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M. Kamaldar, A. Goel, S. A. U. Islam and D. S. Bernstein, "On the Lack of Robustness of Observers for Systems with Uncertain, Unstable Dynamics," 2023 American Control Conference (ACC), San Diego, CA, USA, 2023, pp. 1643-1648, doi: 10.23919/ACC55779.2023.10156076.

<https://doi.org/10.23919/ACC55779.2023.10156076>

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On the Lack of Robustness of Observers for Systems with Uncertain, Unstable Dynamics

Mohammadreza Kamalidar¹, Ankit Goel², S. A. U. Islam¹, and Dennis S. Bernstein¹

Abstract—We consider the robustness of state estimation for linear, time-invariant systems. Since state estimation is dual to full-state feedback, it may be expected that stability of the error dynamics depends continuously on perturbations of the dynamics matrix. This paper shows, however, that, if the system dynamics are unstable, then—regardless of how the filter gain is chosen—there always exist arbitrarily small perturbations of the system dynamics that give rise to unbounded state-estimation error. Since this phenomenon cannot occur in full-state feedback control, this result reveals a surprising breakdown in the duality between estimation and control.

I. INTRODUCTION

In the classical full-state-feedback control problem for linear, time-invariant plants, the goal is to determine a feedback gain K such that the closed-loop dynamics $A+BK$ are asymptotically stable. Since the eigenvalues of a matrix are continuous functions of its entries, it follows that, if $A+BK$ is asymptotically stable, then, for every perturbation ΔA of sufficiently small norm, the perturbed dynamics $A + \Delta A + BK$ are also asymptotically stable. In other words, by virtue of feedback control, $A + BK$ is inherently robust to plant uncertainty, at least to some extent. For the dual problem of state estimation, $A + BK$ is replaced by $A - FC$, where F is the filter gain. Consequently, one might expect an analogous result to hold, namely, that, if $A - FC$ is asymptotically stable, then, for every perturbation ΔA of sufficiently small norm, the perturbed dynamics $A + \Delta A - FC$ are also asymptotically stable. The present paper shows that this expectation is false.

The reason for this breakdown in duality can be seen as follows. For full-state-feedback control with an uncertain dynamics matrix A , the feedback gain is chosen to ensure that $\hat{A} + BK$ is asymptotically stable, where \hat{A} is the model of A , and thus the physical system is asymptotically stable if and only if $A + BK = \hat{A} + \Delta A + BK$ is asymptotically stable, where $\Delta A \triangleq A - \hat{A}$. For state estimation with known A , in which case $\hat{A} = A$, the error dynamics are given by

$$\dot{e}(t) = (A - FC)e(t). \quad (1)$$

However, in the case where A is uncertain, the error dynamics are not given by

$$\dot{e}(t) = (\hat{A} - FC)e(t), \quad (2)$$

but rather are given by

$$\dot{e}(t) = (\hat{A} - FC)e(t) + \Delta A x(t). \quad (3)$$

In the case where A is unstable and thus x is unbounded, the presence of $x(t)$ in (3) can lead to unbounded error. This observation applies to both estimators and observers.

Robustness of state estimation has been considered extensively in the literature [1], [2], [3], [4]. The case of parameter uncertainty is considered in [5], [6], [7], [8], [9]. In addition, H_2/H_∞ extensions of the Kalman filter are considered in [10], [11], [12], [13], [14], [15]. For the case of uncertain, unstable dynamics, as considered in the present paper, the lack of robustness of observers was noted and analyzed in [1]. In particular, Corollary 1.1 of [1] states that, for systems that are completely unstable, that is, all of whose eigenvalues lie in the closed right-half plane, the state error is nonconvergent for “each and every” perturbation of the system dynamics. The present paper revisits this claim by considering a larger class of systems, namely, systems with unstable dynamics, and by characterizing the class of perturbations under which the state-estimation error is bounded.

It is important to emphasize that state estimation for unstable systems is a problem of significant practical interest. For example, the Kalman filter is routinely used to track the trajectories of ballistic and maneuvering vehicles, whose dynamics are modeled by the (unstable) double-integrator. For systems with asymptotically stable dynamics, however, in the absence of process noise, all states decay to zero, and thus every asymptotically stable estimator correctly predicts the asymptotic states, which are zero. Therefore, the performance of the Kalman filter is especially important when the state is not a priori bounded.

The contents of the paper are as follows. Section II presents the state-estimation problem with uncertain, unstable dynamics. Section III analyzes the error dynamics and investigates the robustness in state estimation with uncertain, unstable dynamics. Finally, Section IV shows that the lack of robustness in state estimation with uncertain, unstable dynamics does not apply to LQG, which is based on certainty equivalence.

The following notation will be used throughout the paper. Let $x_{(i)}$ denote the i th entry of $x \in \mathbb{R}^n$. For $A \in \mathbb{R}^{n \times n}$, let $\text{spec}(A)$ denote the set of eigenvalues of A including multiplicity. The i th row of A and the j th column of A are denoted by $\text{row}_i(A)$ and $\text{col}_j(A)$, respectively. The matrix exponential is written as either e^A or $\exp(A)$. The open left-half plane is denoted by OLHP.

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II. STATE ESTIMATION PROBLEM

Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + D_1w(t), \quad (4)$$

$$y(t) = Cx(t) + D_2v(t), \quad (5)$$

where $A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$, $D_1 \in \mathbb{R}^{n \times l}$, $w \in \mathbb{R}^l$ is Gaussian white noise with intensity I_l , $C \in \mathbb{R}^{p \times n}$, $y \in \mathbb{R}^p$, $D_2 \in \mathbb{R}^{p \times r}$, $v \in \mathbb{R}^p$ is Gaussian white noise with intensity I_l . For convenience, we assume that w and v are uncorrelated, and $V_2 \triangleq D_2D_2^T$ is positive definite. Define $V_1 \triangleq D_1D_1^T$, which is positive semidefinite. Assuming that (A, C) is detectable, the state estimator has the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + F(y(t) - C\hat{x}(t)), \quad (6)$$

where $\hat{x} \in \mathbb{R}^n$ and $F \in \mathbb{R}^{n \times p}$.

The optimal time-varying filter gain $F(t)$ is given by

$$F(t) = Q(t)C^TV_2^{-1}, \quad (7)$$

where the covariance $Q(t)$ of the error

$$e \triangleq \hat{x} - x \quad (8)$$

is given by the Riccati differential equation

$$\dot{Q} = AQ + QA^T - QC^TV_2^{-1}CQ + V_1 \quad (9)$$

with the initial data $Q(0)$. The resulting state estimator yields $\lim_{t \rightarrow \infty} e(t) = 0$. If (A, C) is observable and F is chosen to be the constant matrix

$$F = QC^TV_2^{-1}, \quad (10)$$

where the steady-state error covariance Q is given by the positive-semidefinite solution of algebraic Riccati equation

$$AQ + QA^T - QC^TV_2^{-1}CQ + V_1 = 0, \quad (11)$$

then $A - FC$ is asymptotically stable.

III. ANALYSIS OF THE ERROR DYNAMICS

Subtracting (4) from (6) and using (8) implies that

$$\dot{e}(t) = (A - FC)e(t) - D_1w(t) + FD_2v(t), \quad (12)$$

which is the error dynamics for the case where A is known. If, on the other hand, A is uncertain but an estimate \hat{A} of A is known, then the state estimator (6) can be rewritten by replacing A with \hat{A} as

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + F[y(t) - C\hat{x}(t)] \\ &= \hat{A}\hat{x}(t) - FCe(t) + FD_2v(t). \end{aligned} \quad (13)$$

Subtracting (4) from (13) and using (8) implies that

$$\begin{aligned} \dot{e}(t) &= (A - FC)e(t) + (\hat{A} - A)\hat{x}(t) \\ &\quad - D_1w(t) + FD_2v(t), \end{aligned} \quad (14)$$

which is the error dynamics for the case where A is uncertain and \hat{A} is an estimate of A . Alternatively, using $\hat{x} = e + x$, (14) implies that

$$\begin{aligned} \dot{e}(t) &= (\hat{A} - FC)e(t) + (\hat{A} - A)x(t) \\ &\quad - D_1w(t) + FD_2v(t). \end{aligned} \quad (15)$$

Combining (4) and (15) yields

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ \hat{A} - A & \hat{A} - FC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} D_1 & 0 \\ -D_1 & FD_2 \end{bmatrix} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}. \end{aligned} \quad (16)$$

Henceforth in this paper, and in order to focus on observers, we assume that $w \equiv 0$ and $v \equiv 0$.

Define $H: [0, \infty) \rightarrow \mathbb{R}^{n \times n}$ by

$$H(t) \triangleq \int_0^t e^{\tau(\hat{A}-FC)}(\hat{A} - A)e^{(t-\tau)A} d\tau. \quad (17)$$

Since every entry of the integrand of $H(t)$ is continuous, it follows that, for all $t \geq 0$, $H(t)$ exists.

The following result provides boundedness and convergence properties of the state-estimation error with uncertain dynamics. The statement that a limit exists assumes that the limit is finite.

Proposition 1. For all $t \geq 0$,

$$e(t) = e^{t(\hat{A}-FC)}e(0) + H(t)x(0). \quad (18)$$

If, in addition, $\hat{A} - FC$ is asymptotically stable, then the following statements hold:

- i) H is bounded if and only if, for all $x(0) \in \mathbb{R}^n$, e is bounded.
- ii) $\lim_{t \rightarrow \infty} H(t)$ exists if and only if, for all $x(0) \in \mathbb{R}^n$, $\lim_{t \rightarrow \infty} e(t)$ exists.
- iii) $\lim_{t \rightarrow \infty} H(t) = 0$ if and only if, for all $x(0) \in \mathbb{R}^n$, $\lim_{t \rightarrow \infty} e(t) = 0$.

Proof. To show (18), note that (16) implies that, for all $t \geq 0$,

$$\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \exp \left(t \begin{bmatrix} A & 0 \\ \hat{A} - A & \hat{A} - FC \end{bmatrix} \right) \begin{bmatrix} x(0) \\ e(0) \end{bmatrix},$$

which, using [16, Theorem 1], implies that, for all $t \geq 0$,

$$\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} e^{tA} & 0 \\ H(t) & e^{t(\hat{A}-FC)} \end{bmatrix} \begin{bmatrix} x(0) \\ e(0) \end{bmatrix},$$

which confirms (18).

To show i)–iii), note that, since $\hat{A} - FC$ is asymptotically stable, it follows that $e^{t(\hat{A}-FC)} \rightarrow 0$ as $t \rightarrow \infty$. Thus, (18) implies i). Moreover, (18) implies that $e(t) \sim H(t)x(0)$ as $t \rightarrow \infty$, which confirms ii) and iii). \square

Next, for all integers $i \geq 1$, define $G_i \in \mathbb{R}^{n \times n}$ by

$$G_i \triangleq \sum_{j=1}^i (\hat{A} - FC)^{-j} (\hat{A} - A) A^{j-1}. \quad (19)$$

Proposition 2. The following statements hold:

- i) For all $t \geq 0$,

$$H(t) = \sum_{i=1}^{\infty} \frac{t^i}{i!} (\hat{A} - FC)^i G_i. \quad (20)$$

ii) Assume that $(\hat{A} - A)A = 0$. Then, for all $t \geq 0$,

$$H(t) = [e^{t(\hat{A}-FC)} - I_n]G_1. \quad (21)$$

If, in addition, $\hat{A} - FC$ is asymptotically stable, then H is bounded and

$$\lim_{t \rightarrow \infty} H(t) = -G_1. \quad (22)$$

iii) Assume that A is nilpotent with index k . Then, for all $t \geq 0$,

$$H(t) = [e^{t(\hat{A}-FC)} - I_n]G_k - \sum_{i=1}^{k-1} \frac{t^i}{i!} (\hat{A} - FC)^i (G_k - G_i). \quad (23)$$

If, in addition, $\hat{A} - FC$ is asymptotically stable, then, as $t \rightarrow \infty$,

$$H(t) \sim -G_k - \sum_{i=1}^{k-1} \frac{t^i}{i!} (\hat{A} - FC)^i (G_k - G_i). \quad (24)$$

iv) Assume that A is nilpotent with index $k = 2$. Then, for all $t \geq 0$,

$$H(t) = [e^{t(\hat{A}-FC)} - I_n]G_2 - tG_1A. \quad (25)$$

Assume, in addition, that $\hat{A} - FC$ is asymptotically stable. Then, H is bounded if and only if $(\hat{A} - A)A = 0$.

Proof. To show i), using the definition of matrix exponential, it follows from (17) that, for all $t \geq 0$,

$$\begin{aligned} H(t) &= \int_0^t [I_n + \tau(\hat{A} - FC) + \frac{1}{2}\tau^2(\hat{A} - FC)^2 + \dots] \\ &\quad \cdot (\hat{A} - A)[I_n + (t - \tau)A + \frac{1}{2}(t - \tau)^2A^2 + \dots] d\tau \\ &= \int_0^t [\hat{A} - A + (t - \tau)(\hat{A} - A)A \\ &\quad + \tau(\hat{A} - FC)(\hat{A} - A) + \frac{1}{2}(t - \tau)^2(\hat{A} - A)A^2 \\ &\quad + \tau(t - \tau)(\hat{A} - FC)(\hat{A} - A)A \\ &\quad + \frac{1}{2}\tau^2(\hat{A} - FC)^2(\hat{A} - A) + \dots] d\tau \\ &= t(\hat{A} - A) + \frac{1}{2}t^2[(\hat{A} - FC)(\hat{A} - A) \\ &\quad + (\hat{A} - A)A] + \dots \\ &= \sum_{i=1}^{\infty} \frac{t^i}{i!} (\hat{A} - FC)^i G_i. \end{aligned}$$

To show ii), note that $(\hat{A} - A)A = 0$ implies that, for all $i \geq 2$, $G_i = 0$. It thus follows from i) that, for all $t \geq 0$,

$$H(t) = \sum_{i=1}^{\infty} \frac{t^i}{i!} (\hat{A} - FC)^i G_1,$$

which, using the definition of the matrix exponential, confirms ii).

To show iii), note that since, for all $j \geq k$, $A^j = 0$, it follows from i) that, for all $t \geq 0$,

$$\begin{aligned} H(t) &= \sum_{i=1}^{\infty} \frac{t^i}{i!} (\hat{A} - FC)^i G_{\min\{i,k\}} \\ &= \sum_{i=1}^{k-1} \frac{t^i}{i!} (\hat{A} - FC)^i G_i + \sum_{i=k}^{\infty} \frac{t^i}{i!} (\hat{A} - FC)^i G_k \\ &= \sum_{i=1}^{k-1} \frac{t^i}{i!} (\hat{A} - FC)^i G_i \\ &\quad + [e^{t(\hat{A}-FC)} - I_n - \sum_{i=1}^{k-1} \frac{t^i}{i!} (\hat{A} - FC)^i] G_k, \end{aligned}$$

which confirms iii).

To show iv), first, note that (25) follows from iii) with $k = 2$. Next, to show sufficiency, note that $(\hat{A} - A)A = 0$ implies $G_1 = G_2$ and $G_1A = 0$. Since, in addition, $\hat{A} - FC$ is asymptotically stable, it follows from (25) that $\lim_{t \rightarrow \infty} H(t) = -G_1$, which, since H is continuous, implies that H is bounded. To show necessity, note that since $\hat{A} - FC$ is asymptotically stable, it follows from (25) that $G_1A = (\hat{A} - FC)^{-1}(\hat{A} - A)A = 0$. Since, in addition, $(\hat{A} - FC)^{-1}$ is nonsingular, it follows that $(\hat{A} - A)A = 0$. \square

Note that parts iii) and iv) of Proposition 1 consider nilpotent dynamics A . A special case of nilpotent dynamics is given by the chain of integrators dynamics, for which

$$A = \begin{bmatrix} 0_{(n-1) \times 1} & I_{n-1} \\ 0 & 0_{1 \times (n-1)} \end{bmatrix}. \quad (26)$$

The following result provides convergence properties of H for the case where $n = 1$. In particular, this result shows that estimation for an unstable scalar system has a total lack of robustness.

Proposition 3. Let $n = 1$, and assume that $\hat{A} - FC < 0$. Then,

$$\lim_{t \rightarrow \infty} H(t) = \begin{cases} 0, & A < 0 \text{ or } \hat{A} = A, \\ -\frac{\hat{A}}{\hat{A} - FC}, & A = 0, \\ [\text{sign}(\hat{A} - A)]\infty, & \hat{A} \neq A > 0. \end{cases} \quad (27)$$

Now, assume that $A > 0$ and $x(0) \in \mathbb{R}$ is nonzero. Then, the following statements are equivalent:

- i) $\hat{A} = A$.
- ii) H is bounded.
- iii) $\lim_{t \rightarrow \infty} H(t) = 0$.
- iv) e is bounded.
- v) $\lim_{t \rightarrow \infty} e(t) = 0$.

Proof. It follows from (17) that

$$\begin{aligned} H(t) &= (\hat{A} - A)e^{tA} \int_0^t e^{\tau(\hat{A}-FC-A)} d\tau \\ &= \frac{\hat{A} - A}{\hat{A} - FC - A} (e^{t(\hat{A}-FC)} - e^{tA}). \end{aligned} \quad (28)$$

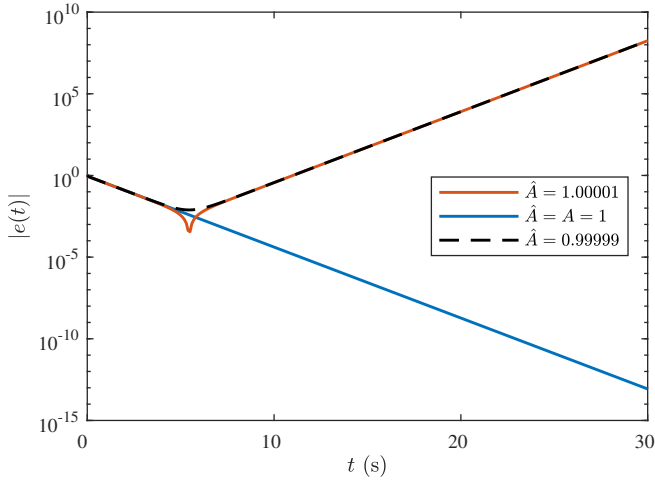


Fig. 1: Example 2. State-estimation error with uncertain, unstable scalar dynamics, where $A = 1$. With $\hat{A} = A$, e converges to 0 exponentially. With $\hat{A} \neq A$, however, e diverges exponentially. As implied by Proposition 3, all perturbations of the system dynamics give rise to unbounded state-estimation error.

Since $\hat{A} - FC < 0$, (28) implies that, as $t \rightarrow \infty$,

$$H(t) \sim -\frac{\hat{A} - A}{\hat{A} - FC - A} e^{tA},$$

which confirms (27).

The equivalence of i - v) follows from (18) and (27). \square

The following example considers scalar dynamics and uses Proposition 3 to show that the state-estimation error converges for all values of \hat{A} assuming that F is chosen such that $\hat{A} - FC < 0$. This example shows that Corollary 1.1 of [1] is incorrect.

Example 1. Uncertain Integrator Dynamics. Consider the integrator dynamics $A = 0$ (i.e., (26) with $n = 1$), and let $C = 1$, $x(0) = 3$, $\hat{x}(0) = 2$, $\hat{A} = -1$, and $F = 2$. Since, $\hat{A} - FC < 0$, it follows from Proposition 3 that $\lim_{t \rightarrow \infty} H(t) = -\hat{A}/(\hat{A} - FC) = -1/3$. It thus follows from Proposition 1 that $\lim_{t \rightarrow \infty} e(t) = -x(0)/3 = -1$. Note that this conclusion contradicts Corollary 1.1 of [1], which implies that the error does not converge. \triangle

Note that Proposition 3 implies that, if $n = 1$ and $A > 0$, then $\hat{A} = A$ is a necessary condition for boundedness and convergence of the state-estimation error. Therefore, in the case of uncertain, unstable scalar dynamics, all perturbations of the system dynamics yield unbounded state-estimation error. The following example illustrates this property.

Example 2. Uncertain, Unstable Scalar Dynamics. Let $A = 1$, $C = 1$, $x(0) = 3$, $\hat{x}(0) = 2$, and $F = 2$. We consider three cases, namely, $\hat{A} = 1.00001$, $\hat{A} = A = 1$, and $\hat{A} = 0.99999$. Note that, for all cases, $\hat{A} - FC$ is asymptotically stable. For both $\hat{A} = 1.00001$ and $\hat{A} = 0.99999$, Proposition 3 implies that, for all nonzero $x(0) \in \mathbb{R}$, e is unbounded. In these two cases, e diverges exponentially, as illustrated in Figure 1. With $\hat{A} = A$, Proposition 3 implies that, for all $x(0) \in \mathbb{R}$, $e \rightarrow 0$ as $t \rightarrow \infty$, as illustrated in Figure 1. \triangle

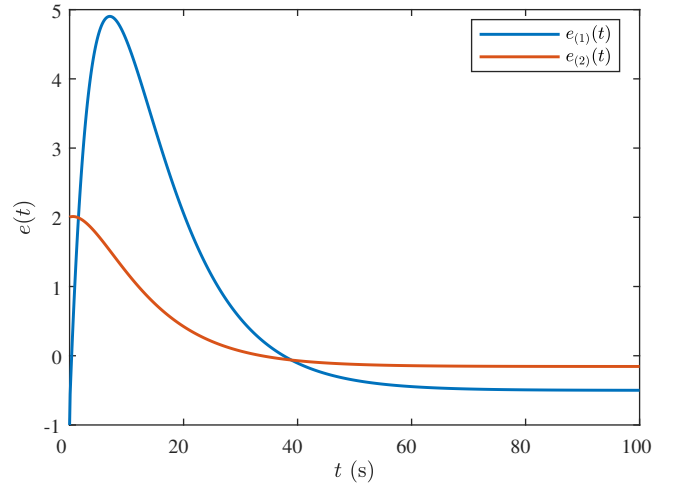


Fig. 2: Example 3. State-estimation error with uncertain double-integrator dynamics, where \hat{A} is chosen such that $(\hat{A} - A)A = 0$. In this case, in contrast to the case of scalar dynamics in Example 2, e converges, as implied by Proposition 4.

For $n \geq 2$, the following result provides sufficient conditions for convergence of the state-estimation error.

Proposition 4. Assume that $(\hat{A} - A)A = 0$ and $\hat{A} - FC$ is asymptotically stable. Then,

$$\lim_{t \rightarrow \infty} e(t) = -G_1 x(0). \quad (29)$$

Proof. Since $\hat{A} - FC$ is asymptotically stable, it follows that $e^{t(\hat{A} - FC)} \rightarrow 0$ as $t \rightarrow \infty$. Thus, (18) and part *ii*) of Proposition 2 confirm (29). \square

Note that condition $(\hat{A} - A)A = 0$ of Proposition 4 is satisfied if and only if each row of $\hat{A} - A$ is in the left null space of A . This condition can be used to characterize a set of perturbed dynamics \hat{A} that yields bounded state-estimation error (e.g., See Proposition 6 in this paper).

The following example illustrates Proposition 4 for an uncertain double-integrator.

Example 3. Uncertain Double-Integrator. Consider the double-integrator dynamics (i.e., (26) with $n = 2$), and let

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0.01 \end{bmatrix}, \quad C = [1 \ 0]. \quad (30)$$

Let $x(0) = [3 \ -1]^T$, $\hat{x}(0) = [2 \ 1]^T$, and $F = [0.3 \ 0.02]^T$. It follows that $\hat{A} - FC$ is asymptotically stable. Note that, since $(\hat{A} - A)A = 0$, it follows from Proposition 4 that

$$\lim_{t \rightarrow \infty} e(t) = -G_1 x(0) = \begin{bmatrix} 0 & 0.5 \\ 0 & 0.16 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = -\begin{bmatrix} 0.5 \\ 0.16 \end{bmatrix}. \quad (31)$$

In this case, in contrast to the case of unstable scalar dynamics, e converges, as illustrated in Figure 2. Nevertheless, this example does not imply that there do not exist arbitrarily small perturbations of the double-integrator dynamics that give rise to unbounded state-estimation error. In fact, Corollary 1 and Example 5 show that, for the double-integrator

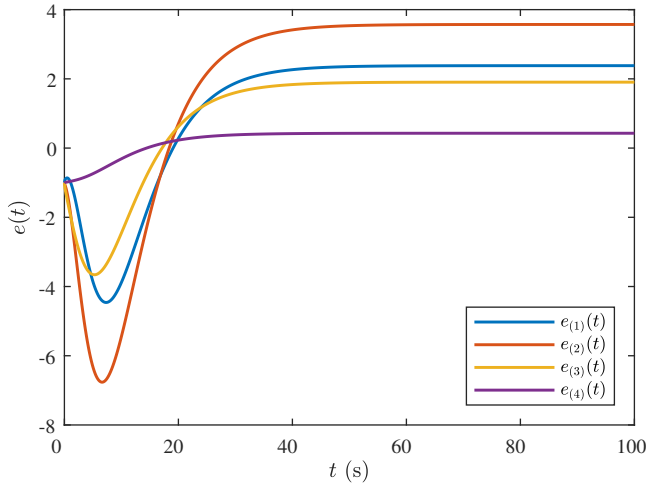


Fig. 3: Example 4. State-estimation error with uncertain, 4th-order chain of integrators dynamics, where \hat{A} is chosen such that $(\hat{A} - A)A = 0$. In this case, in contrast to the case of scalar dynamics in Example 2, e converges, as implied by Proposition 4.

dynamics, there exist arbitrarily small perturbations that yield unbounded state-estimation error. \triangle

The following example illustrates Proposition 4 for an uncertain chain of integrators.

Example 4. Uncertain Chain of Integrators. Consider the chain of integrators dynamics (26) with $n = 4$, and let

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0.01 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}. \quad (32)$$

Let $x(0) = [3 \ 3 \ 3 \ 3]^T$, $\hat{x}(0) = [2 \ 2 \ 2 \ 2]^T$, and $F = [1.5 \ 0.8 \ 0.18 \ 0.01]^T$. It follows that $\hat{A} - FC$ is asymptotically stable. Note that, since $(\hat{A} - A)A = 0$, it follows from Proposition 4 that $\lim_{t \rightarrow \infty} e(t) = [2.38 \ 3.57 \ 1.90 \ 0.43]^T$. In this case, in contrast to the case of unstable scalar dynamics in Example 2, e converges, as illustrated in Figure 3. \triangle

The next result, which relies on Proposition 1, provides a necessary and sufficient condition for the boundedness of the state-estimation error for the case where A is nilpotent with index 2 (e.g., double-integrator dynamics).

Proposition 5. Assume that A is nilpotent with index 2 and $\hat{A} - FC$ is asymptotically stable. Then, for all $x(0) \in \mathbb{R}^n$, e is bounded if and only if $(\hat{A} - A)A = 0$.

Proof. Since $\hat{A} - FC$ is asymptotically stable, it follows from part iv) of Proposition 2 that H is bounded if and only if $(\hat{A} - A)A = 0$. Thus, part i) of Proposition 1 implies that, for all $x(0) \in \mathbb{R}^n$, e is bounded if and only if $(\hat{A} - A)A = 0$. \square

The following result provides the set of all perturbations in the chain of integrator dynamics (26) that yield bounded state-estimation error.

Proposition 6. Let A be a chain of integrators dynamics given by (26). Then, $(\hat{A} - A)A = 0$ if and only if, for all

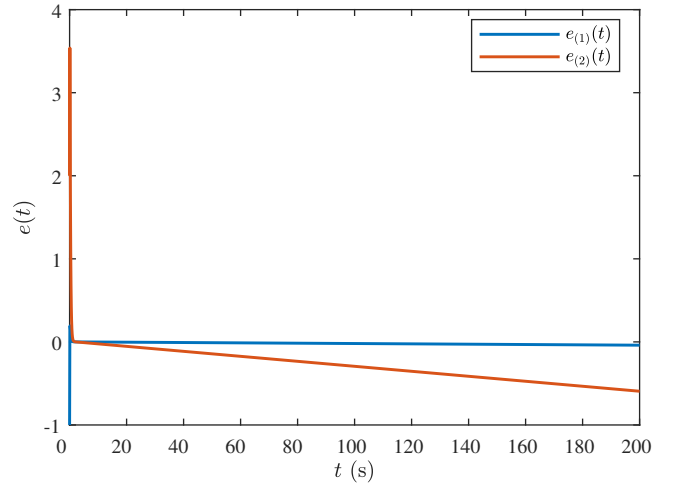


Fig. 4: Example 5. State-estimation error with an uncertain double-integrator dynamics, where \hat{A} is chosen such that $\text{col}_1(\hat{A}) \neq 0$. In contrast to the cases of Examples 3 and 4 where $\text{col}_1(A) = 0$, in this case, e diverges, as implied by Corollary 1.

$i \in \{1, \dots, n-1\}$,

$$\text{col}_i(\hat{A}) = \text{col}_i(A). \quad (33)$$

Proof. To show sufficiency, note that

$$\hat{A} - A = \begin{bmatrix} 0_{n \times (n-1)} & \text{col}_n(\hat{A}) - \text{col}_n(A) \end{bmatrix},$$

which implies that

$$(\hat{A} - A)A = [\text{col}_n(\hat{A}) - \text{col}_n(A)] \otimes \text{row}_n(A) = 0.$$

To show necessity, note that

$$\begin{aligned} 0 &= (\hat{A} - A)A \\ &= \begin{bmatrix} 0_{n \times 1} & \text{col}_1(\hat{A} - A) & \cdots & \text{col}_{n-1}(\hat{A} - A) \end{bmatrix} \\ &= \begin{bmatrix} 0_{n \times 1} & \text{col}_1(\hat{A}) & \cdots & \text{col}_{n-1}(\hat{A}) \end{bmatrix} \\ &\quad - \begin{bmatrix} 0_{n \times 1} & \text{col}_1(A) & \cdots & \text{col}_{n-1}(A) \end{bmatrix}, \end{aligned}$$

which confirms (33). \square

The following result, which follows from Propositions 4–6, provides a necessary and sufficient condition for boundedness and convergence of state-estimation error with uncertain double-integrator dynamics.

Corollary 1. Let A be double-integrator dynamics given by (26) with $n = 2$. Assume that $\hat{A} - FC$ is asymptotically stable. Then, the following statements hold:

- i) e is bounded if and only if $\text{col}_1(\hat{A}) = 0$.
- ii) Assume that $\text{col}_1(\hat{A}) \neq 0$. Then, $\lim_{t \rightarrow \infty} e(t) = -G_1 x(0)$.

The following example illustrates Corollary 1.

Example 5. We reconsider Example 3 but with

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ 0.01 & 0.01 \end{bmatrix}, \quad F = \begin{bmatrix} 15 \\ 50 \end{bmatrix}. \quad (34)$$

Note that, in contrast to Example 3, $\text{col}_1(\hat{A}) \neq 0$. It thus follows from Corollary 1 that e is unbounded. In this case, e diverges, as illustrated in Figure 4.

IV. STATE ESTIMATION WITH FEEDBACK

In this section, we show how the lack of robustness in estimation with uncertain, unstable dynamics disappears in the presence of feedback. In particular, we reconsider the linear time-invariant system (4) and (5) with control input $u \in \mathbb{R}^m$, that is,

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1w(t), \quad (35)$$

$$y(t) = Cx(t) + Du(t) + D_2v(t), \quad (36)$$

where $B \in \mathbb{R}^{n \times m}$ and $D \in \mathbb{R}^{p \times m}$. Assuming that (A, C) is detectable and (A, B) is stabilizable, the state estimation with feedback has the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + F[y(t) - C\hat{x}(t) - Du(t)], \quad (37)$$

$$u(t) = K\hat{x}(t), \quad (38)$$

where $K \in \mathbb{R}^{m \times n}$ is feedback gain. Subtracting (35) from (37) implies that

$$\dot{e}(t) = (A - FC)e(t) - D_1w(t) + FD_2v(t), \quad (39)$$

which is the error dynamics for the case where A is known. If, on the other hand, A is uncertain but an estimate \hat{A} of A is known, then the state estimator (37) can be rewritten by replacing A with \hat{A} as

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + Bu(t) + F[y(t) - C\hat{x}(t) - Du(t)] \\ &= \hat{A}\hat{x}(t) + Bu(t) - FCe(t) + FD_2v(t). \end{aligned} \quad (40)$$

Subtracting (35) from (40) and using (8) implies that

$$\begin{aligned} \dot{e}(t) &= (A - FC)e(t) + (\hat{A} - A)\hat{x}(t) \\ &\quad - D_1w(t) + FD_2v(t), \end{aligned} \quad (41)$$

which is the error dynamics for the case where A is unknown but an estimate \hat{A} of A is known. Alternatively, using $\hat{x} = e + x$, (41) implies

$$\begin{aligned} \dot{e}(t) &= (\hat{A} - FC)e(t) + (\hat{A} - A)x(t) \\ &\quad - D_1w(t) + FD_2v(t). \end{aligned} \quad (42)$$

Combining (35), (38), and (42) yields

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} &= \begin{bmatrix} A + BK & BK \\ \hat{A} - A & \hat{A} - FC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} D_1 & 0 \\ -D_1 & FD_2 \end{bmatrix} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix}. \end{aligned} \quad (43)$$

The following result shows that, in contrast to the case of state estimation without feedback in Section III, the case of state estimation with feedback is robust to sufficiently small perturbations of \hat{A} from A .

Proposition 7. Consider (43), and assume that $A + BK$ is asymptotically stable. There exists $\varepsilon > 0$ such that, if $\|\hat{A} - A\| < \varepsilon$, then, for all $x(0) \in \mathbb{R}^n$, $\lim_{t \rightarrow \infty} e(t) = 0$ if and only if $\hat{A} - FC$ is asymptotically stable.

Proof. Since $A + BK$ is asymptotically stable, it follows that

$$\text{spec} \left(\begin{bmatrix} A + BK & BK \\ 0 & \hat{A} - FC \end{bmatrix} \right) \subset \text{OLHP}, \quad (44)$$

if and only if $\hat{A} - FC$ is asymptotically stable. Since eigenvalues of a matrix are continuously dependent on its entries, it follows from (44) that there exists $\varepsilon > 0$ such that $\|\hat{A} - A\| < \varepsilon$ implies that

$$\text{spec} \left(\begin{bmatrix} A + BK & BK \\ \hat{A} - A & \hat{A} - FC \end{bmatrix} \right) \subset \text{OLHP},$$

if and only if $\hat{A} - FC$ is asymptotically stable. Thus, (43) confirms the result. \square

V. CONCLUSIONS

This paper showed that, for systems with unstable dynamics, modeling errors of arbitrarily small magnitude can result in observers that possess unbounded estimation errors. This severe lack of robustness, which was pointed out in [1], suggests that observers and estimators for real-world systems with unstable dynamics must be implemented and analyzed with extreme caution. Future research will focus on unstable, nonlinear systems, for which we expect that nonlinear observers and estimators suffer from the same severe lack of robustness. The real-world implications of this phenomenon deserve careful attention.

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