partial_{yy}v_2\,dxdy.$$

Now,

$$\begin{aligned}\int_\Omega u_1\partial_x v_2\partial_{yy}v_2\,dxdy &= -\int_\Omega u_1\partial_x v_2\partial_y\partial_x v_1\,dxdy \\ &= \int_\Omega \partial_x u_1\partial_x v_2\partial_y v_1\,dxdy + \int_\Omega u_1\partial_{xx}v_2\partial_y v_1\,dxdy,\end{aligned}$$

so

$$\left|\int_\Omega u_1\partial_x v_2\partial_{yy}v_2\,dxdy\right| \le (c_B+c_T)\|\nabla \boldsymbol{u}\|_{L^2}\|\Delta \boldsymbol{v}\|_{L^2}\|\nabla v_1\|_{L^2}\left(1+\ln\frac{\|\Delta v_1\|_{L^2}}{2\pi\|\nabla v_1\|_{L^2}}\right)^{1/2}. \tag{A.26a}$$

For the other term, we have

$$\int_\Omega u_1\partial_x v_2\partial_{xx}v_2\,dxdy = -\int_\Omega \partial_x u_1\partial_x v_2\partial_x v_2\,dxdy - \int_\Omega u_1\partial_{xx}v_2\partial_x v_2\,dxdy,$$

so,

$$\int_\Omega u_1\partial_x v_2\partial_{xx}v_2\,dxdy = -\frac{1}{2}\int_\Omega \partial_x u_1\partial_x v_2\partial_x v_2\,dxdy.$$

Next,

$$
\begin{aligned}
-\frac{1}{2}\int_{\Omega} \partial_x u_1 \partial_x v_2 \partial_x v_2 \, dxdy &= \frac{1}{2}\int_{\Omega} \partial_y u_2 \partial_x v_2 \partial_x v_2 \, dxdy \\
= -\int_{\Omega} u_2 \partial_x \partial_y v_2 \partial_x v_2 \, dxdy &= \int_{\Omega} u_2 \partial_{xx} v_1 \partial_x v_2 \, dxdy \\
= -\int_{\Omega} \partial_x u_2 \partial_x v_1 \partial_x v_2 \, dxdy &- \int_{\Omega} u_2 \partial_x v_1 \partial_{xx} v_2 \, dxdy.
\end{aligned}
$$

Therefore,

$$
\left| \int_{\Omega} u_1 \partial_x v_2 \partial_{xx} v_2 \, dxdy \right| \leq (c_B + c_T) \|\nabla \boldsymbol{u}\|_{L^2} \|\nabla v_1\|_{L^2} \|\Delta \boldsymbol{v}\|_{L^2} \left(1 + \ln \frac{\|\Delta v_1\|_{L^2}}{2\pi \|\nabla v_1\|_{L^2}} \right)^{1/2}. \tag{A.26b}
$$

Hence, by combining (A.23), (A.24), (A.25), (A.26a), and (A.26b), we obtain:

$$
\left| \int_{\Omega} (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{v} \cdot \Delta \boldsymbol{v} \, dxdy \right| \leq (2c_B + 5c_T) \|\nabla \boldsymbol{u}\|_{L^2} \|\nabla v_1\|_{L^2} \|\Delta \boldsymbol{v}\|_{L^2} \left(1 + \ln \frac{\|\Delta v_1\|_{L^2}}{2\pi \|\nabla v_1\|_{L^2}} \right)^{1/2},
$$

as claimed. □

Bibliography

[1] M.M. Adams, D.R. Stone, D.S. Zimmerman and D.P. Lathrop, Liquid sodium models of the Earth's core, *Progress in Earth and Planetary Science* 2(1) (2015), 1–18,

[2] R. Agapito, M. Schonbek, Non-uniform decay of MHD equations with and without magnetic diffusion, Communications in Partial Differential Equations 32 (2007), 1791-1812.

[3] D.A. Albanez, H.J. Nussenzveig Lopes and E.S. Titi, Continuous data assimilation for the three-dimensional Navier–Stokes-α model, *Asymptotic Anal.* 97(1–2) (2016), 139–164.

[4] S. Alinhac and G. Métivier. Propagation de lÕanalyticit locale pour les solutions de lÕquation dÕEuler. *Arch. Ration. Mech. Anal.* 92 (4) (1986) 287-296.

[5] M.U. Altaf, E.S. Titi, O.M. Knio, L. Zhao, M.F. McCabe and I. Hoteit, Downscaling the 2D Benard Convection Equations Using Continuous Data Assimilation, *ArXiv e-prints* (2015).

[6] A. Azouani, E. Olson and E.S. Titi, Continuous data assimilation using general interpolant observables, *J. Nonlinear Sci.* 24(2) (2014), 277–304,

[7] A. Azouani and E.S. Titi, Feedback control of nonlinear dissipative systems by finite determining parameters—a reaction-diffusion paradigm, *Evol. Equ. Control Theory* 3(4) (2014), 579–594,

[8] H. Bae and A. Biswas. Gevrey regularity for a class of dissipative equations with analytic nonlinearity. *Methods and Applications of Analysis,* 22:4 (2015), 377-408.

[9] D. Le Bail. Analyticité locale pour les solutions de l'équation d'Euler. *Arch. Ration. Mech. Anal.* 95 (2) (1986) 117-136.

[10] C. Bardos and S. Benachour. Domaine d'analycité des solutions de l'équation d'Euler dans un ouvert de $\mathbb{R}^n$. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (4) 4 (4) (1977) 647-687.

[11] C. Bardos, S. Benachour and M. Zerner. Analyticité des solutions périodiques de l'équation d'Euler en deux dimensions. *C. R. Acad. Sci. Paris, Sér. A-B* 282 (17:Aiii) (1976) A995-A998.

[12] C. Bardos, C. Sulem and P.-L. Sulem. Longtime dynamics of a conductive fluid in the presence of a strong magnetic field. *Trans. Amer. Math. Soc.*, 305, 175–191 (1988)

[13] C. Bardos and E. S. Titi. Euler equations for incompressible ideal fluids. *Russian Math. Surveys,* 62:3 (2007), 409-451.

[14] J. T. Beale, T. Kato and A. Majda Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.* 94:1 (1984), 61-66.

[15] A. J. Majda and A. L. Bertozzi. Vorticity and incompressible flow *Cambridge Texts Appl. Math., vol. 27, Cambridge Univ. Press, Cambridge*, 2002.

[16] H. Bessaih, E. Olson and E.S. Titi, Continuous data assimilation with stochastically noisy data, *Nonlinearity* 28(3) (2015), 729–753,

[17] A. Biswas. Gevrey regularity for a class of dissipative equations with applications to decay. *Journal of Differential Equations,* 253:10 (2012), pp. 2739-2764.

[18] A. Biswas, Local existence and Gevrey regularity of 3-D Navier-Stokes equations with ℓ_p initial data, J. Differential Equations 215 (2005), 429-447.

[19] A. Biswas, J. Hudson, Eventual decay of solutions to the magnetohydrodynamic equations in Besov spaces, *(in preparation).*

[20] A. Biswas, J. Hudson, *Space and time analyticity for inviscid equations of fluid dynamics. Nonlinearity* (submitted). (2017).

[21] A. Biswas, J. Hudson, A. Larios, and Y. Pei, Continuous data assimilation for the magnetohydrodynamic equations in 2D using one component of the velocity and magnetic fields, *Asymptot. Anal.* (accepted).

[22] A. Biswas and V.R. Martinez, Higher-order synchronization for a data assimilation algorithm for the 2D Navier–Stokes equations, *Nonlinear Anal. Real World Appl.* 35 (2017), 132–157,

[23] A. Biswas and D. Swanson. Existence and generalized Gevrey regularity of solutions to the Kuramoto-Sivashinsky equation in $\mathbb{R}^n$. *Journal of Differential Equations,* 240:1 (2007), pp. 145-163.

[24] A. Biswas, D. Swanson, Navier-Stokes Equations and Weighted Convolution Inequalities in Groups, Communications in Partial Differential Equations 35 (2010), 559-589.

[25] J. Bloxham, D. Gubbins and A. Jackson, Geomagnetic secular variation, *Phil. Trans. R. Soc. A* 329(1606) (1989), 415–502.

[26] Z. Bradshaw and Z. Grujić, Energy cascades in physical scales of 3D incompressible magnetohydrodynamic turbulence, *J. Math. Phys.* 54(9) (2013), 093503–18,

[27] Z. Bradshaw and Z. Grujić, On the transport and concentration of enstrophy in 3D magnetohydrodynamic turbulence, *Nonlinearity* 26(8) (2013), 2373–2390,

[28] Z. Bradshaw, Z. Grujić and I. Kukavica. Local analyticity radii of solutions to the 3D Navier-Stokes equations with locally analytic forcing. *J. Differential Equations* 259:8 (2015), 3955-3975.

[29] H.-P. Bunge, M.A. Richards and J.R. Baumgardner, Mantle-circulation models with sequential data assimilation: inferring present-day mantle structure from plate-motion histories, *Phil. Trans. R. Soc. A* 360(1800) (2002), 2545–2567,

[30] F. Cheng and C.-J. Xu. On the gevrey regularity of solutions to the 3d ideal mhd equations. ArXiv:1702.06840, February 2017.

[31] L. Caffarelli and A. Vasseur. Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. *Annals of Math.* , 171(3), 1903-1930, 2010.

[32] R.E Caflisch, I Klapper and G. Steele. Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD. *Comm. Math. Phys.*, 184, (1997), 443-455.

[33] C. Cao, I.G. Kevrekidis and E.S. Titi, Numerical criterion for the stabilization of steady states of the Navier–Stokes equations, *Indiana Univ. Math. J.* 50(Special Issue) (2001), 37–96, Dedicated to Professors Ciprian Foias and Roger Temam (Bloomington, IN, 2000),

[34] J. Chandrasekar, A.J. Ridley and D.S. Bernstein, A comparison of the extended and unscented Kalman filters for discrete-time systems with nondifferentiable dynamics, *Proc. of the 2007 American Control Conference* 2 (2007), 4431–4436.

[35] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, The International Series of Monographs on Physics, Clarendon Press, Oxford, 1961, pp. 654–16.

[36] J. Charney, M. Halem, and R. Jastrow, *Use of incomplete historical data to infer the present state of the atmosphere*, Journal of the Atmospheric Sciences, 26 (1969), pp. 1160–1163.

[37] B. Cockburn, D.A. Jones and E.S. Titi, Determining degrees of freedom for nonlinear dissipative equations, *C. R. Acad. Sci. Paris Sér. I Math.* 321(5) (1995), 563–568,

[38] C.G. Constable, C.L. Johnson and S.P. Lund, Global geomagnetic field models for the past 3000 years: transient or permanent flux lobes?, *Phil. Trans. R. Soc. A* 358(1768) (2000), 991–1008.

[39] P. Constantin, C. Fefferman, Direction of vorticity and the problem of global regularity for the Navier-Stokes equations, Indiana Univ. Math. J. 42 (1993), no. 3, 775-789.

[40] P. Constantin and C. Foias, *Navier-Stokes Equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988, p. 190.

[41] P. Constantin, I. Kukavica and V. Vicol. Contrast between Lagrangian and Eulerian analytic regularity properties of Euler equations. *Ann. I. H. Poincaré - AN* 33 (2016), 1569-1588.

[42] P. Constantin, A. Majda, E. Tabak. Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar. *Nonlinearity*, 7, 1495-1533, 1994.

[43] P. Constantin, V. Vicol and J. Wu. Analyticity of Lagrangian trajectories for well posed inviscid incompressible fluid models. *Advances in Mathematics.* 285 (2015), 352-393.

[44] P. Constantin and J. Wu. Behavior of solutions of 2d quasi-geostrophic equations. *SIAM journal on mathematical analysis*, 30:937, 1999.

[45] R. Daley, *Atmospheric Data Analysis*, Cambridge Atmospheric and Space Science Series, Cambridge University Press, 1993.

[46] Keke Zhang and David Gubbins. Scale disparities and magnetohydrodynamics in the earth's core. *Philosophical Transactions: Mathematical, Physical and Engineering Sciences*, 358(1768):899–920, 2000.

[47] P.A. Davidson, *An Introduction to Magnetohydrodynamics*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001, p. 431.

[48] J. M. Delort. Estimations fines pour des oprateurs pseudo-diffrentiels analytiques sur un ouvert à bord de $\mathbb{R}^n$. Application aux équations d'Euler *Comm. Partial Differential Equations* 10 (12) (1985) 1465-1525.

[49] A. Doelman and E. S. Titi. Regularity of solutions and the convergence of the Galerkin method in the complex GinzburgÐLandau equation. *Numer. Func. Opt. Anal.,* 14 (1993), pp. 299-321.

[50] C. Doering and E. S. Titi. Exponential decay rate of the power spectrum for solutions of the NavierÐStokes equations. *Physics of Fluids,* 7 (1995), pp. 1384-1390.

[51] G. Duvaut and J.-L. Lions, Inéquations en thermoélasticité et magnétohydrodynamique, *Arch. Rational Mech. Anal.* 46 (1972), 241–279,

[52] W.M. Elsasser, The hydromagnetic equations, *Phys. Rev.* 79 (1950), 183.

[53] L. Euler. Principes généraux du mouvement des fluides. *Académie Royale des Sciences et des Belles Lettres de Berlin, Mémoires* 11 (1757).

[54] A. Farhat, M.S. Jolly and E.S. Titi, Continuous data assimilation for the 2D Bénard convection through velocity measurements alone, *Phys. D* 303 (2015), 59–66,

[55] A. Farhat, M. S. Jolly, E. S. Titi, Continuous data assimilation for the 2D Bénard convection through velocity measurements alone, Physica D 303 (2015), 59-66.

[56] A. Farhat, E. Lunasin and E.S. Titi, Abridged Continuous Data Assimilation for the 2D Navier–Stokes Equations Utilizing Measurements of Only One Component of the Velocity Field, *J. Math. Fluid Mech.* 18(1) (2016), 1–23.

[57] A. Farhat, E. Lunasin and E.S. Titi, Data assimilation algorithm for 3D Bénard convection in porous media employing only temperature measurements, *Journal of Mathematical Analysis and Applications* 438(1) (2016), 492–506.

[58] A. Farhat, E. Lunasin and E.S. Titi, On the Charney Conjecture of Data Assimilation Employing Temperature Measurements Alone: The Paradigm of 3D Planetary Geostrophic Model, *arXiv:1608.04770* (2016).

[59] A. Farhat, E. Lunasin and E.S. Titi, Continuous Data Assimilation for a 2D Bénard Convection System Through Horizontal Velocity Measurements Alone, *Journal of Nonlinear Science* (2017), 1–23,

[60] A. B. Ferrari and E. S. Titi. Gevrey regularity for nonlinear analytic parabolic equations. *Communications in Partial Differential Equations,* 23:1 (1998), pp. 424-448.

[61] C. Foias, M. S. Jolly, R. Kravchenko, and E. S. Titi. A unified approach to determining forms for the 2D Navier–Stokes equations — the general interpolants case, *Russian Math. Surveys* 69:2 (2014), pp. 359–381.

[62] C. Foias, M. S. Jolly, and W-S Li. Nevanlinna-Pick interpolation of attractors. *Nonlinearity* 15 (2002), 1881-1903.

[63] C. Foias and G. Prodi, Sur le comportement global des solutions non-stationnaires des équations de Navier-Stokes en dimension 2, *Rend. Sem. Mat. Univ. Padova* 39 (1967), 1–34,

[64] C. Foias, O. Manley, R. Rosa and R. Temam, *Navier-Stokes Equations and Turbulence*, Encyclopedia of Mathematics and its Applications, Vol. 83, Cambridge University Press, Cambridge, 2001, p. 347.

[65] C. Foias, O. Manley, R. Temam and Y.M. Trève, Asymptotic analysis of the Navier-Stokes equations, *Physica D: Nonlinear Phenomena* 9(1–2) (1983), 157–188.

[66] C. Foias, C.F. Mondaini and E.S. Titi, A discrete data assimilation scheme for the solutions of the two-dimensional Navier-Stokes equations and their statistics, *SIAM J. Appl. Dyn. Syst.* 15(4) (2016), 2109–2142,

[67] C. Foias and R. Temam, Determination of the solutions of the Navier-Stokes equations by a set of nodal values, *Math. Comp.* 43(167) (1984), 117–133,

[68] C. Foias and R. Temam. Some analytic and geometric properties of the solutions of the Navier-Stokes equations. *J. Math. Pures Appl.,* 58 (1979), pp. 339-368.

[69] C. Foias and R. Temam. Gevrey class regularity for the solutions of the Navier-Stokes equations. *Journal of Functional Analysis,* 87 (1989), pp. 359-369.

[70] C. Foias and E.S. Titi, Determining nodes, finite difference schemes and inertial manifolds, *Nonlinearity* 4(1) (1991), 135–153,

[71] A. Fournier, C. Eymin and T. Alboussière, A case for variational geomagnetic data assimilation: insights from a one-dimensional, nonlinear, and sparsely observed MHD system, *Nonlinear Proc. Geoph.* 14 (2007), 163–180.

[72] A. Fournier, G. Hulot, D. Jault, W. Kuang, A. Tangborn, N. Gillet, E. Canet, J. Aubert and F. Lhuillier, An Introduction to Data Assimilation and Predictability in Geomagnetism, *Space Sci. Rev.* 155(1) (2010), 247–291,

[73] K. Foyash, M.S. Dzholli, R. Kravchenko and È.S. Titi, A unified approach to the construction of defining forms for a two-dimensional system of Navier-Stokes equations: the case of general interpolating operators, *Uspekhi Mat. Nauk* 69(2(416)) (2014), 177–200,

[74] V. Zheligovsky and U. Frisch. Time-analyticity of Lagrangian particle trajectories in ideal fluid flow. *J. Fluid Mech.* 749 (2014), 404-430.

[75] M. Gesho, E. Olson and E.S. Titi, A computational study of a data assimilation algorithm for the two-dimensional Navier-Stokes equations, *Commun. Comput. Phys.* 19(4) (2016), 1094–1110,

[76] N. Glatt-Holtz, I. Kukavica, V. Vicol and M. Ziane, Existence and regularity of invariant measures for the three dimensional stochastic primitive equations, *J. Math. Phys.* 55(5) (2014), 051504–34,

[77] C. Groth, D.L. De Zeeuw, T.I. Gombosi and K.G. Powell, Global three-dimensional MHD simulation of a space weather event: CME formation, interplanetary propagation, and interaction with the magnetosphere, *J. Geophys. Res.: Space Phys.* 105(A11) (2000), 25053–25078.

[78] Z. Grujić. The geometric structure of the super-level sets and regularity for 3D Navier-Stokes equations. *Indiana Univ. Math. J.,* 50:3 (2001), pp. 1309-1317.

[79] Z. Grujić and I. Kukavica. Space Analyticity for the Navier-Stokes and Related Equations with Initial Data in L_p. *Journal of Functional Analysis,* 152:2 (1998), pp. 447-466.

[80] P. Han, C. He, Decay properties of solutions to the incompressible magnetohydrodynamics equations in a half space, Mathematical Methods in the Applied Sciences 35 (2012), 1472-1488.

[81] K. Hayden, E. Olson and E.S. Titi, Discrete data assimilation in the Lorenz and 2D Navier-Stokes equations, *Phys. D* 240(18) (2011), 1416–1425,

[82] C. He, Y. Wang, Remark on the regularity for weak solutions to the magnetohydrodynamic equations, Math. Meth. Appl. Sci. (2008), 31:1667-1684

[83] J. Hudson, M. Jolly, A computational study of feedback control based data assimilation algorithms for the magnetohydrodynamic equations in 2D, *(in preparation).*

[84] M.S. Jolly, V.R. Martinez and E.S. Titi, A Data Assimilation Algorithm for the Subcritical Surface Quasi-Geostrophic Equation, *Adv. Nonlinear Stud.* 17(1) (2017), 167–192,

[85] M.S. Jolly, T. Sadigov and E.S. Titi, A determining form for the damped driven nonlinear Schrödinger equation—Fourier modes case, *J. Differential Equations* 258(8) (2015), 2711–2744,

[86] D.A. Jones and E.S. Titi, Upper bounds on the number of determining modes, nodes, and volume elements for the Navier-Stokes equations, *Indiana Univ. Math. J.* 42(3) (1993), 875–887,

[87] D.A. Jones and E.S. Titi, Determining finite volume elements for the 2D Navier-Stokes equations, *Phys. D* 60(1–4) (1992), 165–174, Experimental mathematics: computational issues in nonlinear science (Los Alamos, NM, 1991),

[88] D.A. Jones and E.S. Titi, On the number of determining nodes for the 2D Navier-Stokes equations, *J. Math. Anal. Appl.* 168(1) (1992), 72–88,

[89] E. Kalnay, *Atmospheric Modeling, Data Assimilation and Predictability*, Cambridge University Press, 2003.

[90] S. Kim, Gevrey class regularity of the Magnetohydrodynamics equations, Anziam J. 43 (2002), 397-408.

[91] A. Kiselev. Some recent results on the critical surface quasi-geostrophic equation: a review. *Proc. Sympos. Appl. Math.*, 67(Part 1):105-122, 2009.

[92] A. Kiselev, F. Nazarov, and A. Volberg. Global well-posedness for the critical 2D dissipative quasi-geostrophic equation. *Inventiones mathematicae*, 167, 445-453, 2007.

[93] I. Kukavica. Level sets of the vorticity and the stream function for the 2-D periodic Navier- Stokes equations with potential forces. Journal of Differential Equations, 126 (1996), pp. 374-388.

[94] I. Kukavica and J. C. Robinson. Distinguishing smooth functions by a finite number of point values, and a version of the Takens embedding theorem. *Physica D,* 196 (2004), pp. 45-66.

[95] I. Kukavica and V. Vicol. On the radius of analyticity of solutions to the three-dimensional Euler equations. Proc. Amer. Math. Soc. 137 (2) (2009) 669-677.

[96] I. Kukavica and V. Vicol. On the analyticity and Gevrey-class regularity up to the boundary for the Euler equations. *Nonlinearity,* 24 (3) (2011) 765-796.

[97] O. Ladyzhenskaya and V. Solonnikov. Solution of some nonstationary magnetohydrodynamical problems for incompressible fluid. *Tr. Mat. Inst. Steklova*, 59:115–173, 1960.

[98] A. Larios, E. Lunasin and E.S. Titi, Global well-posedness for the Boussinesq-Voigt equations, (preprint) :1010.5024.

[99] A. Larios and Y. Pei, Nonlinear continuous data assimilation, *arXiv:1703.03546* (2017).

[100] K. Law, A. Stuart and K. Zygalakis, *A Mathematical Introduction to Data Assimilation*, Texts in Applied Mathematics, Vol. 62, Springer, Cham, 2015, p. 242.

[101] C. D. Levermore and M. Oliver. Analyticity of solutions for a generalized Euler equation. *J. Differential Equations* 133 (2) (1997) 321-339.

[102] K. Li, A. Jackson and P.W. Livermore, Variational data assimilation for a forced, inertia-free magnetohydrodynamic dynamo model, *Geophys. J. Int.* 199(3) (2014), 1662–1676.

[103] O. Lielausis. *Liquid Metal in a Strong Magnetic Field*, pages 3–12. Springer Netherlands, Dordrecht, 1989.

[104] E. Lunasin and E.S. Titi, Finite determining parameters feedback control for distributed nonlinear dissipative systems - a computational study, arXiv:1506.03709 [math.AP], (2015).

[105] P.A. Markowich, E.S. Titi and S. Trabelsi, Continuous data assimilation for the three-dimensional Brinkman–Forchheimer-extended Darcy model, *Nonlinearity* 29(4) (2016), 1292.

[106] K. Masuda. On the analyticity and the unique continuation theorem for Navier-Stokes equations. *Proc. Japan Acad. Ser. A Math. Sci.,* 43 (1967), pp. 827-832.

[107] O.B. Mendoza, B. De Moor and D.S. Bernstein, Data assimilation for magnetohydrodynamics systems, *J. Comput. Appl. Math.* 189(1–2) (2006), 242–259,

[108] Michel Sermange and Roger Temam. Some mathematical questions related to the MHD equations. *Comm. Pure Appl. Math.*, 36(5):635–664, 1983.

[109] C.F. Mondaini and E.S. Titi, Postprocessing Galerkin method applied to a data assimilation algorithm: a uniform in time error estimate (2016), (arXiv 1612.06998).

[110] M. Oliver and E. S. Titi. Remark on the Rate of Decay of Higher Order Derivatives for Solutions to the Navier-Stokes Equations in $\mathbb{R}^n$. *Journal of Functional Analysis,* 172:1 (2000), pp. 1-18.

[111] E. Olson and E.S. Titi, Determining modes for continuous data assimilation in 2D turbulence, *J. Statist. Phys.* 113(5–6) (2003), 799–840, Progress in statistical hydrodynamics (Santa Fe, NM, 2002),

[112] J. Pedlosky. *Geophysical Fluid Dynamics.* Springer-Verlag, 1986.

[113] S.G. Resnick. Dynamical problems in non-linear advective partial differential equations. *PhD thesis*, pp. 1-86, August 1995.

[114] J. C. Robinson, W. Sadowski, R. P. Silva, Lower bounds on blow up solutions of the three-dimensional Navier-Stokes equations in homogeneous Sobolev spaces, J.Math. Phys Volume 53 (2012), 115618.

[115] T.J. Sabaka, N. Olsen and R.A. Langel, A comprehensive model of the quiet-time, near-Earth magnetic field: phase 3, *Geophys. J. Int.* 151(1) (2002), 32–68.

[116] R. Salmon. *Lectures on Geophysical Fluid Dynamics.* Oxford Univ. Press, 1998.

[117] R.B. Scott, D.E. McKenzie and D.W. Longcope, Inferring the Magnetohydrodynamic Structure of Solar Flare Supra-Arcade Plasmas from a Data-assimilated Field Transport Model, *Astrophys. J.* 819(1) (2016), 56.

[118] P. Serfati. Equation d'Euler et holomorphies à faible régularité spatiale. *C. R. Acad. Sci. Sér. 1, Math.* 320 (2) (1995) 175-180.

[119] M. Sermange and R. Temam, Some mathematical questions related to the MHD equations, *Comm. Pure Appl. Math.* 36(5) (1983), 635–664,

[120] A. Shnirelman. On the analyticity of particle trajectories in the ideal incompressible fluid. arXiv:1205.5837, 2012.

[121] Z. Sun, A. Tangborn and W. Kuang, Data assimilation in a sparsely observed one-dimensional modeled MHD system, *Nonlinear Proc. Geoph.* 14(2) (2007), 181–192.

[122] B.O. Teixeira, A. Ridley, L.A. Tôrres, L.A. Aguirre and D.S. Bernstein, Data assimilation for magnetohydrodynamics with a zero-divergence constraint on the magnetic field, in: *2008 American Control Conference*, IEEE, 2008, pp. 2534–2539.

[123] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, AMS Chelsea Publishing, Providence, RI, 2001, p. 408, Theory and numerical analysis, Reprint of the 1984 edition.

[124] E.S. Titi, On a criterion for locating stable stationary solutions to the Navier-Stokes equations, *Nonlinear Analysis: Theory, Methods & Applications* 11(9) (1987), 1085–1102.

[125] Y. Wang, Asymptotic decay of solutions to 3D MHD equations, Nonlinear Analysis 132 (2016), 115-125.