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Evaluation of Magnetic Helicity in Homogeneous Turbulence

William H. Matthaeus and Melvyn L. Goldstein

*Laboratory for Extraterrestrial Physics, National Aeronautics and Space Administration
Goddard Space Flight Center, Greenbelt, Maryland 20771*

and

Charles Smith

Department of Physics, The College of William and Mary, Williamsburg, Virginia 23185

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A technique is presented for the measurement of magnetic helicity from values of the two-point magnetic-field correlation matrix under the assumption of spatial homogeneity. Knowledge of a single scalar function of space, derivable from the correlation matrix, suffices to determine the magnetic helicity. The technique is illustrated by a report of the first measurement of the magnetic helicity of the solar wind.

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A considerable body of theoretical plasma physics literature over the last twenty-five years has emphasized the importance of magnetic helicity.¹⁻⁷ The total magnetic helicity may be defined as

$$K \equiv \int \vec{A} \cdot \vec{B} d^3x, \quad (1)$$

where \vec{B} and \vec{A} are the magnetic field and the vector potential, respectively. This integral extends over all field-containing regions, and \vec{A} is subject to the gauge condition $\nabla \cdot \vec{A} = 0$. The value of K is unique for boundary conditions such as \vec{A} and $\vec{B} \rightarrow 0$ at infinity. For finite or homogeneous plasmas the magnetic helicity density H_m [the mean value of the integrand in (1)] is of interest. The magnetic helicity measures the departure of a turbulent magnetic field from mirror symmetry, or equivalently, the degree of topological linkage of magnetic flux tubes.¹

Woltjer² noticed that under fairly general assumptions H_m is an integral invariant of the incompressible, one-fluid, ideal, magnetohydrodynamic (MHD) equations. Woltjer,³ Taylor,⁴ and Montgomery, Turner, and Vahala⁵ have implemented variational formulations involving H_m as a global constraint to calculate MHD equilibria. The behavior of reversed-field pinch plasma-confinement devices⁶ is thought to be, at least in part, explained by "relaxation theories" such as these.

Turbulence theory has also addressed the dynamical role of magnetic helicity. Frisch *et al.*⁷ conjectured that an inverse cascade of magnetic helicity may be characteristic of forced dissipative three-dimensional MHD flows. Montgomery and co-workers^{5,8} have discussed a class of selective decay hypotheses which for MHD flows leads to the conjecture that H_m decays more slowly

than energy. This prediction is based on the suggestion that turbulence preferentially transfers H_m to the large scales which are only weakly affected by resistivity.

Our understanding of the dynamical importance of H_m has been limited by the lack of interplay between theory and experiment. We are not aware of a single direct measurement of magnetic helicity or its spectrum. Thus, it is desirable to develop procedures for obtaining magnetic helicity from experiments. In this Letter we present a straightforward procedure for evaluating magnetic helicity from values of the two-point magnetic-field correlation matrix under the assumption that the statistical properties are spatially homogeneous. Following the usual conventions of homogeneous turbulence theory,⁹ the quantity we consider is the magnetic helicity density $H_m = \langle \vec{A} \cdot \vec{B} \rangle$ where the brackets denote an ensemble average⁹ and the fields \vec{A} and \vec{B} are the fluctuating fields. Thus H_m is the helicity of the fluctuations.

We begin by defining the correlation matrix

$$R_{ij}(\vec{r}) = \langle B_i(\vec{x}) B_j(\vec{x} + \vec{r}) \rangle, \quad (2)$$

where the assumption of "weak" homogeneity renders R_{ij} a function only of the spatial separation \vec{r} ; thus $R_{ij}(\vec{r}) = R_{ji}(-\vec{r})$. The energy spectrum tensor is related to $R_{ij}(\vec{r})$ by

$$S_{ij}(\vec{k}) = \left(\frac{1}{2\pi} \right)^3 \int d^3r e^{-i\vec{k} \cdot \vec{r}} R_{ij}(\vec{r}). \quad (3)$$

The ability to measure H_m is dependent on two properties of homogeneous tensors.^{10,11} The first of these¹⁰ is that $R_{ij}(\vec{r})$ may always be additively decomposed into a symmetric matrix $T_{ij}(\vec{r})$ which transforms as a "proper" tensor and is of even spatial parity and an antisymmetric matrix $P_{ij}(\vec{r})$

which is a pseudotensor with odd spatial parity. It is this pseudotensor P_{ij} from which H_m can be extracted.

The second relevant property is that the most general form of $P_{ij}(\vec{r})$ is

$$P_{ij}(\vec{r}) = \epsilon_{ijm} \partial \Phi(\vec{r}) / \partial r_m, \quad (4)$$

where Φ is a scalar satisfying $\Phi(\vec{r}) = \Phi(-\vec{r})$. This result may be established most easily in Fourier space. Some simple algebra together with the solenoidal constraint leads to the result that there is only one linearly independent homogeneous pseudotensor possible, *viz.*, $\epsilon_{ijm} k_m G(\vec{k})$, where G is an even scalar function of \vec{k} . Equation (4) follows directly by inverse transformation. Additional symmetries imposed on the system (such as axisymmetry or isotropy) or dependence on any number of preferred directions can only modify the way in which G depends on \vec{k} (and Φ depends on \vec{r}), but cannot introduce additional functions into the form of the pseudotensor. The fact that P_{ij} depends only on the gradient of a single scalar function of \vec{r} provides the basis for determining the magnetic helicity.

Let the ij th component of the Fourier transform of the symmetric part of the $\langle \vec{B} \cdot \vec{A} \rangle$ correlation be denoted H_{ij} . Then, with use of Eq. (3), $H_{ij}(\vec{k})$ is

$$H_{ij}(\vec{k}) = i \epsilon_{jrs} k_r S_{is}(\vec{k}) / k^2 \quad (5)$$

and

$$H_m = \int d^3k H_{jj}(\vec{k}), \quad (6)$$

where from Eq. (5) $H_{jj}(\vec{k})$ is the spectrum of magnetic helicity, $H_m(\vec{k})$. Furthermore, because $H_m(\vec{k})$ depends solely on the antisymmetric part of S_{ij} , $G(\vec{k}) = i H_m(\vec{k}) / 2$. Thus,

$$H_m = \langle \vec{A} \cdot \vec{B} \rangle = 2\Phi(\vec{r}=0). \quad (7)$$

It should be emphasized that Eq. (7) is true for arbitrary homogeneous turbulence, and can therefore be utilized even when the symmetries of the fluctuations are *a priori* unknown.

The function $\Phi(\vec{r})$, which determines both the total magnetic helicity and its spectrum, may be evaluated by performing a line integral over separation values:

$$2\Phi(\vec{r}) = 2 \int_{\infty}^{\vec{r}} \nabla \Phi \cdot d\vec{l} = \int_{\infty}^{\vec{r}} dl_i \epsilon_{ijm} R_{jm}(\vec{l}). \quad (8)$$

Equation (8) is valid provided that the correlations vanish rapidly as $|\vec{l}| \rightarrow \infty$. H_m is then given by Eq. (7).

To determine $H_m(\vec{k})$, one requires exhaustive knowledge of Φ for all \vec{r} . However, a *reduced* helicity spectrum is available from knowledge of

$R_{ij}(r_1, 0, 0)$, the correlation tensor for a sequence of collinear separations in the \hat{r}_1 direction. From the definition of the reduced energy spectrum tensor,⁹ $S_{ij}^r(k_1) = \int dk_2 dk_3 S_{ij}(k_1, k_2, k_3)$, one has

$$H_m^r(k_1) = 2 \text{Im} S_{23}^r(k_1) / k_1 \quad (9)$$

and

$$H_m = \int dk_1 H_m^r(k_1) = 2\Phi(r_1=0, 0, 0). \quad (10)$$

The reduced spectrum contains less information than the full three-dimensional spectrum, except in special cases, e.g., slab or isotropic symmetries.

Application of these results to fusion plasmas may be limited by strong inhomogeneities present in a bounded laboratory device. For example, the presence of a mean electric current density implies that the mean field is not uniform, which violates the homogeneity property as usually formulated.⁹ However, in the laboratory one can interpret the ensemble average as an average over identically prepared "shots" of a containment device. This may permit useful values of helicity to be extracted, particularly if the mean fields can be defined in such a way as to establish homogeneity of the high-wave-number fluctuations.

Magnetic helicity can also be measured in space plasmas where R_{ij} has been routinely extracted from measurements of interplanetary magnetic fields. To illustrate the technique for measuring H_m described above, we have used magnetometer data from the Voyager 2 spacecraft¹² in the solar wind near 2.8 astronomical units (AU). The results reported here are from a single span of 64 h on days 95–97 of 1978 during which the spacecraft was in the trailing edge of a small amplitude corotating stream.¹³ Because the mean solar wind flow is super-Alfvénic and very nearly in the heliocentric radial direction, one can use the MHD analogue of the frozen-in flow approximation to obtain $S_{ij}^r(k)$, where k is the radial wave number (the subscripts on k are hereafter omitted). The mean magnetic field during the interval is in the ecliptic plane and 66° from radial. We have assumed that (4)–(10) are valid even though the homogeneity of solar wind fluctuations has not been unambiguously established. (E.g., one-point measurements do not allow mean currents to be measured.)

We evaluated $S_{ij}^r(k)$ via two independent means: the Blackman-Tukey mean lagged product technique with 10 degrees of freedom, and the fast Fourier transform technique smoothed to have

equivalent statistical validity. The two techniques gave essentially identical results. Standard tests were made to eliminate the possibility that aliasing or leakage affected the analysis.¹⁴ The results shown in Figs. 1 and 2 were obtained with the fast Fourier transform approach.

Figure 1 displays both the reduced magnetic energy density $E(k) \equiv S_{ii}^r(k)$ and $|kH_m^r(k)|$ [cf. Eq. (10)] in units of Γ^2 ($1 \Gamma = 10^{-5}$ G). (The solar wind speed of 442 km/s is used to convert from frequency to wave number.)

As has often been reported, $E(k)$ closely approximates a power law (the slope in this case is -1.7 ± 0.1). The notable feature in Fig. 1 is that the envelope of $|kH_m^r(k)|$ closely traces the same power law. Note that there is no tendency for $|kH_m^r(k)|/E(k)$ to become small at large k . In fact, $kH_m^r(k)$ oscillates between $\approx \pm 0.4$ of its maximum possible value $[E(k)]$ throughout much of the spectrum. Nevertheless, the net helicity density ($\approx 18 \text{ G}^2 \text{ cm}$) is entirely due to the helicity in the largest scale fluctuations because at those scales the oscillations of $H_m^r(k)$ do not cancel. The correlation length⁹ ($\int R_{ii} dr/E$) for this data

set is $2.13 \times 10^{12} \text{ cm}$. The analogous helicity correlation length (H_m/E) is $2.0 \times 10^{12} \text{ cm}$. The helicity-containing length $[2\pi H_m / \sum k H_m^r(k)]$ is $1.13 \times 10^{13} \text{ cm}$ compared with the similarly defined energy-containing length $[2\pi E / \sum k E(k)]$, which is $\approx 3.44 \times 10^{11} \text{ cm}$.

The fact that the helicity-containing length is larger than the energy-containing length is consistent with the expectations of the inverse cascade and selective decay conjectures^{5,7,8} and suggests that at some point in their history the magnetic fluctuations in the solar wind have undergone turbulent evolution. Whether this has occurred *in situ*, or is a relic of coronal processes which have been convected outward due to the super-Alfvénic flow, is as yet unclear.

The high degree of correlation and anticorrelation between $E(k)$ and $H_m^r(k)$ is illustrated in Fig. 2 where $k^{5/3}E(k)$ and $k^{8/3}H_m^r(k)$ are plotted against both frequency and wave number. Because of the linear scale, Fig. 2 emphasizes the high-frequency behavior of $H_m^r(k)$. These results suggest that the solar wind during this period is rich in helical structures, exhibiting a significant twist to the magnetic field at very large scales. Large clumps of positive and negative helicity are also found at all smaller scales, but with an average value close to zero. The fact that it is only the average value of the helicity spectrum in the inertial range that is close to zero was apparently not anticipated, and represents one of the new experimental results of this Letter. For example, a recent statistical theory of homogeneous magnetic fluctuations¹⁵ assumes that $kH_m(k)/E(k) = \text{const}$, which appears to be at odds with the present observations.

A second span of magnetometer data from Voyager 1 at 1 AU has been similarly analyzed, and while details will be presented elsewhere, the

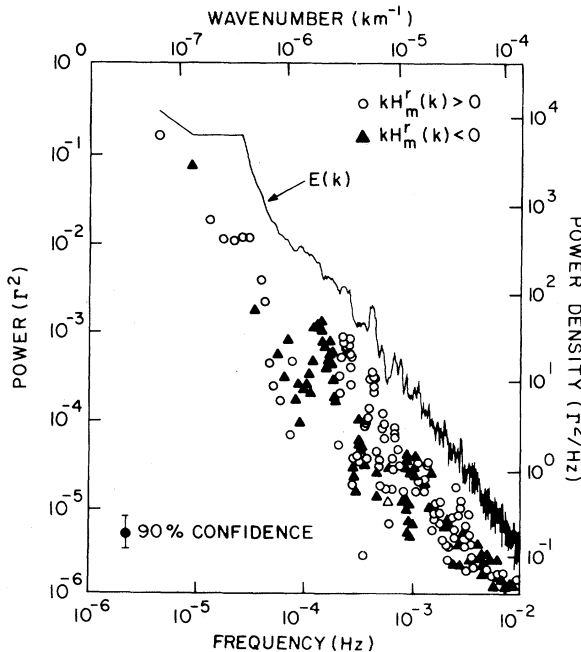


FIG. 1. The reduced magnetic energy density $E(k)$ and the reduced helicity spectrum $|kH_m^r(k)|$ (in energy units) of the solar wind at 2.8 AU. The solar wind velocity is 442 km/s, and the total fluctuation energy is $4.8 \times 10^{-12} \text{ erg/cm}^3$. $E(k)$ has a power-law slope of $k^{-1.7 \pm 0.1}$. For clarity, not all values of $H_m^r(k)$ are plotted at high frequencies.

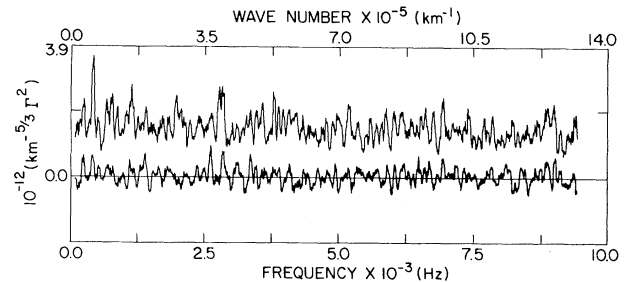


FIG. 2. $E(k)$ and $H_m^r(k)$ plotted on a linear scale for the data shown in Fig. 1. The top trace is $k^{5/3}E(k)$; the bottom trace is $k^{8/3}H_m^r(k)$.

general results are similar: The characteristic lengths of helicity are larger than those of the energy, and the helicity again oscillates between positive and negative values at all smaller scales. The data at 1 AU were taken in a stream interaction region¹³ and included both slow stream and fast stream intervals. Thus these results suggest that there is nothing atypical in the data shown in Figs. 1 and 2. A detailed analysis of the solar wind magnetic field structure at several additional locations in the heliosphere during different time periods and in different types of morphological environments is in preparation and will be presented in a more complete paper.

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Direct Measurement of Desorption Kinetics of ⁴He at Low Temperatures

M. Sinvani, P. Taborek,^(a) and D. Goodstein

California Institute of Technology, Pasadena, California 91125

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A direct method for measuring the desorption time constant of flash-desorbed ⁴He films ($\lesssim 1$ monolayer), adsorbed on Nichrome or Constantan heaters, is described. A time constant τ is found which behaves as $\tau = \tau_0 \exp[E/T_s]$, where T_s is the heater temperature. The value for the characteristic lifetime τ_0 is 10^{-9} – 10^{-10} sec, orders of magnitude shorter than that previously reported. The measured energy parameter E was found to be $\sim \frac{2}{3}$ of the chemical potential.

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If the temperature of a solid surface on which a film is adsorbed is suddenly increased, some adatoms are desorbed and the film reaches a new steady state in a characteristic time τ . Relating the desorption rate τ^{-1} to the surface temperature and the surface-adatom interaction potential is a difficult but basic problem of surface physics which has been the objective of many theoretical

treatments using a variety of approaches.¹ For a weakly bound physisorbed film, theories based on simple thermodynamics and kinetic theory,² transition state theory,³ as well as detailed quantum mechanical calculations⁴⁻⁶ all predict a rate law of the so-called Frenkel form:

$$\tau = \tau_0 \exp(E/T_s) \quad (1)$$