

© 2016, Society for Industrial and Applied Mathematics.

<https://doi.org/10.1137/15M1040529>

Access to this work was provided by the University of Maryland, Baltimore County (UMBC) ScholarWorks@UMBC digital repository on the Maryland Shared Open Access (MD-SOAR) platform.

Please provide feedback

Please support the ScholarWorks@UMBC repository by emailing scholarworks-group@umbc.edu and telling us what having access to this work means to you and why it's important to you. Thank you.

FEEDBACK STABILIZATION OF A FLUTTERING PANEL IN AN INVISCID SUBSONIC POTENTIAL FLOW*

IRENA LASIECKA[†] AND JUSTIN T. WEBSTER[‡]

The authors would like to dedicate this work to the memory of Professor A. V. Balakrishnan

Abstract. Asymptotic-in-time feedback control of a panel interacting with an inviscid, subsonic flow is considered. The classical model [E. Dowell, *AIAA*, 5 (1967), pp. 1857–1862] is given by a clamped nonlinear plate strongly coupled to a convected wave equation on the half space. In the absence of imposed energy dissipation the plate dynamics converge to a compact and finite dimensional set [I. Chueshov, I. Lasiecka, and J. T. Webster, *Comm. Partial Differential Equations*, 39 (2014), pp. 1965–1997]. With a sufficiently large velocity feedback control on the structure we show that *the full flow-plate system exhibits strong convergence* to the stationary set in the natural energy topology. To accomplish this task, a novel decomposition of the nonlinear plate dynamics is utilized: a smooth component (globally bounded in a higher topology) and a uniformly exponentially decaying component. Our result implies that *flutter* (a periodic or chaotic end behavior) can be eliminated (in subsonic flows) with sufficient frictional damping in the structure. While such a result has been proved in the past for regularized plate models (with rotational inertia terms or thermal considerations [I. Chueshov and I. Lasiecka, *Springer Monogr. Math.*, Springer-Verlag, Berlin, 2010; I. Lasiecka and J. T. Webster, *Comm. Pure Appl. Math.*, 13 (2014), pp. 1935–1969; I. Ryzhkova, *J. Math. Anal. Appl.*, 294 (2004), pp. 462–481; I. Ryzhkova, *Z. Angew. Math. Phys.*, 58 (2007), pp. 246–261], this is the first treatment which does not incorporate smoothing effects for the structure.

Key words. strong stability, nonlinear plates, flow-structure interaction, PDE with delay, attractors

AMS subject classifications. 74F10, 74K20, 76G25, 35B40, 35G25, 37L15

DOI. 10.1137/15M1040529

1. Introduction. This work is motivated by physical problems in aeroelasticity. *Flutter* of a thin structure immersed in a flow field occurs (as a bifurcation at a particular flow velocity) when the structural modes couple with the dynamic loading of the flow, creating a feedback [4, 21, 22]. The canonical example of *panel flutter* occurs in aircraft paneling: panel deformations perturb the flow field, which then determines the dynamic loading on the surface of the plate (resulting in structural deformation, etc.). The principal challenge in aeroelasticity is the control, suppression, or prevention of the flutter of mechanical structures in flight [23, 24, 25]. Though the *flutter point* may be predicted from linear dynamics, accurately describing post-flutter displacements of a panel requires a nonlinear model [21, 22, 23].

From a mathematical point of view the flutter problem may be described as *stabilization to a given stationary set* for a coupled evolutionary PDE system: a nonlinear clamped plate interacting with a potential flow [4, 21, 22, 29]. Here, we consider a Berger plate in the absence of *rotational inertia* (consistent with the conventions of

*Received by the editors September 21, 2015; accepted for publication (in revised form) February 4, 2016; published electronically June 2, 2016.

<http://www.siam.org/journals/sima/48-3/M104052.html>

[†]Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152 and IBS, Polish Academy of Sciences, Warsaw, Poland (lasiecka@memphis.edu). The research of this author was partially supported by the National Science Foundation with grant NSF-DMS-0606682 and the United States Air Force Office of Scientific Research with grant AFOSR-FA99550-9-1-0459.

[‡]Mathematics, The College of Charleston, Charleston, SC 29424 (websterj@cofc.edu). The research of this author was partially supported by the National Science Foundation with grant NSF-DMS-1504697.

aeroelasticity; see [4, 21, 22, 23, 24] and references therein), where the plate is thin and in-plane accelerations are not accounted for [31]. The task at hand, then, is to determine conditions under which the PDE system is *strongly stable*, in the sense that full flow-plate trajectories converge (in the strong topology of the underlying finite energy space) to a set of interest (e.g., the equilibria set for the dynamical system generated by the solutions). Such a strong convergence to equilibria does not seem possible in the absence of dissipative mechanisms. (See [9] for a detailed discussion and illustrative example.) Thus, we seek an interior velocity feedback control (of sufficient size—depending on the intrinsic parameters of the problem, not on the particular initial data). Under some physical assumptions this will imply that flutter is not present asymptotically in time. Our result demonstrates that flutter can be eliminated (in the subsonic regime), provided a sufficiently large damping is present in the plate. In agreement with the physics of the problem, the linear model for the flow dynamics remains undamped.

In this treatment, we primarily focus on the Berger plate model. Other work on flutter dynamics have considered Berger-like beam and plate models (see [13] and references therein), as well as the scalar von Karman equations [14, 16, 18, 45]. Many of our supporting results here are valid for both the von Karman *and* Berger plate, but the principal results presented below in section 1.6 concern Berger plate dynamics. The Berger nonlinearity is nonlocal and of cubic-type. It is arrived at via a simplification of the scalar von Karman equation [2, 36, 44] (with the scalar von Karman equation itself a simplification of the *full von Karman system* [30, 31]).¹

Physically, when the plate has no portion of its boundary *free*, the Berger approximation is taken to be valid [36, 44] (and especially in the case of clamped conditions, as we take here). The von Karman plate dynamics were considered in [33] (and in the treatments leading up to [33], including [12, 40, 41] and [14]). In [33], large *static and viscous* damping was considered to obtain results analogous to our main result here (which is valid for both the nonrotational von Karman and Berger dynamics). Here, we consider the case where *only viscous damping is active* in the plate dynamics (though we cannot dispense with the size requirement).

1.1. Previous literature in relation to the present work. The study in [16] provides long-time asymptotic properties of (finite energy) solutions for the model herein (for Berger, von Karman, and other nonlinearities of *physical type* [18]) *with no restrictions on the damping size*—it can be taken to be zero. In this case, the originally “rough” plate dynamics become asymptotically regular and finite dimensional in nature. Thus, the flow has the ability to dissipate mechanical plate energy and thereby induce a degree of “stability” to the structural dynamics. However, this stability, in general, may be of a chaotic character. It is one of the principal purposes of this treatment to show that, in the subsonic case, sufficiently large dissipation in the structure alone prevents chaotic behavior and leads to strong stability to the equilibria set.

We now mention three other closely related scenarios in the literature to date: (i) the addition of thermoelastic dynamics to the nonlinear plate, (ii) the presence of rotational inertia terms *and strong* mechanical damping, and (iii) large *static and viscous* damping in the model presented here. (Each of these treatments address von Karman plates, though the results also hold for Berger plates.) The treatments in [40, 41] consider the plate with the addition of a heat equation in the plate dynamics. In this case no damping is needed, as the analytic smoothing and stability properties

¹In fact, in many cases—not here—the analysis of the scalar von Karman plate subsumes that of Berger [14].

of thermoelasticity provide ultimate compactness of the plate dynamics and, furthermore, provide a stabilizing effect for the flow to equilibria (in subsonic flows). As for (ii) mentioned above, results on ultimate compactness of plate dynamics, as well as convergence (for subsonic flows) of full flow-plate trajectories to points of equilibrium, were announced in [12], with a proof sketch appearing in [14]. The recent treatment [33] addresses (iii); it is shown that by considering “large” static damping and “large” viscous damping—of the form $D \cdot [u_t + u]$ in the plate equation, D sufficiently large—flow-plate trajectories with finite energy initial data converge to the set of equilibria. The techniques in [33] are rooted in Lyapunov methods and utilize *critically* the prior work in [13, 14, 40, 41]. The key point there is that for large static and viscous damping, exponential decay of the plate velocity provides global-in-time Hadamard continuity for the flow-plate dynamics, which permits approximation by smooth data (for which the desired result holds—Theorem 4.9 here).

The main goal of the present work is to obtain strong stability results (strong convergence to the set of equilibria) *without accounting for smoothing effects in plate dynamics* and without *assuming large static damping* imposed on the structure (plate). The primary mathematical issue to contend with is low regularity of the hyperbolic Neumann map (boundary into the interior)—i.e., the failure of the uniform Lopatinski conditions in dimensions higher than one. This precludes direct analysis of the coupling via the compactness properties of the aforementioned Neumann map. (Such an approach was used critically in previous analyses [5, 6, 12, 14, 40, 41].) In fact, while we still rely on these past developments, the key point of departure is that we can no longer afford to treat the problem componentwise. Rather, we must rely on global analysis involving relaxed, long-time compactness [20, 42], which depends on uniform long-time invariance of higher energies [33]. This latter property is highly nontrivial due to the effects of the plate nonlinearity combined with the lack of damping exhibited by the flow. In order to overcome this obstacle we shall rely on the idea introduced in [14] and used in [16], where the structural equation under the influence of a flow is represented (for large times) by a delay system. It is then shown in [16] that the corresponding reduced plate admits a *smooth* global attracting set. In order to extract a long-time convergence valid for the *entire flow-structure system* it would suffice to have a control of the L_1 rate of convergence of delayed plate solutions to such an attractor. This brings forward a concept of *exponential attractor*, which by definition provides exponential convergence rates, but it may lose “smoothness” properties. This discussion leads to a well known dichotomy between smoothness and rates of convergence to attracting sets. Resolving this dichotomy lies at the heart of the problem presented.

A delicate argument based on a careful analysis of delayed dynamics, along with a uniform-in-time Hadamard continuity property exhibited by a suitable decomposed part of the full dynamics only, allows us to reconcile this dilemma. As a consequence, we show that, with a sufficiently large *viscous damping parameter*, the entire finite energy flow-plate dynamics exhibits strong convergence properties. We emphasize that our results require *only viscous damping* in the structure and *do not make use of any advantageous parabolic (smoothing) effects* or additional static damping. One may argue with the requirement of sufficiently large size of the damping (not needed in parabolic like problems); however, the fact that we de facto stabilize a delayed plate equation (with inherent instability) justifies mathematically [37, 39] the need for large viscous damping.

Our result provides mathematical insight into the panel flutter problem; in [16] the flow, while driving the plate dynamics, also contributes a stabilizing effect to the

plate dynamics. Our present result indicates that for a panel in a subsonic flow, strong stability of the plate (due to mechanical damping) can be *transferred* to the flow, in some sense. (We note that in all real panels there is some degree of internal damping.) These results are in agreement with experimental and numerical studies wherein *divergence* (or “buckling”) of panels is observed for subsonic flows, in contrast to chaotic or periodic end behavior (i.e., *flutter*) in the case of supersonic flow velocities. We quote from a recent survey article authored by foremost aeroelastician Dowell [25] with regard to subsonic flows: “...if the trailing edge, as well as the leading edge, of the panel is fixed then divergence (static aeroelastic instability) will occur rather than flutter.” In the analysis below, the subsonic nature of the flow is critical, as it provides a viable energy balance equation; this energy balance is not available for supersonic flows [18].

The engineering literature in [2, 36, 44] analyzes the validity of Berger’s plate approximation (originally appearing in [2]) and concludes that it is accurate in the case of clamped and hinged boundary conditions. The approximation is based on the assumption that the second strain invariant is negligible. Mathematically, the Berger dynamics are addressed in [8, 13, 19] (in the context of aeroelasticity) and also in the works [11, 27, 28]. Much of the abstract work in [14] (which is focused on the scalar von Karman nonlinearity) also applies to the Berger dynamics. The main results appearing in [16, 17, 18, 45] for the model presented above discuss and analyze both the Berger *and* von Karman nonlinearities, as well as other “physical” nonlinearities satisfying a specific set of bounds (which are referred to as *nonlinearities of physical type*).

1.2. Outline. The rest of the paper is organized as follows:

- The remainder of the introduction is devoted to a discussion of the mathematical model, including the relationship between our analysis and previous literature. We discuss the fundamental notions of well-posedness and associated energies and describe some key results from [16, 33] which motivate this analysis. We then describe the main results.
- Section 2 describes the technical preliminaries needed to read the proof of the main results. This section can also be read as a detailed summary of the preceding work on the specific flow-plate model described herein.
- Section 3 describes the supporting results from [16], used here, which deal with the existence of a compact global attractor for the plate dynamics, as well as some weak stability properties.
- In section 4.2 we prove the primary *supporting* result upon which our later analysis is based: Theorem 4.3, which says that a global-in-time bound on the plate dynamics in higher norms yields the desired convergence to equilibria result for the full flow-plate dynamics. This section outlines a strategy of passing convergence results from *smooth* initial data onto finite energy initial data, and we describe in section 4.5 the primary technical obstacles in showing the convergence to equilibria result for finite energy initial data in the absence of smooth effects or large static damping.
- Section 5 begins by providing an overview of the main steps in the proof and their relation to the established results in section 2. (We believe this to be helpful to the reader, owing to the technical and multistep nature of the proof.) The proof of the main result follows in substeps: a decomposition of plate solutions (sections 5.3 and 5.4), a key decomposition for the full flow-plate dynamics, an analysis of the decomposed dynamics, and a concluding argument which yields the final result.

- In section 6 we discuss the open problem of eliminating the size requirements on the damping.
- Readers will find the authors' acknowledgements in the final section.

1.3. Notation. For the remainder of the text we write \mathbf{x} for $(x, y, z) \in \mathbb{R}_+^3$ or $(x, y) \in \Omega \subset \mathbb{R}_{(x,y)}^2$, as dictated by context. Norms $\|\cdot\|$ are taken to be $L_2(D)$ for a domain D . The symbols ν and τ will be used to denote the unit normal and tangent vectors to a given domain, again, dictated by context. Inner products in $L_2(\mathbb{R}_+^3)$ are written (\cdot, \cdot) , while inner products in $L_2(\mathbb{R}^2 \equiv \partial\mathbb{R}_+^3)$ are written $\langle \cdot, \cdot \rangle$. Also, $H^s(D)$ will denote the Sobolev space of order s , defined on a domain D , with $H_0^s(D)$ denoting the closure of $C_0^\infty(D)$ in the $H^s(D)$ norm (which we denote by $\|\cdot\|_{H^s(D)}$ or $\|\cdot\|_{s,D}$). When $s = 0$ we may abbreviate the notation to $\|\cdot\|$. We will also utilize the homogenous Sobolev space

$$W^k(\mathbb{R}_+^3) \equiv \left\{ \phi(\mathbf{x}) \in L_2^{loc}(\mathbb{R}_+^3) : \|\phi\|_{W^k}^2 \equiv \sum_{j=0}^{k-1} \|\nabla^j \phi\|_{j, \mathbb{R}_+^3}^2 \right\}.$$

We make use of the standard notation for the trace of functions defined on \mathbb{R}_+^3 , i.e., for $\phi \in H^1(\mathbb{R}_+^3)$, $tr[\phi] = \phi|_{z=0}$ is the restriction of ϕ on the plane $\{\mathbf{x} : z = 0\}$. (We use analogous notation for $tr[w]$ as the map from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$.) Finally, we will denote a ball of radius R centered at c in a topological vector space X by $B_X(c, R)$; when the center is the zero element we simply write $B_X(R)$.

1.4. Mathematical model and natural energies. The gas-flow environment is modeled by $\mathbb{R}_+^3 = \{(x, y, z) : z > 0\}$. The plate is immersed in an inviscid potential flow (over body) with velocity $U < 1$ in the x -direction. (Here we (i) normalize $U = 1$ to be Mach 1, i.e., $0 \leq U < 1$ corresponds to a subsonic flow, and (ii) consider only the dynamics in the upper half space.) The panel is modeled by a bounded domain $\Omega \subset \mathbb{R}_{(x,y)}^2 = \{\mathbf{x} : z = 0\}$ with smooth boundary $\partial\Omega = \Gamma$, and the scalar function $u : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ represents the (transverse) displacement of the plate in the z -direction at (x, y) at the moment t . For the flow component of the model we make use of linear potential theory [4, 24], and the (perturbed) flow potential is given by $\phi : \mathbb{R}_+^3 \times \mathbb{R}_+ \rightarrow \mathbb{R}$. The strong coupling occurs (i) in the dynamic pressure term (RHS of the plate equation), which contains the acceleration potential of the flow, and (ii) in the downwash of the flow (Neumann condition), the latter including the material derivative of the structure.

$$(1.1) \quad \begin{cases} u_{tt} + \Delta^2 u + ku_t + f(u) = p_0 + r_\Omega tr[(\partial_t + U\partial_x)\phi] & \text{in } \Omega \times (0, T), \\ u(0) = u_0; \quad u_t(0) = u_1, \\ u = \partial_\nu u = 0 & \text{on } \Gamma \times (0, T), \\ (\partial_t + U\partial_x)^2 \phi = \Delta \phi & \text{in } \mathbb{R}_+^3 \times (0, T), \\ \phi(0) = \phi_0; \quad \phi_t(0) = \phi_1, \\ \partial_z \phi = [(\partial_t + U\partial_x)u]_{\text{ext}} & \text{on } \mathbb{R}_{(x,y)}^2 \times (0, T). \end{cases}$$

The notation r_Ω corresponds to the restriction (of the function's domain) to Ω (in the appropriate functional sense) for functions supported on $\partial\mathbb{R}_+^3$; correspondingly, we have the extension by zero for functions supported on Ω to the entirety of $\partial\mathbb{R}_+^3$ (in particular, for elements of $H_0^2(\Omega)$), denoted by u_{ext} . The term p_0 is a static pressure

on the surface of the plate. The nonlinearity of principal interest here is the Berger nonlinearity:

$$(1.2) \quad f_B(u) = [b - \|\nabla u\|^2] \Delta u.$$

The parameter $b \geq 0$ is a physical parameter [36, 44] which corresponds to in-plane bending or stretching.²

The *plate energy* is defined as usual [14, 31]:

$$(1.3) \quad E_{pl}(u) = \frac{1}{2} [\|u_t\|_\Omega^2 + \|\Delta u\|_\Omega^2] + \Pi(u).$$

$\Pi(u)$ is a potential of the nonlinear and nonconservative forces, given by

$$(1.4) \quad \Pi(u) = \Pi_B(u) = \frac{1}{4} \|\nabla u\|_\Omega^4 - \frac{b}{2} \|\nabla u\|_\Omega^2 - \langle p_0, u \rangle_\Omega.$$

The natural energies associated with the subsonic wave and interactive dynamics are given below:

$$(1.5) \quad E_{fl}(\phi) = \frac{1}{2} [\|\phi_t\|_{\mathbb{R}_+^3}^2 - U^2 \|\partial_x \phi\|_{\mathbb{R}_+^3}^2 + \|\nabla \phi\|_{\mathbb{R}_+^3}^2];$$

$$(1.6) \quad E_{int}(u, \phi) = 2U \langle tr[\phi], u_x \rangle_\Omega.$$

The total (unsigned) energy is then defined to be

$$(1.7) \quad \mathcal{E}(u(t), \phi(t)) = \mathcal{E}(t) = E_{pl}(u(t)) + E_{fl}(\phi(t)) + E_{int}(u(t), \phi(t)).$$

We will also need to consider *positive energies*, so we define

$$(1.8) \quad \Pi_*(u) = \frac{1}{4} \|\nabla u\|_\Omega^4, \quad E_*(u) = \frac{1}{2} [\|u_t\|_\Omega^2 + \|\Delta u\|_\Omega^2] + \Pi_*(u),$$

$$(1.9) \quad \mathcal{E}_*(u, \phi) = E_*(u) + E_{fl}(\phi).$$

According to these norms, the natural energy space is then

$$(1.10) \quad Y = Y_{pl} \times Y_{fl} \equiv (H_0^2(\Omega) \times L_2(\Omega)) \times (W^1(\mathbb{R}_+^3) \times L_2(\mathbb{R}_+^3)),$$

taken with norm

$$(1.11) \quad \|(u, v; \phi, \psi)\|_Y^2 = \|\Delta u\|_\Omega^2 + \|v\|_\Omega^2 + \|\nabla \phi\|_{\mathbb{R}_+^3}^2 + \|\psi\|_{\mathbb{R}_+^3}^2.$$

We can also consider a stronger space:

$$(1.12) \quad Y_s \equiv (H_0^2(\Omega) \times L_2(\Omega)) \times (H^1(\mathbb{R}_+^3) \times L_2(\mathbb{R}_+^3)).$$

1.4.1. Damping. The damping coefficient is given by $k \geq 0$, representing viscous damping (or energy dissipation) effects in the model. This can be viewed as intrinsic damping for the structure, or imposed as a velocity feedback control. For certain results (see [16]) we can consider $k = 0$, but, for the main results below we will need specific assumptions on the size of the damping coefficient k .

²In full generality, b can be any real number; we choose the more mathematically interesting nondissipative case, with $b \geq 0$.

Additionally, in some auxiliary results presented below (especially those related to [33]) we will refer to *static damping*—this occurs when the viscous damping ku_t is accompanied by a term of the form βu . The resulting system, which will be referred to below, is

$$(1.13) \quad \begin{cases} u_{tt} + \Delta^2 u + \boxed{ku_t + \beta u} + f(u) = p_0 + r_\Omega \operatorname{tr}[(\partial_t + U\partial_x)\phi] & \text{in } \Omega \times (0, T), \\ u(0) = u_0; \quad u_t(0) = u_1, \\ u = \partial_\nu u = 0 & \text{on } \Gamma \times (0, T), \\ (\partial_t + U\partial_x)^2 \phi = \Delta \phi & \text{in } \mathbb{R}_+^3 \times (0, T), \\ \phi(0) = \phi_0; \quad \phi_t(0) = \phi_1, \\ \partial_z \phi = [(\partial_t + U\partial_x)u]_{\text{ext}} & \text{on } \mathbb{R}_{(x,y)}^2 \times (0, T). \end{cases}$$

Note that when $\beta = 0$ this system collapses to (1.1), the principal model under consideration here. In the case $\beta > 0$, we adapt the plate energy E_{pl} and augment it by the quantity $\beta \|u\|^2$:

$$E_{pl,\beta}(u) = \frac{1}{2} [\|u_t\|^2 + \beta \|u\|^2 + \|\Delta u\|^2] + \Pi(u) = E_{pl}(u) + \frac{\beta}{2} \|u\|^2.$$

In fact, the primary results in [33] consider *large* static and viscous damping for the von Karman plate. The main aim of the present paper is to obtain strong stability result *without* any static damping, i.e., with $\beta = 0$. This will be achieved by demonstrating that sufficient amount of stability can be harvested from the flow itself. However, a rigorous proof of this physically plausible fact is quite involved and delicate.

1.5. Well-posedness and fundamental notions. The dynamics discussed above, when cast in the appropriate framework, are well-posed [5, 6, 14, 15, 18, 45]. (For precise definitions of *strong*, *mild*, and *weak* solutions for the flow-plate dynamics consult [14, 15, 33, 45].) The following well-posedness result is established in [45] (and in [15]).

THEOREM 1.1 (nonlinear semigroup). *Assume $U < 1$, $p_0 \in L_2(\Omega)$. Then for any $T > 0$, (1.1) has a unique strong (resp., generalized—and hence weak) solution on $[0, T]$, denoted by $S_t(y_0)$, for initial data $y_0 = (u_0, u_1; \phi_0, \phi_1) \in Y$. (In the case of strong solutions, the natural compatibility condition must be in force on the data: $\partial_z \phi_0 = (u_1 + Uu_{0x})_{\text{ext}}$.) Moreover, (S_t, Y) is a (nonlinear) dynamical system. Additionally, for $y_0 \in Y_s$, (S_t, Y_s) is also a dynamical system. Weak (and hence generalized and strong) solutions satisfy the following energy equality:*

$$(1.14) \quad \mathcal{E}(t) + k \int_s^t \int_\Omega |u_t(\tau)|^2 d\Omega d\tau = \mathcal{E}(s)$$

for $t > s$. Moreover, the solution $S_t(y_0)$ is stable in the norm of Y , i.e., there exists a constant $C(\|y_0\|_Y)$ such that for all $t \geq 0$ we have

$$(1.15) \quad \|S_t(y_0)\|_Y \leq C(\|y_0\|_Y).$$

In addition, the semigroup S_t is locally Lipschitz on Y :

$$(1.16) \quad \|S_t(y_1) - S_t(y_2)\|_Y \leq C(R, T) \|y_1 - y_2\|_Y \quad \forall \|y_i\|_Y \leq R, \quad t \leq T.$$

Remark 1.1. The proof of Theorem 1.1 is based on a suitable rescaled variant of monotone operator theory and is given in [45]. See also [15] for a different approach based on a viscosity method. In the case when $U > 1$ the situation is more delicate and dealt with in [18]. In this latter case (1.14) no longer holds, and the inequality in (1.15) is valid only locally, for $t \leq T$. This is the result of the loss of dissipativity for the flow dynamics occurring in the supersonic case.

Remark 1.2. The above result and the discussion which follows below is unaffected by the presence of static damping; this is to say that the same well-posedness results hold (*mutatis mutandis*) for (1.13).

For the above semigroup we introduce the dynamics operator $\mathbb{T} : \mathcal{D}(\mathbb{T}) \subset Y_s \rightarrow Y_s$. For its precise structure, we refer to [15, 18, 45], which provide the details of the abstract model.³ A key property needed in this treatment is

$$(1.17) \quad \mathcal{D}(\mathbb{T}) \subset (H^4 \cap H_0^2)(\Omega) \times H_0^2(\Omega) \times H^2(\mathbb{R}_+^3) \times H^1(\mathbb{R}_+^3).$$

Remark 1.3. As per [14, 15, 45] the natural *invariance* of the dynamics is with respect to the norm $\|\cdot\|_Y$. However, via the elementary estimate

$$(1.18) \quad \|\phi(t)\|_{L_2(\mathbb{R}_+^3)} \leq \|\phi_0\|_{L_2(\mathbb{R}_+^3)} + \int_0^t \|\phi_t(\tau)\|_{L_2(\mathbb{R}_+^3)} d\tau,$$

invariance in Y_s can be recovered *on finite time intervals*.

In order to describe the dynamics of the flow in the context of long-time behavior it is necessary to introduce local spaces, denoted by $Y_{fl,\rho}$:

$$\|(\phi_0, \phi_1)\|_{Y_{fl,\rho}} \equiv \int_{K_\rho} |\nabla \phi_0|^2 + |\phi_1|^2 d\mathbf{x},$$

where $K_\rho \equiv \{\mathbf{x} \in \mathbb{R}_+^3; |\mathbf{x}| \leq \rho\}$. We denote by Y_ρ the space $Y_{pl} \times Y_{fl,\rho}$, and we will consider convergence (in time) in Y_ρ for any $\rho > 0$. Such convergence will be denoted by *convergence in the sense of \tilde{Y}* .

By virtue of the Hardy inequality [14, p. 301]

$$\|\phi_0\|_{L_2(K_\rho)}^2 \leq C_\rho \|\nabla \phi_0\|_{L_2(\mathbb{R}_+^3)}^2$$

and hence

$$\|(\phi_0, \phi_1)\|_{Y_{fl,\rho}}^2 \leq \|(\phi_0, \phi_1)\|_{H^1(K_\rho) \times L_2(K_\rho)}^2 \leq \|(\phi_0, \phi_1)\|_{Y_{fl}}^2.$$

Remark 1.4. We clarify the relation between the various flow topologies: $H^1(\mathbb{R}_+^3) \times L_2(\mathbb{R}_+^3)$, Y_{fl} , and $Y_{fl,\rho}$. Initial flow data (ϕ_0, ϕ_1) are typically chosen in $H^1(\mathbb{R}_+^3) \times L_2(\mathbb{R}_+^3)$. Solutions $(u(t), u_t(t); \phi(t), \phi_t(t))$ are global-in-time bounded in the topology of Y but not necessarily global-in-time bounded in the full Y_s norm (owing to the contribution of the L_2 norm from $\phi(t)$). Hence, for considerations involving $t \rightarrow \infty$, we require a spatially-localized flow perspective. By restricting to any ball $K_\rho \subset \mathbb{R}_+^3$ for the flow dynamics we obtain global-in-time boundedness in the topology of Y , as well as local compactness results.

³We suffice to say that Ball's method provides the generator of the nonlinear semigroup with appropriate dense domain $\mathcal{D}(\mathbb{T})$.

By the boundedness in Theorem 1.1, the topology corresponding to Y_ρ (for any $\rho > 0$) becomes a viable measure of long-time behavior which we will refer to as the *local energy sense*, and it is appropriate to take limits of the form $\lim_{t \rightarrow \infty} \|S_t(y_0)\|_{Y_\rho}$ for any $\rho > 0$, where

$$\|\cdot\|_{Y_\rho} \equiv \|\cdot\|_{Y_{pl} \times Y_{fl, \rho}},$$

and S_t is the flow on Y associated to the well-posedness in Theorem 1.1 above.

We now note the boundedness (from below) of the nonlinear energy. Such a bound is necessary to obtain the boundedness of the semigroup quoted in Theorem 1.1 above. First, we have [45, Lemma 5.2, p. 3136] the following.

LEMMA 1.2. *Let the hypotheses of Theorem 1.1 be in force. Then for generalized solutions to (1.1), there exist positive constants c, C , and M positive such that*

$$(1.19) \quad c\mathcal{E}_*(t) - M_{p_0, b} \leq \mathcal{E}(t) \leq C\mathcal{E}_*(t) + M_{p_0, b}.$$

The proof of Lemma 1.2 given in [45] relies on two estimates controlling lower frequencies. Since these estimates will be used frequently in what follows, we shall formulate the relevant results. The first estimate controls interactive energy E_{int} on the strength of Hardy inequality (see [14, p. 301]).

LEMMA 1.3.

$$(1.20) \quad |E_{int}(t)| \leq U\delta \|\nabla \phi(t)\|_{\mathbb{R}_+^3}^2 + \frac{C \cdot U}{\delta} \|u_x(t)\|_{\Omega}^2, \quad \delta > 0.$$

The second one controls lower frequencies [14, p. 49] by exploiting nonlinear effects of potential energy.

LEMMA 1.4. *For any $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\epsilon > 0$ there exists M_ϵ such that*

$$\|u\|_{\Omega}^2 \leq \epsilon [\|\Delta u\|_{\Omega}^2 + \Pi_*(u)] + M_\epsilon.$$

Remark 1.5. In fact, in the case of Berger's nonlinearity Lemma 1.4 holds without the term $\|\Delta u\|_{\Omega}^2$. However, in the case of other nonlinearities, such as von Karman, the presence of Δ is indispensable. For this reason we cite a more general version of this lemma which can be used for von Karman models as well.

Note. Global-in-time boundedness of solutions cannot be obtained without accounting for nonlinear effects.

From the above lemmas and energy inequality we also have (see [15, 45]) the following.

LEMMA 1.5. *Let the hypotheses of Theorem 1.1 be in force. Then any generalized (and hence weak) solution to (1.1) satisfies the bound*

$$(1.21) \quad \sup_{t \geq 0} \left\{ \|u_t\|_{\Omega}^2 + \|\Delta u\|_{\Omega}^2 + \|\phi_t\|_{\mathbb{R}_+^3}^2 + \|\nabla \phi\|_{\mathbb{R}_+^3}^2 \right\} \leq C(\|y_0\|_Y) < +\infty.$$

Moreover, from energy inequality (1.14) and (1.21) the so-called dissipation integral is finite (see [14, 33]).

COROLLARY 1.6. *Let the hypotheses of Theorem 1.1 be in force and assume $k > 0$; then the dissipation integral is finite. Namely, for a generalized solution to (1.1), we have*

$$\int_0^\infty \|u_t(t)\|_{L_2(\Omega)}^2 dt \leq K < \infty,$$

where K depends on the particular trajectory.

We now state and discuss the stationary problem associated to (1.1), which has the form

$$(1.22) \quad \begin{cases} \Delta^2 u + f(u) = p_0(\mathbf{x}) + U r_\Omega \operatorname{tr}[\partial_x \phi], & \mathbf{x} \in \Omega, \\ u = \partial_\nu u = 0, & \mathbf{x} \in \Gamma, \\ \Delta \phi - U^2 \partial_x^2 \phi = 0, & \mathbf{x} \in \mathbb{R}_+^3, \\ \partial_z \phi = U \partial_x u_{\text{ext}}, & \mathbf{x} \in \partial \mathbb{R}_+^3. \end{cases}$$

This problem has been studied in long-time behavior considerations for flow-plate interactions, most recently in [14, section 6.5.6]; in this reference, the following theorem is shown for subsonic flows (this is given as [14, Theorem 6.5.10]).

THEOREM 1.7. *Suppose $0 \leq U < 1$, $k \geq 0$, with $p_0 \in L_2(\Omega)$. Then weak solutions $(u(\mathbf{x}), \phi(\mathbf{x}))$ to (1.22) exist and satisfy the additional regularity property*

$$(u, \phi) \in (H^4 \cap H_0^2)(\Omega) \times W^2(\mathbb{R}_+^3).$$

Moreover, the stationary solutions mentioned above correspond to the extreme points of the potential energy functional

$$D(u, \phi) = \frac{1}{2} \|\Delta u\|_\Omega^2 + \Pi(u) + \frac{1}{2} \|\nabla \phi\|_{\mathbb{R}_+^3}^2 - \frac{U^2}{2} \|\partial_x \phi\|_{\mathbb{R}_+^3}^2 + U \langle \partial_x u, \operatorname{tr}[\phi] \rangle_\Omega,$$

considered for $(u, \phi) \in H_0^2(\Omega) \times W^1(\mathbb{R}_+^3)$.

The potential energy $D(u, \phi)$ is smooth on the space $H_0^2(\Omega) \times W^1(\mathbb{R}_+^3)$ and moreover

$$D(u, \phi) \geq c \left[\|\Delta u\|_\Omega^2 + \|\nabla \phi\|_{\mathbb{R}_+^3}^2 \right] - C.$$

This latter property is a consequence of Lemma 1.4. Thus, it achieves its minimum, and the extremal set of the functional D is nonempty. We denote the set of all stationary solutions (weak solutions to (1.22) above) as \mathcal{N} , that is,

$$\mathcal{N} \equiv \{(\hat{u}, \hat{\phi}) \in H_0^2(\Omega) \times W^1(\mathbb{R}_+^3) : (\hat{u}, \hat{\phi}) \text{ satisfy (1.22) variationally}\}.$$

1.6. Statement of main result.

THEOREM 1.8 (principal result). *Consider (1.1). Let $0 \leq U < 1$ and assume $p_0 \in L_2(\Omega)$ and $b \geq 0$. Assume $y_0 = (u_0, u_1; \phi_0, \phi_1) \in Y$. Then there is a minimal damping coefficient $k_{\min} > 0$ (not depending on the particular solution) so that for $k \geq k_{\min} > 0$ any generalized solution $(u(t), \phi(t))$ to the system (1.1) with localized initial flow data (i.e., the supports of ϕ_0 and ϕ_1 are contained in some K_{ρ_0}) has the property that*

$$\lim_{t \rightarrow \infty} \inf_{(\hat{u}, \hat{\phi}) \in \mathcal{N}} \left\{ \|u(t) - \hat{u}\|_{H^2(\Omega)}^2 + \|u_t(t)\|_{L_2(\Omega)}^2 + \|\phi(t) - \hat{\phi}\|_{H^1(K_\rho)}^2 + \|\phi_t(t)\|_{L_2(K_\rho)}^2 \right\} = 0 \text{ for any } \rho > 0.$$

Remark 1.6. In this result, and all results below, the minimal damping coefficient k_{\min} depends on the loading p_0 and b , as well as the domain Ω , the flow support parameter ρ_0 , and the unperturbed flow velocity U , but are independent of the particular initial data of the system.

We also mention that our proof relies on the analysis of a *delay-type* system (as an intermediate step)—see Theorem 2.5. It has been established that stabilizing

delay-type systems requires a scaling of the damping with respect to the time of delay [37, 38, 39]; as the delay parameter in this problem depends on the quantities mentioned in the previous paragraph, this would seem consistent with [37, 38, 39]. (See Remark 5.2 below.)

If we make a further physical assumption that \mathcal{N} is an isolated set (e.g., finite), we have the following second main result as a corollary.

COROLLARY 1.9. *Let the hypotheses of Theorem 1.8 be in force. Assume that \mathcal{N} is an isolated set; then for any generalized solution (u, ϕ) to (1.1) (with localized flow data, as above), there exists a solution $(\hat{u}, \hat{\phi})$ satisfying (1.22) such that*

$$\lim_{t \rightarrow \infty} \left\{ \|u(t) - \hat{u}\|_{H^2(\Omega)}^2 + \|u_t(t)\|_{L_2(\Omega)}^2 + \|\phi(t) - \hat{\phi}\|_{H^1(K_\rho)}^2 + \|\phi_t(t)\|_{L_2(K_\rho)}^2 \right\} = 0 \text{ for any } \rho > 0.$$

Remark 1.7. For given loads b and p_0 , the set of solutions is generically finite. This is to say that there is an open dense set $\mathcal{R} \subset L_2(\Omega) \times \mathbb{R}$ such that if $(p_0, b) \in \mathcal{R}$, then the corresponding set of stationary solutions \mathcal{N} is finite. This follows from the Sard–Smale theorem, as shown in [14, Theorem 1.5.7 and Remark 6.5.11].

In addition to the primary results, we mention some results obtained along the way which are of interest in their own right. Owing to the necessary background information to precisely state these results, at this stage we describe them only informally.

First, in section 4.2 (following a few technical descriptions and lemmas) we prove the main supporting result: Theorem 4.3. This result says that global-in-time bounds on plate dynamics in a higher topology yield the desired convergence of the flow-plate dynamics to equilibria at the finite-energy level (in Y_ρ for all $\rho > 0$). It appeared in [33] as a step in the proof of the main result but was not presented independently.

In section 4.3 we state and prove Theorem 4.7. This result shows that for “smooth” initial data for the full flow-plate dynamics we can propagate regularity on the finite time horizon. This result (or slightly modified versions of it) will be used many times below. For the plate dynamics alone this regularity can be propagated on the infinite time horizon (with corresponding uniform bound). Moreover, the result *does not depend on the size of the damping*.

2. Technical preliminaries. In this section we discuss a few key technical results and constructions which will be used critically later. References are provided for each of the considerations below.

2.1. Flow potentials with given Neumann data. In what follows it will be necessary to consider the flow equation with *given* Neumann data. Consider the problem:

$$(2.1) \quad \begin{cases} (\partial_t + U\partial_x)^2 \phi = \Delta \phi & \text{in } \mathbb{R}_+^3, \\ \partial_\nu \phi|_{z=0} = h(\mathbf{x}, t) & \text{in } \mathbb{R}^2, \\ \phi(t_0) = \phi_0; \quad \phi_t(t_0) = \phi_1. \end{cases}$$

We have the following theorem from [7, 14, 35].

THEOREM 2.1. *Assume $U \geq 0$, $U \neq 1$; take $(\phi_0, \phi_1) \in H^1(\mathbb{R}^3) \times L_2(\mathbb{R}^3)$. If $h \in C([t_0, \infty); H^{1/2}(\mathbb{R}^2))$, then (2.1) is well-posed (in the weak sense) with*

$$\phi \in C([t_0, \infty); H^1(\mathbb{R}_+^3)), \quad \phi_t \in C([t_0, \infty); L_2(\mathbb{R}_+^3)).$$

Remark 2.1. In fact, a stronger regularity result is available. Finite energy $(H^1(\Omega) \times L_2(\Omega))$ solutions are obtained with $h \in H^{1/3}((0, T) \times \mathbb{R}^2)$ [32, 43]. However, the corresponding estimate does not have sufficiently good control of the dependence on time T and hence is of limited applicability in the present (stabilization) context.

2.2. Flow superposition and compactness criteria. We may decompose the flow problem from (2.1) into two pieces corresponding to zero Neumann data and zero initial data, respectively:

$$(2.2) \quad \begin{cases} (\partial_t + U\partial_x)^2 \phi^* = \Delta \phi^* & \text{in } \mathbb{R}_+^3 \times (0, T), \\ \phi^*(0) = \phi_0; \quad \phi_t(0) = \phi_1, \\ \partial_\nu \phi^* = 0 & \text{in } \partial\mathbb{R}_+^3 \times (0, T), \end{cases}$$

$$(2.3) \quad \begin{cases} (\partial_t + U\partial_x)^2 \phi^{**} = \Delta \phi^{**} & \text{in } \mathbb{R}_+^3 \times (0, T), \\ \phi^{**}(0) = 0; \quad \phi_t^{**}(0) = 0, \\ \partial_\nu \phi^{**} = h(\mathbf{x}, t) & \text{in } \partial\mathbb{R}_+^3 \times (0, T), \end{cases}$$

where we will later take

$$(2.4) \quad h(\mathbf{x}, t) \equiv -[u_t + Uu_x]_{\text{ext}} \in C(L_2(\mathbb{R}^2)).$$

Following the analyses in [5, 7, 14, 40, 41] we denote the solution to (2.2) as ϕ^* and the solution to (2.3) as ϕ^{**} ; then, the full flow solution ϕ coming from (2.1) has the form

$$\phi(t) = \phi^*(t) + \phi^{**}(t).$$

Remark 2.2. The analysis of ϕ^* below is identical to that given in [5, 7, 14, 40]. However, the treatment of the second part $\phi^{**}(t)$, which corresponds to the hyperbolic Neumann map, is very different than these references due to the loss of a derivative in the Neumann map. Indeed, with rotational inertia present in the model ($u_t \in H^1(\Omega)$) one has (for finite energy solutions) $h \in C(0, T; H^1(\Omega))$. On the other hand, from [35], $h \in L_2(H^{1/2}(\mathbb{R}^2)) \mapsto \phi^{**} \in C(H^1(\mathbb{R}_+^3) \cap C^1(L_2(\mathbb{R}_+^3)))$ —where the latter is of finite energy. Thus the Neumann map is *compact* in this case.

In the absence of rotational inertia one has only that $h \in C(L_2(\mathbb{R}^2))$. Such boundary regularity does not reproduce finite energy solutions, with a maximal regularity $\phi^{**} \in C(H^{2/3}(\mathbb{R}_+^3)) \cap C^1(H^{-1/3}(\mathbb{R}_+^3))$, yielding the loss of $1/3$ derivative. This loss is optimal and cannot be improved [43]. This fact underscores that the entirely componentwise analysis of finite energy solutions for the flow-plate model successfully performed in the past literature [14, 40, 41] cannot be utilized here.

For the analysis of ϕ^* we use the tools developed in [5, 7]. Using the Kirchhoff type representation for the solution $\phi^*(\mathbf{x}, t)$ in \mathbb{R}_+^3 (see, e.g., Theorem 6.6.12 in [14]), we can conclude that if the initial data ϕ_0 and ϕ_1 are localized in the ball K_ρ , then by finite dependence on the domain of the signal in three dimensions (Huygen's principle), one obtains $\phi^*(\mathbf{x}, t) \equiv 0$ for all $\mathbf{x} \in K_\rho$ and $t \geq t_\rho$. Thus we have that

$$(\partial_t + U\partial_x) \text{tr}[\phi^*] \equiv 0, \quad \mathbf{x} \in \Omega, \quad t \geq t_\rho.$$

Thus ϕ^* tends to zero in the sense of the local flow energy, i.e.,

$$(2.5) \quad \|\nabla \phi^*(t)\|_{L_2(K_\rho)}^2 + \|\phi_t^*(t)\|_{L_2(K_\rho)} \rightarrow 0, \quad t \rightarrow \infty,$$

for all fixed $\rho > 0$.

We now introduce a compactness criterion in the *local flow energy sense*.

LEMMA 2.2. Let $\{(\phi_0^m, \phi_1^m)\}_m^\infty$ be a bounded sequence in $Y_{fl} = W_1(\mathbb{R}_+^3) \times L_2(\mathbb{R}_+^3)$ and let $\eta > 0$. If for any $\rho > 0$ there exists an $N(\rho)$ and $C(\rho)$ so that

$$\|\nabla \phi_0^m\|_{\eta, K_\rho}^2 + \|\phi_1^m\|_{\eta, K_\rho}^2 \leq C(\rho)$$

for all $m > N(\rho)$, then the sequence $\{(\phi_0^m, \phi_1^m)\}_m^\infty$ is compact in \tilde{Y}_{fl} .

This is given as Lemma 10 in [40, p. 472] (where it is proved) and is utilized in a critical way in [41] as well.

For the term ϕ^{**} we have the following theorem which provides us with an explicit form of the solution. (For a proof, see, for instance, [14, Theorem 6.6.10].) Below, we utilize the notation

$$f^\dagger(\mathbf{x}, t, s, \theta) = f(x - \kappa_1(\theta, s, z), y - \kappa_2(\theta, s, z), t - s)$$

and

$$\kappa_1(\theta, s, z) = Us + \sqrt{s^2 - z^2} \sin \theta, \quad \kappa_2(\theta, s, z) = \sqrt{s^2 - z^2} \cos \theta.$$

THEOREM 2.3. Considering the problem in (2.3) with zero initial flow data and considering

$$h(\mathbf{x}, t) = -[u_t(x, y, t) + Uu_x(x, y, t)]_{ext},$$

there exists a time $t^*(\Omega, U)$ such that we have the following representation for the weak solution $\phi^{**}(t)$ for $t > t^*$:

$$(2.6) \quad \phi^{**}(\mathbf{x}, t) = -\frac{\chi(t-z)}{2\pi} \int_z^t \int_0^{2\pi} (u_t^\dagger(\mathbf{x}, t, s, \theta) + Uu_x^\dagger(\mathbf{x}, t, s, \theta)) d\theta ds,$$

where $\chi(s)$ is the Heaviside function. The time t^* is given by

$$t^* = \inf\{t : \mathbf{x}(U, \theta, s) \notin \Omega \ \forall (x, y) \in \Omega, \ \theta \in [0, 2\pi], \text{ and } s > t\}$$

with $\mathbf{x}(U, \theta, s) = (x - (U + \sin \theta)s, y - s \cos \theta) \in \mathbb{R}^2$ (not to be confused with \mathbf{x} having no dependencies noted, which is simply $\mathbf{x} = (x, y)$).

Moreover, we have the following pointwise formula for the derivative in t [41, p. 480] (which is justified for plate solutions with initial data in $\mathcal{D}(\mathbb{T})$ and can be taken distributionally for data in Y):

$$(2.7) \quad \begin{aligned} \phi_t^{**}(\mathbf{x}, t) &= \frac{1}{2\pi} \left\{ \int_0^{2\pi} u_t^\dagger(\mathbf{x}, t, t^*, \theta) d\theta - \int_0^{2\pi} u_t^\dagger(\mathbf{x}, t, z, \theta) d\theta \right. \\ &\quad + U \int_z^{t^*} \int_0^{2\pi} [\partial_x u_t^\dagger](\mathbf{x}, t, s, \theta) d\theta ds \\ &\quad \left. + \int_z^{t^*} \int_0^{2\pi} \frac{s}{\sqrt{s^2 - z^2}} [M_\theta u_t^\dagger](\mathbf{x}, t, s, \theta) d\theta ds \right\} \end{aligned}$$

with $M_\theta = \sin \theta \partial_x + \cos \theta \partial_y$.

We note that with $(\phi_0, \phi_1) \in H^1(\mathbb{R}_+^3) \times L_2(\mathbb{R}_+^3)$ one obtains [18, 35]: $(\phi^*(t), \phi_t^*(t)) \in H^1(\mathbb{R}_+^3) \times L_2(\mathbb{R}_+^3)$. Thus, by Theorem 1.1 we also have that

$$(\phi^{**}(t), \phi_t^{**}(t)) \in H^1(\mathbb{R}_+^3) \times L_2(\mathbb{R}_+^3).$$

Remark 2.3. Note that *this last property is not valid for a flow solution with $h \in L_2(\mathbb{R}^2)$ boundary Neumann data.* The general theory will provide at most $H^{2/3}(\mathbb{R}_+^3 \times [0, T])$ [43]. However, the improved regularity is due to the interaction with the plate and the resulting cancellations on the interface. Moreover, we also obtain a meaningful “hidden trace regularity” for the aeroelastic potential on the boundary of the structure [17, 18]:

$$(2.8) \quad (\partial_t + U\partial_x)tr[\phi] \in L_2\left(0, T; H^{-1/2}(\mathbb{R}^2)\right),$$

where T is arbitrary.

Additionally, the following key inequality has been shown in [40, Lemma 8, p. 469 and (56), p. 479].

LEMMA 2.4. For (2.3) taken with $h(\mathbf{x}, t) = -(u_t + Uu_x)_{ext}$, we have

$$(2.9) \quad \begin{aligned} & \|\nabla\phi^{**}(t)\|_{\eta, K_\rho}^2 + \|\phi_t^{**}(t)\|_{\eta, K_\rho}^2 \\ & \leq C(\rho) \left\{ \|u(\cdot)\|_{H^{s+\eta}(t-t^*, t; H_0^{2+\eta}(\Omega))}^2 + \|u_t(\cdot)\|_{H^{s+\eta}(t-t^*, t; H_0^{1+\eta}(\Omega))}^2 \right\} \end{aligned}$$

for $s, \eta \geq 0$, $0 < s + \eta < 1/2$ and $t > t^*(U, \Omega)$.

2.3. Nonlinear plates with delay. In this section, the full flow-plate problem is reduced to a delayed plate problem. This delayed plate is subject three conflicting actions contributed by flow:

1. the delay term itself: $q^u(t)$,
2. the term Uu_x appearing in the plate equation—nonconservative, potentially leading to chaos,
3. the term u_t (with “good” sign) in the plate equation.

Above, 3 is stabilizing, but 1 and 2 are “bad” and do not stabilize the plate. It is also known that a delay term can produce destabilizing effects on the dynamics, *unless* the damping $(k+1)u_t$ is sufficiently large [38] and also [37, 39]. This is one contributing factor for our need of a large damping— k value.

On the other hand, the term u_x can typically be dealt with via a large static damping [33]. Here, we do not assume any static damping for the overall model. Our challenge, then, in this paper is to show that this potentially chaotic term is “thrown” to the equilibria set and that the dynamics converge to it.

The fundamental work in [16] focuses on the reduction of the model in (1.1) to a delayed plate equation.

THEOREM 2.5. Let the hypotheses of Theorem 1.1 be in force, and $(u_0, u_1; \phi_0, \phi_1) \in H_0^2(\Omega) \times L_2(\Omega) \times H^1(\mathbb{R}_+^3) \times L_2(\mathbb{R}_+^3)$. Assume that there exists a ρ_0 such that $\phi_0(\mathbf{x}) = \phi_1(\mathbf{x}) = 0$ for $|\mathbf{x}| > \rho_0$. Then there exists a time $t^\#(\rho_0, U, \Omega) > 0$ such that for all $t > t^\#$ the weak solution $u(t)$ in (1.1) satisfies the following equation:

$$(2.10) \quad u_{tt} + \Delta^2 u + ku_t + f(u) = p_0 - (\partial_t + U\partial_x)u - q^u(t)$$

with

$$(2.11) \quad q^u(t) = \frac{1}{2\pi} \int_0^{t^*} ds \int_0^{2\pi} d\theta [M_\theta^2 u_{ext}](x - (U + \sin \theta)s, y - s \cos \theta, t - s).$$

Owing to this reduction we can choose to study a general delayed plate model (as was done in [16]):

$$(2.12) \quad \begin{cases} u_{tt} + \Delta^2 u + k_0 u_t + f(u) = p_0 - q(u^t) + Lu & \text{in } \Omega, \\ u = \partial_\nu u = 0 & \text{on } \Gamma, \\ u(0) = u_0, \quad u_t(0) = u_1, \quad u|_{(-t^*, 0)} = \eta. \end{cases}$$

The bounded linear operator $L : H_{2-\sigma}(\Omega) \rightarrow H^{-\sigma}(\Omega)$, $\sigma \in [0, \sigma_0]$ for some $\sigma_0 > 0$, corresponds to nondissipative lower order terms: $Lu = -Uu_x$ and $k_0 = k + 1$. We have utilized the standard notation that $u^t(s) = u(t + s)$, so u^t denotes $u(s)$ for all $s \in (t - t^*, t)$. The operator $q(u^t)$ represents a general delay-type operator (depending on u^t), of which $q^u(t)$ is a specific example. (This notation is consistent with [16].)

We will also need the following formula for the time derivative of $q^u(t)$ appearing above in (2.11):

$$(2.13) \quad \begin{aligned} \partial_t[q^u](t) = & \int_0^{2\pi} \frac{1}{2\pi} [M_\theta^2 u]_{\text{ext}}(\mathbf{x}(U, \theta, 0), t) d\theta \\ & - \int_0^{2\pi} \frac{1}{2\pi} [M_\theta^2 u]_{\text{ext}}(\mathbf{x}(U, \theta, t^*), t - t^*) d\theta \\ & + \left(\int_0^{t^*} \int_0^{2\pi} (U + \sin \theta) \frac{1}{2\pi} [M_\theta^2 u_x]_{\text{ext}}(\mathbf{x}(U, \theta, s), t - s) d\theta ds \right) \\ & + \left(\int_0^{t^*} \int_0^{2\pi} (\cos \theta) \frac{1}{2\pi} [M_\theta^2 u_y]_{\text{ext}}(\mathbf{x}(U, \theta, s), t - s) d\theta ds \right). \end{aligned}$$

The proofs of Theorems 3.1 and 3.2 cited below are based on a quasistability analysis [10, 14] of the delay evolution (T_t, \mathbf{H}) with $\mathbf{H} \equiv H_0^2(\Omega) \times L_2(\Omega) \times L_2(-t^*, 0; H_0^2(\Omega))$, corresponding to the reduced plate with given data $x_0 \in \mathbf{H}$; we have that $T_t(x_0) = (u(t), u_t(t); u^t)$ with $x_0 = (u_0, u_1, \eta)$. The relevant is taken to be

$$\|(u, v; \eta)\|_{\mathbf{H}}^2 \equiv \|\Delta u\|^2 + \|v\|^2 + \int_{-t^*}^0 \|\Delta \eta(t + s)\|^2 ds.$$

In application, we will consider an initial datum $y_0 \in Y$ corresponding to the dynamics in (1.1) (the full flow-plate dynamics) at time $t = 0$. We employ the reduction result Theorem 2.5, and we may consider the “initial time” ($t = t_0$) for the delay dynamics corresponding to any time after the reduction time $t^\#(\rho_0, U, \Omega)$ above. At such a time, the data which is fed into (2.10) is $x_0 = (u(t_0), u_t(t_0), u^{t_0})$, where this data is determined by the full dynamics of (1.1) on $(t_0 - t^*, t_0)$. (That (1.1) is well-posed on Y and (2.12) is well-posed on \mathbf{H} —see [16]—allows us to move back and forth between the two descriptions of the system.)

3. Supporting results. For self-containedness of exposition we now provide a summary of the key results of [16] which will be used critically in this treatment.

3.1. Plate dynamics. The first theorem we present is the main result of [16], given there as Theorem 3.4.

THEOREM 3.1. *Consider (1.1). Suppose $0 \leq U \neq 1$ and $p_0 \in L_2(\Omega)$. Take any $k \geq 0$. Then there exists a compact set $\mathcal{U} \subset H_0^2(\Omega) \times L_2(\Omega)$ of finite fractal dimension such that*

$$\lim_{t \rightarrow \infty} d_{Y_{pt}}((u(t), u_t(t)), \mathcal{U}) = \lim_{t \rightarrow \infty} \inf_{(\nu_0, \nu_1) \in \mathcal{U}} (\|u(t) - \nu_0\|_2^2 + \|u_t(t) - \nu_1\|^2) = 0$$

for any weak solution $(u, u_t; \phi, \phi_t)$ to (1.1) with initial data $(u_0, u_1; \phi_0, \phi_1) \in Y_{\rho_0}$, i.e., which are localized in $\mathbb{R}_+^3 - \phi_0(\mathbf{x}) = \phi_1(\mathbf{x}) = 0$ for $|\mathbf{x}| > \rho_0$ for some $\rho_0 > 0$. We have the extra regularity $\mathcal{U} \subset (H^4(\Omega) \cap H_0^2(\Omega)) \times H^2(\Omega)$, and any plate trajectory (u, u_t) on the plate attracting set (namely, with initial $(u_0, u_1) \in \mathcal{U}$) has the additional property that

$$(u, u_t, u_{tt}) \in C(\mathbb{R}; (H^4 \cap H_0^2)(\Omega) \times H_0^2(\Omega) \times L_2(\Omega)).$$

Remark 3.1. We note that the above result holds for any nonlinearity of physical type, which includes the Berger, von Karman, and Kirchhoff type nonlinearities (see [16]).

In the setting of [33] and the present analysis we are interested in obtaining a sufficiently large damping so as to guarantee strong convergence of full flow-plate trajectories. It is imperative here that there exists a uniform absorbing ball for the plate dynamics which is independent of the damping parameter (when it is sufficiently large). The following result is given Lemma 4.6 in [33] (there applying to (1.13) for $\beta \geq 0$).

LEMMA 3.2. *With the same hypotheses as Theorem 3.1 above, there is an absorbing set $\mathcal{B} \subset H_0^2(\Omega) \times L_2(\Omega)$ which is (i) uniform with respect to the initial data y_0 (though it may depend on ρ_0) and (ii) uniform in the damping parameter $k \geq k_* > 0$ for some k_* .*

Remark 3.2. When we include the effects of static damping—of the form βu , as in (1.13)—in the plate dynamics, we also have the existence of the attractor (and corresponding absorbing set) whose size also can be made independent of both $k \geq k_*$ and $\beta \geq 0$ —see [33].

The results in [16] perform analysis on the abstract delay equation (2.12) on \mathbf{H} , including energy methods which show the existence of a true global attractor \mathcal{U} (as well as an absorbing set $\mathcal{B} \supset \mathcal{U}$) for the the delay evolution (T_t, \mathbf{H}) . These results depend on taking $k_0 > 0$ (where k_0 is the damping coefficient in (2.12)). (Note that for the specific delayed plate of interest here (2.10), we have $k_0 = k + 1$, and thus we could take $k \in (-1, \infty]$.) Then, to achieve the results in Theorems 3.1 and 3.2 projections are taken which suppress the delay component of the evolution.

Remark 3.3. In what follows, the notation $\mathcal{U} \subset \mathcal{B}$ correspond to the global attractor and (a fixed) uniform absorbing set for the dynamics (T_t, \mathbf{H}) . The notation $\mathcal{U} \subset \mathcal{B}$ will refer to the attracting set and absorbing ball for the *projected dynamics* (u, u_t) of (S_t, Y) (no delay or flow components).

3.2. Weak stability. Given the existence of a compact global attracting set for the plate dynamics, as well as the finiteness of the dissipation integral (Corollary 1.6), we have the following theorem (given as Theorem 7.1 in [33] in the context of (1.13) for $\beta \geq 0$).

THEOREM 3.3. *Consider (1.1). Assume $k > 0$. For any initial data $y_0 = (u_0, u_1; \phi_0, \phi_1) \in Y$ and any sequence of times $t_n \rightarrow \infty$ there is a subsequence t_{n_j} and a point $\hat{y} = (\hat{u}, \hat{w}; \hat{\phi}, \hat{\psi})$ so that the following hold:*

1. $S_{t_{n_j}}(y_0) \rightharpoonup \hat{y}$.
2. $\|u(t_{n_j}) - \hat{u}\|_{2, \Omega} \rightarrow 0$.
3. $\|u_t(t)\|_0^2 \rightarrow 0$, $t \rightarrow \infty$, and hence $\hat{w} = 0$.

4. Along the sequence of times $t_{n_j} \rightarrow \infty$ we have

$$(3.1) \quad \sup_{\tau \in [-c, c]} \|u(t_{n_j} + \tau) - \hat{u}\|_{2-\delta, \Omega} \rightarrow 0 \quad \text{for any fixed } c > 0, \delta \in (0, 2).$$

In view of the above result, it is clear that the mathematical challenge in resolving the flutter problem is to improve the convergence of the flow component of the dynamics from *weak* to *strong*. This is a highly nontrivial task, taking into consideration the lack of compactness properties of the dynamics, along with the lack of dissipation acting on the flow equation.

4. From weak to strong. We present (in a streamlined way) the key past results in [40, 41], which yield supporting results that can then be applied in the proof of our main results here. Additionally, we restate in the present context certain ideas and results which were critically used in [33].

4.1. Hadamard continuity. The key results below address the *continuous dependence* property of the dynamics, namely, that $S_t(\cdot)$ is Hadamard continuous on Y (and Y_s on *finite time intervals*). This was first addressed within the proof of Theorem 3.1 [33, pp. 1963–1964], and the proof of both Lemma 4.1 and Theorem 4.2 can be found in [33]. (In fact, the Hadamard continuity results below are true for the von Karman dynamics.)

LEMMA 4.1. *Let $0 \leq U < 1$ and assume $p_0 \in L_2(\Omega)$. Assume $y_0 = (u_0, u_1; \phi_0, \phi_1) \in Y$. Consider the dynamics generated by generalized solutions to (1.1) (or (1.13) if $\beta > 0$), denoted $S_t(y_0)$. For any $k \geq 0$, $\beta \geq 0$, and for any fixed $T > 0$, we have*

$$S_t(y_0^m) \rightarrow y(t) = S_t(y_0) \in Y,$$

when $y_0^m \rightarrow y_0 \in Y$, uniformly for all $t < T$; this convergence may depend—possibly—on $T > 0$.

Remark 4.1. The corresponding delay system (T_t, \mathbf{H}) exhibits the Hadamard continuity property on any $[t_0, T]$ as well. This follows from Lemma 4.1 and Theorem 2.5. Moreover, in [16] the *continuous dependence* property is shown directly from energy estimates on the delay system (2.12).

Remark 4.2. At present, it is not clear that the full dynamics for (1.1) are *global-in-time* Hadamard continuous. Whether this is the case is related to the decay rate of $\|u_t(t)\|_{L_2(\Omega)} \rightarrow 0$; indeed, to obtain this global-in-time property it is sufficient for $u_t \in L_1(T, \infty; L_2(\Omega))$. With $k > 0$, we do have $u_t \in L_2(0, \infty; L_2(\Omega))$; however, improving $L_2(0, \infty)$ to $L_1(0, \infty)$ in general requires very substantial work. See Remark 4.10.

If we consider the addition of *large static damping* (see Remark 3.2), as well as large viscous damping, we see that the Hadamard continuity can be improved to global-in-time (this was one of the primary results in [33]).

THEOREM 4.2. *Let $0 \leq U < 1$ and assume $p_0 \in L_2(\Omega)$. Assume $y_0 = (u_0, u_1; \phi_0, \phi_1) \in Y$. Consider the dynamics generated by generalized solutions to (1.13) (i.e., with $\beta > 0$), denoted $S_t(y_0)$. Assuming that both damping parameters are sufficiently large $k \geq k_c > 0$, $\beta \geq \beta_c > 0$, the semigroup $S_t(\cdot)$ is uniform-in-time Hadamard continuous, i.e., for any sequence $y_0^m \rightarrow y_0$ in Y and any $\epsilon > 0$ there is an M so that for $m > M$*

$$\sup_{t > 0} \|S_t(y_0^m) - S_t(y_0)\|_{Y_\rho} < \epsilon.$$

This theorem and its proof are given in [33, section 7]. Theorem 4.2 will be used below in an auxiliary fashion on a decomposed portion of the full flow-plate dynamics which resembles (1.13) (i.e., has static damping added)—Theorem 5.7.

Remark 4.3. The minimal damping coefficients k_c and β_c for the result above depend on the invariant set \mathcal{B} for the delayed plate dynamics, which itself depends on the loading p_0 and b , as well as the domain Ω and other physical constants such as U, ρ_0 , but *is independent on the particular initial data of the system*.

4.2. Bounds in higher norms. We now state and prove the critical supporting theorem for this treatment (and also [33]), Theorem 4.3:

For finite energy solutions to the flow-plate system, a uniform-in-time bound on plate solutions in a higher topology yields the desired convergence to equilibria result (as in Theorem 1.8) in the topology Y_ρ for any $\rho > 0$.

This result is independent of the particular structure of the plate dynamics—only global-in-time bounds in higher norms of the plate solution (u, u_t) are needed; indeed, Theorem 4.3 follows from the structure of the (linear) flow-plate coupling via the Neumann flow condition, written in terms of $[u_t + Uu_x]_{\text{ext}}$, and appearing in Theorem 2.3.

THEOREM 4.3. *Suppose $0 \leq U \neq 1$ and $p_0 \in L_2(\Omega)$, and take any $k > 0$. Let (u, ϕ) be a weak solution to (1.1) (or (1.13) if $\beta > 0$) with finite energy (flow-localized) initial data in $y_0 \in Y_{\rho_0}$. If there is a time T^* so that*

$$(4.1) \quad \sup_{t \in [T^*, \infty)} \{ \|u(t)\|_4^2 + \|\Delta u_t(t)\|_0^2 + \|u_{tt}(t)\|_0^2 \} \leq C_1,$$

then for any sequence of times $t_n \rightarrow +\infty$, there is a subsequence of times t_{n_k} and a point $\hat{y} = (\hat{u}, 0; \hat{\phi}, 0)$ with $(\hat{u}, \hat{\phi}) \in \mathcal{N}$ so that

$$\lim_{k \rightarrow \infty} d_{Y_\rho}(S_{t_{n_k}}(y_0), \hat{y}) = 0$$

for any $\rho > 0$. This implies the result of Theorem 1.8.

We present this theorem as an independent result, in the spirit of what is used in [40, 41]. However, a word of caution is in order: the assumed bound (4.1) will be valid only for “certain” solutions to the plate problem and certainly *not* for every weak solution. This is in striking contrast with [40, 41], where the smoothing properties of thermoelasticity provide the additional boundedness of *all* plate solutions in higher topologies. Here, we can obtain the global-in-time bounds in higher topologies (as in (4.1) below) by *considering smooth initial data* for the flow-plate system and *propagating* this regularity—see sections 4.3 and 4.4 below. Alternatively, a key point of our proof below relies on a decomposition of the plate dynamics (see section 5.2) in which one of the decomposed portions of the plate dynamics will satisfy a bound like (4.1) (and the other portion will be uniformly exponentially stable). The nondecomposed plate dynamics will not necessarily have such a global-in-time bound. See Remark 4.4 below for further discussion of how Theorem 4.3 and its proof are utilized later.

Proof of Theorem 4.3. We first note that the solution to (2.2) (via classical scattering theory—see [40, 41] and references therein) is stable, so

$$(\phi^*(t_n), \phi_t^*(t_n)) \rightarrow (0, 0)$$

in the local flow energy topology \tilde{Y}_{fl} .

We now utilize the assumption (4.1), along with the bound in Lemma 2.4 to bound, uniformly, the flow in higher norms. Specifically, we note that by assumption

$$(4.2) \quad u \in C^2([T^*, \infty); L_2(\Omega)) \cap C^1([T^*, \infty); H_0^2(\Omega)) \cap C([T^*, \infty); H^4(\Omega) \cap H_0^2(\Omega)).$$

(Below, we utilize the notation $H^s(H^r(\Omega))$; H^s regularity refers to the time variable.) This implies, in particular, that $u \in H^2(L_2(\Omega)) \cap H^1(H^2(\Omega)) \cap L_2(H^4(\Omega))$. With $u \in L_2(H^4(\Omega)) \cap H^1(H^2(\Omega))$, via interpolation [3, 34], we have

$$(4.3) \quad u \in H^{\alpha_1} \left(H^{2+2(1-\alpha_1)}(\Omega) \right), \quad \alpha_1 \in [0, 1].$$

Similarly, from $u_t \in L_2(H^2(\Omega)) \cap H^1(L_2(\Omega))$ we obtain

$$(4.4) \quad u_t \in H^{\alpha_2} \left(H^{2(1-\alpha_2)}(\Omega) \right), \quad \alpha_2 \in [0, 1].$$

To apply inequality in Lemma 2.4 with $s = 0$, $\eta > 0$ we will take $\alpha_1 = \eta = \alpha_2$

$$4 - 2\alpha_1 \geq 2 + \alpha_1, \quad \text{so,} \quad 2 \geq 3\alpha_1 \quad \text{and} \quad 2 - 2\alpha_2 \geq 1 + \alpha_2, \quad \text{so,} \quad 1 \geq 3\alpha_2.$$

Hence, for $\eta \leq 1/3$ we have

$$\begin{aligned} & \|\nabla \phi^{**}(t)\|_{\eta, K_\rho}^2 + \|\phi^{**}(t)\|_{\eta, K_\rho}^2 \\ & \leq \left\{ \|u(\cdot)\|_{H^{s+\eta}(t-t^*, t; H_0^{2+\eta}(\Omega))}^2 + \|u_t(\cdot)\|_{H^{s+\eta}(t-t^*, t; H_0^{1+\eta}(\Omega))}^2 \right\} \leq C_1, \end{aligned}$$

where C_1 corresponds to the global-in-time bound on plate solutions in the assumption of the theorem. Then, applying the compactness criterion in Lemma 2.2 we have shown the following.

LEMMA 4.4. *With assumption (4.1) in place, for any sequence of times $t_n \rightarrow +\infty$, there is a subsequence (also denoted t_n) and a limit point $(\hat{u}, 0; \hat{\phi}, \hat{\psi}) \in Y$ such that*

$$(u(t_n), u_t(t_n); \phi(t_n)^{**}, \phi_t^{**}(t_n)) \rightarrow (\hat{u}, 0; \hat{\phi}, \hat{\psi})$$

in \tilde{Y} as $n \rightarrow \infty$. (Note that we know, a priori, that $u_t \rightarrow 0$ in $L_2(\Omega)$ —see Theorem 3.3.)

Now, to further characterize the flow limit point, we return to the formula in Theorem 2.3.

LEMMA 4.5. *The limit point $\hat{\psi} \in Y$ in Lemma 4.4 above is identified as 0 in $L_2(K_\rho)$ for any $\rho > 0$.*

Proof of Lemma 4.5. The flow solution $(\phi^*(t), \phi_t^*(t))$ tends to zero (again, we emphasize that this convergence is in the local flow energy sense); hence, we have that $\phi(t_n) \rightarrow \hat{\phi}$. To identify $\hat{\psi}$ with zero, we note that (by assumption) $u_t(t) \in H^1(\Omega)$. This allows to differentiate the flow formula in Theorem 2.3 in time (as was done in [14] and [41]) to arrive at that in (2.7).

For a fixed $\rho > 0$, we multiply $\phi_t^{**}(\mathbf{x}, t)$ in (2.7) by a smooth function $\zeta \in C_0^\infty(K_\rho)$ and integrate by parts in space. This results in the bound

$$(4.5) \quad |(\phi_t^{**}, \zeta)_{L_2(K_\rho)}| \leq C(\rho) \sup_{\tau \in [0, t^*]} \|u_t(t - \tau)\|_{0, \Omega} \|\zeta\|_{1, K_\rho}.$$

From here, we utilize the fact that $u_t(t) \rightarrow 0$ in $L_2(\Omega)$ when $t \rightarrow \infty$ (Theorem 3.3), and hence $(\phi_t^{**}(t), \zeta)_{K_\rho} \rightarrow 0$ and $\phi_t^* \rightarrow 0$ in $L_2(K_\rho)$ for any fixed $\rho > 0$. This gives

that $\phi_t(t) \rightarrow 0$ in $L_2(\Omega)$, and we identify the strong limit point $\hat{\psi}$ with 0 in $L_2(\Omega)$ so that $(u(t_n), u_t(t_n); \phi^{**}(t_n), \phi_t^{**}(t_n)) \rightarrow (\hat{u}, 0; \hat{\phi}, 0)$ in \tilde{Y} as $n \rightarrow \infty$. \square

Now, we show that the limit points obtained above satisfy the static problem in a weak sense.

LEMMA 4.6. *The pair $(\hat{u}, \hat{\phi})$ as in Lemmas 4.4 and 4.5 satisfies the stationary problem (1.22) in the variational sense.*

Proof of Lemma 4.6. We begin by multiplying the system (1.1) by $\eta \in C_0^\infty(\Omega)$ (plate equation) and $\psi \in C_0^\infty(\mathbb{R}_+^3)$ (flow equation) and integrate over the respective domains. This yields

$$(4.6) \quad \begin{cases} \langle u_{tt}, \eta \rangle + \langle \Delta u, \Delta \eta \rangle + k \langle u_t, \eta \rangle + \langle f(\hat{u}), \eta \rangle = \langle p_0, \eta \rangle - \langle \text{tr}[\phi_t + U\phi_x], \eta \rangle, \\ \langle \phi_{tt}, \psi \rangle + U(\phi_{tx}, \psi) + U(\phi_{xt}, \psi) - U^2(\phi_x, \psi_x) = -(\nabla \phi, \nabla \psi) + \langle u_t + Uu_x, \text{tr}[\psi] \rangle. \end{cases}$$

Now, we consider the above relations evaluated at the points t_n (identified as a subsequence for which the convergences above hold) and integrate in the time variable from t_n to $t_n + c$. The limit passage on the linear terms is clear, owing to the convergence properties for the plate component in the Theorem 3.3. The locally Lipschitz nature the nonlinearity allows us to pass with the limit on $f(u(t_n))$. (This is by now standard [14].) We then arrive at the following static relations:

$$(4.7) \quad \begin{cases} \langle \Delta \hat{u}, \Delta w \rangle + \langle f(u), w \rangle = \langle p_0, w \rangle + U \langle \text{tr}[\hat{\phi}], w_x \rangle, \\ \langle \nabla \hat{\phi}, \nabla \psi \rangle - U^2(\hat{\phi}_x, \psi_x) = U \langle \hat{u}_x, \text{tr}[\psi] \rangle. \end{cases}$$

This implies that our limiting point $(\hat{u}, 0; \hat{\phi}, 0)$ of the sequence $(u(t_n), u_t(t_n); \phi(t_n), \phi_t(t_n))$ is in fact a solution (in the weak/variational sense) to the static equations, i.e., it is a *stationary solution* to the flow-plate system (1.1). We have thus shown that any trajectory contains a sequence of times $\{t_n\}$ such that, along these times, we observe convergence to a solution of the stationary problem. \square

The above lemmas complete the proof of Theorem 4.3. \square

Remark 4.4. In the proof above we have shown how a bound on the plate dynamics in higher topologies results in the desired convergence to equilibrium result for the full dynamics. We pause to point out that this will be used in two different ways: below we will show that with *smooth initial data* for (1.1), via a propagation of regularity result, such a bound can be obtained. Alternatively, in the proof of our main result (in section 5.5), we will use the proof of Theorem 4.3 for decomposed systems (5.24) and (5.26). We note that the result of Theorem 4.3 is not strictly dependent on the structure of the plate equation but rather the linear nature of the coupling via the Neumann condition for the flow (2.3). Hence, if we modify the structure of the semilinear plate equation, the result of Theorem 4.3 will be unchanged, so long as (i) the modified plate dynamics still converge in the sense of Theorem 3.3, and (ii) the requisite bound in Theorem 4.3 is obtained. Specifically, in what follows we will consider a general system:

$$(4.8) \quad \begin{cases} u_{tt} + (k+1)u_t + \Delta^2 u = p_0 - U\partial_x u - q^u + F(t, u) & \text{in } \Omega, \\ u = \partial_\nu u = 0 & \text{on } \Gamma, \\ u(t_0) = u_0, \quad u_t(t_0) = u_1, \quad u^{t_0} = \eta, \end{cases}$$

where the data will be smooth (as in the proof of Theorem 5.6) or possibly zero (as in the proof of Theorem 5.10). In both of these cases the nonlinearity $F(t, u)$ will be

broader than $f = f(u)$, depending on a *given function*, owing to the decomposition of the dynamics considered below in (5.1) and (5.2).

4.3. Propagation of regularity. In order to utilize the key technical supporting result in Theorem 4.3, we must show a propagation of regularity result for the dynamics. Specifically, we must show that for smooth initial data, we have smooth time dynamics; this is Theorem 4.7. Such a result was correctly stated in [33], but the proof given there requires a stronger assumption on the damping coefficient (which is not needed here). This has been updated in a subsequent version [33] of the same reference. We include a proof here in the case of Berger dynamics (with the clear proof for von Karman dynamics in [33]).

Remark 4.5. To show infinite-time propagation of the plate dynamics, we rely on the full flow-plate dynamics to achieve *propagation of the initial regularity* on any interval $[0, T^*]$. Once this is achieved, we may work on the reduced delay plate (after sufficient time has passed) and utilize sharp bounds to obtain regularity of plate trajectories on the infinite horizon (propagation on $[T^*, \infty)$). This part of the argument depends on (i) uniform exponential decay for the nonlinear plate equation with large static and viscous damping [31] and (ii) specific properties of nonlinearity. (In the von Karman case one applies the *sharp* regularity of Airy stress function [14, p. 44].)

THEOREM 4.7. *Consider the dynamics (S_t, Y) corresponding to (1.1). Consider initial data $y_m \in \mathcal{D}(\mathbb{T})$ such that $y^m \in B_R(Y)$ and take $k > 0$. Then we have that for the trajectory $S_t(y^m) = (u^m(t), u_t^m(t); \phi^m(t), \phi_t^m(t))$*

$$(u^m(\cdot), \phi^m(\cdot)) \in C^1(0, T; H_0^2(\Omega) \times H^1(\mathbb{R}_+^3))$$

for any T , along with the bound

$$(4.9) \quad \sup_{t \in [0, T]} \{ \|\Delta u_t^m\|_\Omega^2 + \|u_{tt}^m\|_\Omega^2 \} \leq C_{m, T} < \infty.$$

Additionally, if we assume the flow initial data are localized and consider the delayed plate trajectory (via the reduction result Theorem 2.5), we will have

$$(4.10) \quad \sup_{t \in [0, \infty)} \{ \|\Delta u_t^m\|_\Omega^2 + \|u_{tt}^m\|_\Omega^2 \} \leq C_1 < \infty.$$

Then, by the boundedness in time of each of the terms in the (2.10), we have

$$\sup_{t \in [0, \infty)} \|\Delta^2 u^m(t)\|_0 \leq C_2,$$

where we critically used the previous bound in (4.10). In particular, this implies that, taking into account the clamped boundary conditions,

$$(4.11) \quad \sup_{t \in [0, \infty)} \|u^m(t)\|_{4, \Omega} \leq C_3.$$

Each of the terms C_i above depends on the intrinsic parameters in the problem (including the respective loading, which depends on the nonlinearity being considered).

Remark 4.6. The superscript m in the above theorem serves to indicate that smooth data (in the sense above) will be used later to approximate given finite energy data.

Remark 4.7. The parameter $\beta \geq 0$ is inert with respect to this result. That is, Theorem 4.7 holds for (1.1) ($\beta = 0$) and (1.13) ($\beta > 0$).

For this proof we consider no static damping ($\beta = 0$ in (1.13)) and any $k > 0$ fixed. The proof of Theorem 4.7 is based on two steps labeled Steps 1 and 2. We first prove finite time propagation of regularity (Step 1). Following this, we shall propagate regularity for all times uniformly (Step 2) *without making assumptions on the damping coefficients*. This is motivated by the fact that for the infinite time propagation we must use the delay representation for plate solutions, which is valid only for sufficiently large times. Thus, the regular initial condition required to do so is obtained via the finite time propagation of regularity for the full flow-plate model.

Proof of Theorem 4.7.

4.3.1. Step 1. To prove (4.10) above (which then implies (4.11)) we will consider the time differentiated version of the entire flow-plate dynamics in (1.1), which is permissible in $\mathcal{D}(\mathbb{T})$. We label $\bar{u} = u_t^m$ and $\bar{\phi} = \phi_t^m$, multiply the plate equation by \bar{u}_t and the flow equation by $\bar{\phi}_t$, and integrate in time. We consider the “strong” energies

$$\begin{aligned} E_{pl}^s(t) &= \frac{1}{2} [\|\Delta \bar{u}(t)\|^2 + \|\bar{u}_t(t)\|^2], \\ E_{fl}^s(t) &= \frac{1}{2} [\|\nabla \bar{\phi}(t)\|_{\mathbb{R}_+^3}^2 - U^2 \|\partial_x \bar{\phi}(t)\|_{\mathbb{R}_+^3}^2 + \|\bar{\phi}_t(t)\|_{\mathbb{R}_+^3}^2], \\ E_{int}^s(t) &= 2U \langle \bar{u}_x(t), \bar{\phi}(t) \rangle_\Omega, \\ (4.12) \quad \mathcal{E}^s(t) &= E_{pl}^s(t) + E_{fl}^s(t) + E_{int}^s(t), \quad \mathbf{E}^s(t) = E_{pl}^s(t) + E_{fl}^s(t). \end{aligned}$$

From the standard analysis (done at the energy level) we have that [14, 45]

$$(4.13) \quad |E_{int}^s(t)| \leq \delta \|\nabla \phi_t^m(t)\|^2 + \frac{C}{\delta} \|u_{xt}^m(t)\|_0^2 \leq \delta \|\nabla \phi_t^m(t)\|^2 + \epsilon \|\Delta u_t^m(t)\|^2 + C_{\epsilon, \delta} \|u_t^m(t)\|^2.$$

Using the boundedness of trajectories at the energy level $\sup_{t>0} \|u_t^m(t)\|_0 \leq C_R$, we then have that for all $\epsilon > 0$

$$(4.14) \quad |E_{int}^s(t)| \leq \epsilon \mathbf{E}^s(t) + K_\epsilon,$$

and this results in the existence of constants $c, C, K > 0$ such that

$$(4.15) \quad c\mathbf{E}^s(t) - K \leq \mathcal{E}^s(t) \leq C\mathbf{E}^s(t) + K \quad \forall t > 0.$$

Then the energy balance for the time-differentiated equations is

$$(4.16) \quad \mathcal{E}^s(t) + k \int_0^t \|\bar{u}_t(\tau)\|^2 d\tau = \mathcal{E}^s(0) + B(t),$$

where $B(t)$ represents the contribution of the nonlinearity in the time-differentiated dynamics:

$$(4.17) \quad B(t) \equiv \int_0^t \left\langle \frac{d}{dt} (f(u^m)), \bar{u}_t \right\rangle_\Omega d\tau.$$

Using (4.15), we have

$$(4.18) \quad \mathbf{E}^s(t) \leq C\mathbf{E}^s(0) + K + B(t).$$

We note immediately that if we can bound $B(t)$, we will obtain the estimate in (4.10).

Continuing, we use the key decomposition for $f(u) = [b - \|\nabla u\|^2]\Delta u$:

$$\begin{aligned} (4.19) \quad & \left(\frac{d}{dt} ([\|\nabla u^m\|^2 - b]\Delta u^m), u_{tt}^m \right)_\Omega \\ &= 2(\nabla u^m, \nabla u_t^m)(\Delta u^m, u_{tt}^m) + [\|\nabla u^m\|^2 - b](\Delta u_t^m, u_{tt}^m) \\ (4.20) \quad &= \frac{d}{dt} Q_1(t) + P_1(t), \end{aligned}$$

where

$$(4.21) \quad Q_1(t) = -\frac{1}{2} [\|\nabla u^m\|^2 - b] \|\nabla u_t^m\|^2 - (\nabla u^m, \nabla u_t^m)^2,$$

$$(4.22) \quad P_1(t) = 3(u^m, \Delta u_t^m)(u_t^m, \Delta u_t^m).$$

Thus we have

$$\begin{aligned} (4.23) \quad B(t) &\leq C \left\{ \left\| [\|\nabla u^m\|^2 - b] \|\nabla u_t^m\|^2 \right\|_0 \Big|_0^t + \int_0^t |(u^m, \Delta u_t^m)(u_t^m, \Delta u_t^m)| d\tau \right\} \\ &\leq \epsilon \|u_t^m(t)\|_2^2 + C_\epsilon(R) \|u_t^m\|^2 + C(\mathbf{E}^s(0)) + C_R \int_0^t \|u_t^m\|_2^2 d\tau. \end{aligned}$$

We again appeal to the estimate,

$$\|u_t^m(t)\|_0^2 \leq \|u_t^m(0)\|_0^2 + \int_0^t \|u_{tt}^m(\tau)\|_0^2 d\tau,$$

which yields

$$(4.24) \quad B(t) \leq \epsilon \|u_t^m(t)\|_2^2 + C_{\epsilon,R}(\mathbf{E}^s(0)) + C_{\epsilon,R} \int_0^t [\|u_t^m\|_2^2 + \|u_{tt}^m\|_0^2] d\tau$$

$$(4.25) \quad \leq \epsilon \|u_t^m(t)\|_2^2 + C_{\epsilon,R}(\mathbf{E}^s(0)) + C_{\epsilon,R} \int_0^t \mathbf{E}^s(\tau) d\tau.$$

Hence the above inequality, when used in conjunction with (4.15), (4.16), and Gronwall's inequality, provide uniform boundedness of the higher norm energy $\mathbf{E}^s(t)$ on any time interval $[0, T]$ for any *finite* T .

4.3.2. Step 2.

Remark 4.8. Using the variation of parameters formula and global-in-time bounds of delayed plate trajectories in the energy topology one can obtain the bounds in (4.10) and (4.11) straightforwardly, but this requires *large viscous damping* (see (5.13)). Here we show below that these bounds (and the final result for strong stability for smooth initial data) can be obtained with minimal viscous damping and *no* static damping.

In order to obtain uniform-in-time boundedness for *plate solutions* (as in 4.10 and 4.11), more work is needed. The needed argument depends on the reduction in Theorem 2.5. This reduction is valid for sufficiently large times—thus the result of Step 1 applies to all the times before the delayed model is valid.

We return to the delay representation of the plate solutions (Theorem 2.5) and utilize our finite time horizon regularity from Step 1 to obtain regular *plate data* at time T . Considering $(\bar{u}, \bar{u}_t) = (u_t^m, u_{tt}^m)$, we write

$$(4.26) \quad \bar{u}_{tt} + \Delta^2 \bar{u} + (k+1)\bar{u}_t = \frac{d}{dt} \{ [\|\nabla u^m\|^2 - b] \Delta u^m \} - U \partial_x \bar{u} - \frac{d}{dt} \{ q(u^m) \},$$

which is valid for any $t > \max\{T, t^\#\}$.

We utilize an approach taken in [14] for the nondelayed plate. We will define a Lyapunov-type function for the time-differentiated, smooth dynamics so that we may use Gronwall's inequality, along with the finiteness of the dissipation integral (Corollary 1.6):

$$\int_0^\infty \|u_t^m\|^2 d\tau = \int_0^\infty \|\bar{u}\|^2 d\tau < +\infty.$$

Let us use some more convenient notation here: $E(t) = \frac{1}{2}[\|\Delta\bar{u}\|^2 + \|\bar{u}_t\|^2]$. Then define

$$Q(t) = E(t) - Q_1(t) + \nu\|\bar{u}\|^2,$$

where Q_1 and P_1 are as above.

Let $\nu, \mu > 0$. We can choose $\nu(R)$ large enough so that $Q(t)$ is positive [14]; we make this choice and then keep ν fixed. We will now construct a Lyapunov-like function for the dynamics $T_t^{\bar{u}}(x_0) = (\bar{u}(t), \bar{u}_t(t); \bar{u}^t) \in \mathbf{H}$ with $x_0 \in \mathbf{H}$. There are constants $a_0(\nu)$ and $a_1(\nu)$ so that

$$a_0[\|\Delta\bar{u}\|^2 + \|\bar{u}_t\|^2] \leq Q(t) \leq a_1[\|\Delta\bar{u}\|^2 + \|\bar{u}_t\|^2].$$

The parametric Lyapunov function is then

$$W(t) = Q(t) + \varepsilon(\bar{u}_t, \bar{u}) + \mu \int_0^{t^*} \int_{t-s}^t E(\tau) d\tau ds, \quad \varepsilon \in (0, \varepsilon_0).$$

The parameter μ will be addressed below. The value of ε_0 is chosen so that

$$(4.27) \quad \frac{1}{2}a_0E(t) \leq W(t) \leq 2a_1E(t) + \mu t^* \int_{t-t^*}^t E(\tau) \quad \forall \quad \varepsilon \in (0, \varepsilon_0),$$

and we note that

$$\mu \int_0^{t^*} \int_{t-s}^t E(\tau) d\tau ds \leq \mu t^* \int_{t-t^*}^t E(\tau).$$

We compute $W'(t)$:

$$(4.28) \quad \frac{d}{dt}W(t) = \langle \Delta\bar{u}, \Delta\bar{u}_t \rangle + \langle \bar{u}_t, \bar{u}_{tt} \rangle - \frac{d}{dt}Q_1(t) + \nu\langle \bar{u}, \bar{u}_t \rangle$$

$$(4.29) \quad + \varepsilon\langle \bar{u}_{tt}, \bar{u} \rangle + \varepsilon\|\bar{u}_t\|^2 + \mu t^*E(t) - \mu \int_{t-t^*}^t E(\tau).$$

Make the replacement:

$$\bar{u}_{tt} = -\Delta^2\bar{u} - (k+1)\bar{u}_t - \frac{d}{dt}q^{u^m} - \frac{d}{dt}[f(u^m)] - U\bar{u}_x.$$

Then

$$\begin{aligned} \frac{d}{dt}W(t) &= (\mu t^* + \varepsilon - k - 1)\|\bar{u}_t\|^2 + (\nu - \varepsilon k - 1)\langle \bar{u}, \bar{u}_t \rangle - \langle [q^{u^m}]', \bar{u}_t \rangle \\ &\quad - \langle [f(u^m)]', \bar{u}_t \rangle - U\langle \bar{u}_x, \bar{u}_t \rangle - \frac{d}{dt}Q_1(\mu t^* - \varepsilon)\|\Delta\bar{u}\|^2 \\ &\quad + \varepsilon\langle [q^{u^m}]', \bar{u} \rangle - \varepsilon\langle [f(u)]', \bar{u} \rangle - \varepsilon U\langle \bar{u}_x, \bar{u} \rangle - \mu \int_{t-t^*}^t E(\tau). \end{aligned}$$

Thus

$$\begin{aligned}
 (4.30) \quad & \frac{d}{dt}W(t) + (k+1)\|\bar{u}_t\|^2 + \varepsilon\|\Delta\bar{u}\|^2 + \mu \int_{t-t^*}^t E(\tau) d\tau \\
 &= (\mu t^* + \varepsilon)\|\bar{u}_t\|^2 + \mu t^*\|\Delta\bar{u}\|^2 + P_1(t) - \varepsilon\langle [f(u^m)]', \bar{u} \rangle \\
 &\quad - \varepsilon\langle [q^{u^m}]', \bar{u} \rangle - \langle [q^{u^m}]', \bar{u}_t \rangle + (\nu - \varepsilon(k+1))\langle \bar{u}, \bar{u}_t \rangle - \varepsilon U\langle \bar{u}_x, \bar{u} \rangle - U\langle \bar{u}_x, \bar{u}_t \rangle.
 \end{aligned}$$

Via (4.27), choosing μ and ε sufficiently small, there is a $\gamma(a_1, \nu, k, \mu, t^*) > 0$ so that for any $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned}
 (4.31) \quad & \frac{d}{dt}W(t) + \gamma \left[E(t) + \int_{t-t^*}^t E(\tau) d\tau \right] \\
 &\leq C \left\{ P_1(t) - \varepsilon\langle [f(u)]', \bar{u} \rangle - \varepsilon\langle [q^{u^m}]', \bar{u} \rangle \right. \\
 &\quad \left. - \langle [q^{u^m}]', \bar{u}_t \rangle + (\nu - \varepsilon k - 1)\langle \bar{u}, \bar{u}_t \rangle - \varepsilon U\langle \bar{u}_x, \bar{u} \rangle - U\langle \bar{u}_x, \bar{u}_t \rangle \right\}.
 \end{aligned}$$

LEMMA 4.8. *There exists a $\tilde{\gamma} > 0$ (with the same dependencies γ above), and constant M (depending on R and the same quantities as $\tilde{\gamma}$) so that*

$$(4.32) \quad W'(t) + \tilde{\gamma}W(t) \leq M[1 + \|\bar{u}\|^2 E(t)].$$

Proof of Lemma 4.8. We recall (2.13). So, along with the global-in-time priori bounds for the lower energy level (u^m, u_t^m) ,

$$\sup_{t \in [t_0, \infty)} \left[\|u^m\|_2^2 + \|\bar{u}\|^2 \right] < \infty$$

and

$$\begin{aligned}
 \frac{d}{dt}f(u^m) &= -2(\nabla u^m, \nabla u_t^m)\Delta u^m + [b - \|\nabla u^m\|^2]\Delta u_t^m \\
 &= 2(\Delta u^m, \bar{u})\Delta u^m + [b - \|\nabla u^m\|^2]\Delta \bar{u},
 \end{aligned}$$

and we obtain

$$(4.33) \quad |P_1| \leq C\|\Delta u^m\| \|\bar{u}\| \|\nabla \bar{u}\|^2,$$

$$(4.34) \quad |\langle [f(u^m)]', \bar{u} \rangle| \leq C(R)[\|\bar{u}\|^2 + \|\bar{u}\| \|\Delta \bar{u}\|],$$

$$\begin{aligned}
 (4.35) \quad & (\nu - \varepsilon k - 1)|\langle \bar{u}, \bar{u}_t \rangle| \leq \delta\|\bar{u}_t\|^2 + C_\delta(\nu, \varepsilon, k, R),
 \end{aligned}$$

$$\begin{aligned}
 (4.36) \quad & \varepsilon U\langle \bar{u}_x, \bar{u} \rangle + U\langle \bar{u}_x, \bar{u}_t \rangle \leq \delta[\|\bar{u}_t\|^2 + \|\Delta \bar{u}\|^2] + C_\delta(\varepsilon, U, R),
 \end{aligned}$$

$$\begin{aligned}
 (4.37) \quad & \varepsilon|\langle [q^{u^m}]', \bar{u} \rangle| \leq C(\varepsilon) \left(\|u^m(t)\|_2^2 + \|u^m(t-t^*)\|_2^2 + \int_{-t^*}^0 \|u^m(t+\tau)\|_2^2 d\tau \right) \\
 & \quad + \delta\|\bar{u}\|_2^2 + C_\delta(\varepsilon)\|\bar{u}\|^2
 \end{aligned}$$

$$(4.38) \quad \leq \delta E(t) + C_\delta(\varepsilon, t^*, R).$$

And, noting the explicit estimates for the term q_t^u performed in [16, (6.1)], we also have

$$(4.39) \quad \langle \partial_t [q^{u^m}], \bar{u}_t \rangle \leq \epsilon \|\bar{u}_t\|^2 + C_\delta \left(\|u^m(t)\|_{2,\Omega}^2 + \|u^m(t-t^*)\|_{2,\Omega}^2 + \int_{-t^*}^0 \|u^m(t+\tau)\|_{3,\Omega}^2 d\tau \right).$$

Then we note

$$\|u\|_3^2 \leq \delta \|u\|_4^2 + C_\epsilon(R) \|u\|_2^2 \leq C_\epsilon + \epsilon \|u\|_4^2,$$

where $\|u\|_4$ is to be calculated from an elliptic equation in terms of $\|(\bar{u}, \bar{u}_t)\|_{Y_{pl}}$. Indeed, we consider biharmonic problem with the clamped boundary conditions.

$$\Delta^2 u^m = -\bar{u}_t - (k+1)\bar{u} + [|\nabla u^m|^2 - b]\Delta u^m - U u_x^m - q^{u^m}$$

with

$$u^m = \partial_\nu u^m = 0 \text{ on } \partial\Omega.$$

This gives

$$\begin{aligned} \|u^m\|_4 &\leq C \|\bar{u}_t - (k+1)\bar{u} + [|\nabla u^m|^2 - b]\Delta u^m - u_x^m - q^{u^m}\| \\ &\leq C [\|(\bar{u}, \bar{u}_t)\|_{Y_{pl}} + \|u^m\|_1 + \|u^m\|_2^3 + \|u^m\|_2^2]. \end{aligned}$$

We have

$$(4.40) \quad \|u^m\|_4^2 \leq C(R)[E(t) + 1],$$

$$\langle [q^{u^m}]', \bar{u}_t \rangle \leq \|\bar{u}_t\|_0 \left\| \frac{d}{dt} q^{u^m} \right\|_0 \leq \delta \left[E(t) + \int_{t-t^*}^t E(\tau) d\tau \right] + C_\delta(t^*, R).$$

Combining these estimates, we return to (4.31):

$$(4.41) \quad \begin{aligned} &\frac{d}{dt} W(t) + \gamma \left[E(t) + \int_{t-t^*}^t E(\tau) d\tau \right] \\ &= C \left\{ P_1(t) - \varepsilon([f(u^m)]', \bar{u}) - \varepsilon([q^{u^m}]', \bar{u}) - ([q^{u^m}]', \bar{u}_t) \right. \\ &\quad \left. + (\nu - \varepsilon(k+1))(\bar{u}, \bar{u}_t) - \varepsilon U(\bar{u}_x, \bar{u}) - U(\bar{u}_x, \bar{u}_t) \right\} \end{aligned}$$

$$(4.42) \quad \begin{aligned} &\leq \delta \left[E(t) + \int_{t-t^*}^t E(\tau) d\tau \right] + C_\delta(\varepsilon, \nu, R) \|\bar{u}\|^2 E(t) \\ &\quad + C_\delta(R, \nu, \varepsilon, t^*, U, k). \end{aligned}$$

□

Scaling δ appropriately and absorbing terms, we have

$$\frac{d}{dt} W(t) + \gamma \left[E(t) + \int_{t-t^*}^t E(\tau) d\tau \right] \leq M + C \|\bar{u}\|^2 E(t).$$

Recalling (4.27), we have some $\tilde{\gamma} > 0$ such that

$$\frac{d}{dt} W(t) + \tilde{\gamma} W(t) \leq M + C \|\bar{u}\|^2 E(t).$$

Noting (4.27), we have

$$\frac{d}{dt}W(t) + \tilde{\gamma}W(t) \leq M + C_1\|\bar{u}\|^2W(t),$$

and utilizing the integrating factor we see that

$$\frac{d}{dt}\left(e^{\tilde{\gamma}t}W(t)\right) \leq Me^{\tilde{\gamma}t} + C_1e^{\tilde{\gamma}t}\|\bar{u}\|^2W(t).$$

Integrating, we have

$$e^{\tilde{\gamma}T}W(T) \leq e^{\tilde{\gamma}t_0}W(t_0) + \frac{M}{\tilde{\gamma}}\left[e^{\tilde{\gamma}T} - e^{\tilde{\gamma}t_0}\right] + C_1\int_{t_0}^Te^{\tilde{\gamma}t}\|\bar{u}(t)\|^2W(t)dt,$$

and thus

$$\begin{aligned} W(T) &\leq e^{-\tilde{\gamma}(T-t_0)}W(t_0) + \frac{M}{\tilde{\gamma}}\left[1 - e^{-\tilde{\gamma}(T-t_0)}\right] + C_1\int_{t_0}^T\|\bar{u}(t)\|^2W(t)dt \\ &\leq \frac{M}{\tilde{\gamma}} + e^{-\tilde{\gamma}(T-t_0)}\left[W(t_0) - \frac{M}{\tilde{\gamma}}\right] + C_1\int_{t_0}^T\|\bar{u}(t)\|^2W(t)dt. \end{aligned}$$

Let

$$\alpha(T) = \frac{M}{\tilde{\gamma}} + e^{-\tilde{\gamma}(T-t_0)}\left[W(t_0) - \frac{M}{\tilde{\gamma}}\right],$$

and note that $\alpha(T) \geq 0$ for $T \geq t_0$. Then, using Grownall, we have

$$W(T) \leq \alpha(t) + \int_{t_0}^T \alpha(t)\|\bar{u}(t)\|^2 \exp\left\{\int_t^T\|\bar{u}(s)\|^2ds\right\}dt.$$

Note that $\alpha(T) \geq 0$ and α is bounded, and thus, owing to the finiteness of the dissipation integral

$$\int_{t_0}^{\infty}\|\bar{u}\|^2dt = \int_{t_0}^{\infty}\|u_t^m\|_0^2dt < +\infty,$$

we obtain global-in-time boundedness of $W(t)$. Then, using (4.27), we have global-in-time boundedness of

$$E(t) = \frac{1}{2}\left[\|\Delta u_t^m\|^2 + \|u_{tt}^m\|^2\right].$$

Via elliptic theory (as utilized above), we also have that $\|u^m(t)\|_4^2$ is globally bounded in time. We note that $W(t_0)$ has the requisite regularity by Step 1 above (where we have shown propagation of regularity on the finite time horizon for the full flow-plate dynamics).

This concludes the proof of Theorem 4.7. \square

Remark 4.9. As mentioned above, the proof steps can be adapted to the case of the von Karman nonlinearity using the *sharp regularity of the Airy stress function* [14, p. 44].

4.4. Strong convergence for smooth data. From Theorem 4.7 we see that for the Berger dynamics (and, in fact for von Karman), in the presence of minimal viscous damping (only requiring $k > 0$), we have a propagation of regularity result on the infinite-time horizon. From this bound, for initial data $y^m \in \mathcal{D}(\mathbb{T})$, we can immediately apply Theorem 4.3. This yields one of the primary results of [33, Theorem 2.1].

THEOREM 4.9. *Consider $y_m \in \mathcal{D}(\mathbb{T}) \cap B_R(Y) \subset (H^4 \cap H_0^2)(\Omega) \times H_0^2(\Omega) \times H^2(\mathbb{R}_+^3) \times H^1(\mathbb{R}_+^3) \subset Y$. Suppose that the initial flow data are supported on a ball of radius ρ_0 (as in Theorem 2.5). Suppose $k > 0$. Then for any sequence of times $t_n \rightarrow \infty$ there is a subsequence of times t_{n_k} identified by t_k and a point $\hat{y} = (\hat{u}, 0; \hat{\phi}, 0) \in Y$ with $(\hat{u}, \hat{\phi}) \in \mathcal{N}$, so that*

$$\|S_{t_k}(y^m) - \hat{y}\|_{Y_\rho} \rightarrow 0, \quad t_k \rightarrow \infty.$$

This implies the primary result: strong convergence of $(u(t), \phi(t))$ to the equilibria set \mathcal{N} for initial data $y_0 \in \mathcal{D}(\mathbb{T}) \cap Y_{\rho_0}$.

Remark 4.10. We note that the situation when stability to the set of equilibria is achieved for smooth initial data is typical in the analysis of strong stability for hyperbolic models with damping. The next step is to extend the desired property via density to all initial data residing in the finite energy space. This can be done (see, e.g., [1]) via a uniform-in-time Lipschitz (or Hadamard) property of the semigroup. As we describe in the next section, this is precisely the key issue to contend with in the analyses in [33] (where large static damping was used in conjunction with large static damping) and here, where a fortuitous decomposition allows us to consider only large viscous damping.

4.5. Strong convergence for finite energy data—Technical difficulties.

With the technical background established in sections 2–4.3, we now address the specific difficulties involved in showing a stabilization to equilibria result without assuming either (i) terms of the form $k_1 \Delta u_{tt} - k_2 \Delta u_t$ (as in [12, 14]) or (ii) exploiting parabolic effects in a thermoelastic plate [40, 41]. In both cases, as well as in [16], the key task is *to first show compact end behavior for the plate dynamics*. In both cases (i) and (ii) above the ultimate character of the nonlinear component of the model is *compact*—owing to the fact that parabolic smoothing *and* rotational inertia both provide $\nabla u_t \in L_2(\Omega)$. This permits compactness of the boundary-to-flow (Lopatinski or hyperbolic Neumann) mapping. The results in [16] were the first to show that dissipation could be harvested from the flow in order to show ultimate compactness of the plate dynamics *without* utilizing the compactness of this mapping.

We note that in [41] the key to the stabilization result lies in a compactness criterion (given as Lemma 10 in [40, p. 472]) for flow trajectories (given here as (2.4)). The flow bound in higher norms is obtained via the thermoelastic character of the structural dynamics.

In [33], the major contribution was the ability to circumvent the seeming lack of natural compactness in the dynamics (for von Karman or Berger) by utilizing large static *and* viscous damping. The requirement for large static and viscous damping was needed to show exponential decay of plate trajectories, which resulted in uniform-in-time Hadamard continuity of the full flow-plate dynamics (see Theorem 4.2 and Remark 4.10 above). With this, and the desired result holding for regular initial data (Theorem 4.9), an approximation argument was used to pass convergence properties from regular data onto finite energy initial data. The principal result there [33,

Theorem 2.3] is the analogue of Theorem 1.8, but *it requires the addition of large static damping*.

THEOREM 4.10. *Consider (1.13). Let $0 \leq U < 1$, and assume $p_0 \in L_2(\Omega)$. Then there are minimal damping coefficients k_{\min} and β_{\min} so that for $k \geq k_{\min} > 0$ and $\beta \geq \beta_{\min} > 0$ any generalized solution $(u(t), \phi(t))$ to the system with localized initial flow data in Y_{ρ_0} has the property that*

$$\lim_{t \rightarrow \infty} \inf_{(\hat{u}, \hat{\phi}) \in \mathcal{N}} \left\{ \|u(t) - \hat{u}\|_{H^2(\Omega)}^2 + \|u_t(t)\|_{L_2(\Omega)}^2 + \|\phi(t) - \hat{\phi}\|_{H^1(K_\rho)}^2 + \|\phi_t(t)\|_{L_2(K_\rho)}^2 \right\} = 0 \text{ for any } \rho > 0.$$

As in Theorem 1.8 here, the minimal damping coefficients k_{\min} and β_{\min} depend on an invariant set for the plate dynamics, which itself depends on the intrinsic quantities in the problem.

Though the results of [33] stand as the culmination of the treatments [16, 45] and critically use the results of [14, 40, 41], static damping seemed necessary to contend with lower order terms (due to the nondissipative nature of the reduced plate dynamics (2.12)) but seemed physically anomalous. The key technical point in this treatment is *the elimination of this need for large static damping imposed on the structure*. We seek the physical result which says that for large viscous damping only (size dependent upon the intrinsic parameters of the problem) the full-flow plate dynamics stabilize to the stationary set. Our approach here is based on a decomposition of plate dynamics into an exponentially stable component and a smooth component. To show smoothness of this latter component in the decomposition, we require working in *higher topologies*.

5. Proper proof of main results. In this section we prove Theorem 1.8. The proof follows through several steps and critically uses the results and arguments presented up to this point.

5.1. Outline of the proof. For the reader's convenience we now give a summary of each of the key steps in the proof:

In section 5.2 we consider a decomposition of the delayed plate dynamics (2.10): we consider z and w components, such that $z + w = u$. We will decompose the nonlinearity in an advantageous way (for t sufficiently large), as well as add static damping βz to the z equation and subtract it from the w dynamics. This allows us to use damping of the form $kz_t + \beta z$ in the z plate dynamics, which will result in uniform exponential stability for (5.1). For the w component, we will consider zero initial data (the initial data being left in the z dynamics), giving (5.2). The w dynamics, with zero initial data, will be smooth, having appropriate global-in-time bounds. We note the following: *this outcome cannot be achieved for von Karman's dynamics*; the decomposition utilized is not amenable to the structure of the nonlinearity in this case.

Sections 5.3 and 5.4 provide the proofs associated to the aforementioned plate decomposition and are the major technical contribution of this work. Section 5.3 follows [33] in utilizing Lyapunov methods to show exponential stability of z component of the plate decomposition; section 5.4 demonstrates the smoothness of the w component of the decomposed plate dynamics and critically uses the null initial data in differentiating the dynamics and obtaining global-in-time bounds.

In section 5.5 we decompose the *flow* according the plate decompositions in the previous step. This allows us to reconstruct a full flow-plate dynamics corresponding

to z , and a full flow-plate dynamics corresponding to w , coupled through a mixed, nonlinear plate term. The (z, ϕ^z) dynamics (corresponding to the exponentially stable z plate dynamics) will exhibit global-in-time Hadamard continuity (depending critically on large static and viscous damping), and thus we can approximate any trajectory by smooth data. (This follows the general structure of the argument in [33].) The (w, ϕ^w) dynamics (corresponding to the plate dynamics with zero initial data) are smooth, and thus the results and techniques for smooth initial data can be reapplied (Theorems 4.3 and 4.7).

5.2. Decomposition of delay plate dynamics. We consider the reduced, delay plate equation in (2.10) (see [8, 14, 16]). The general PDE system with delay (2.12) is well-posed on $\mathbf{H} = H_0^2(\Omega) \times L_2(\Omega) \times L_2(-t^*, 0; H_0^2(\Omega))$. We write $T_t(x_0) = (u, u_t; u^t)$ with $x_0 = (u(t_0), u_t(t_0); u^{t_0}) \in \mathbf{H}$. There is a compact global attractor $\mathcal{U} \subset \mathbf{H}$ [16], and we denote by $\mathcal{B} \subset \mathbf{H}$ a fixed absorbing ball of diameter R for the dynamics, such that $\mathcal{U} \subset \mathcal{B}$. We note that for $k > k_* > 0$ the size of the absorbing ball \mathcal{B} can be taken uniform in k . In what follows, we can reduce the above delay plate dynamics (T_t, \mathbf{H}) to those on the set \mathcal{B} by waiting for absorption; this is to say that for any initial data $\|(u(t_0), u_t(t_0); u^{t_0})\|_{\mathbf{H}} \leq P$ there is a time of absorption t_P such that $T_{t_P}(B_{\mathbf{H}}(P)) \subset \mathcal{B}$. Thus, by waiting sufficiently long (i.e., choosing $t_0 > t_P$ —valid, as we are considering strong stability as $t \rightarrow +\infty$), we can reduce dynamics corresponding to any initial data to dynamics with initial data chosen from \mathcal{B} . We will denote this dependence by R —the diameter of \mathcal{B} .

Owing to the fact that q is linear, we will apply the following decomposition of the solution (inspired by [26]).

Take $(u(t_0), u_t(t_0); u^{t_0}) \in \mathcal{B} \subset \mathbf{H}$. We write $u = z + w$, where z and w correspond to the systems:

$$(5.1) \quad \begin{cases} z_{tt} + (k+1)z_t + \beta z + \Delta^2 z + \mathcal{F}(z, u) = q^z - Uz_x & \text{in } \Omega, \\ \mathcal{F}(z, u) = [b - \|\nabla u\|^2]\Delta z, \\ z = \partial_\nu z = 0 & \text{on } \Gamma, \\ z(t_0) = u(t_0), \quad z_t(t_0) = u_t(t_0), \quad z|_{(t_0-t^*, t_0)} = u^{t_0}. \end{cases}$$

$$(5.2) \quad \begin{cases} w_{tt} + (k+1)w_t + \Delta^2 w = p_0 + q^w - Uw_x - \mathcal{F}(w, u) + \beta z & \text{in } \Omega, \\ \mathcal{F}(w, u) = [b - \|\nabla u\|^2]\Delta w, \\ w = \partial_\nu w = 0 & \text{on } \Gamma, \\ w(t_0) = 0, \quad w_t(t_0) = 0, \quad w|_{(t_0-t^*, t_0)} = 0. \end{cases}$$

Remark 5.1. Since we are taking null initial data for the w portion of the decomposition, we note that the presence of u in the term $\mathcal{F}(w, u)$ is the driver of the dynamics.

Given u , the z/w system is well-posed with respect to finite energy topology \mathbf{H} . Indeed, in light of the stability estimates shown below for the z dynamics (with u given) the analysis in [16] applies; from there, the w dynamics are well-posed as a difference $w = u - z$. We state this as a lemma.

LEMMA 5.1. *Consider the dynamics in (5.1) at time t_0 with initial data $(u(t_0), u_t(t_0); u^{t_0}) \in \mathcal{B} \subset \mathbf{H}$. Given $u \in C([t_0 - t^*, T]; H_0^2(\Omega))$ the dynamics corresponding to (5.1) are well-posed on \mathbf{H} [16]. Then, considering $w = u - z$ with u corresponding*

to the solution in (2.10) on \mathbf{H} , the problem (5.2) is well posed on \mathbf{H} with $(w, w_t; w^t)$ as a solution.

Let

$$E_\beta(z(t)) \equiv \frac{1}{2} [||\Delta z||^2 + ||z_t||^2 + \beta ||z||^2].$$

For the decomposed plate dynamics, we will show the following supporting lemmas.

LEMMA 5.2. *There exists $k_e(R) > 0$ and $\beta_e(R) > 0$ such that for $k > k_e$ and $\beta > \beta_e$, the quantity $E_\beta(z(t))$ decays exponentially to zero. In fact, $T_t^z(x_0) = (z, z_t; z^t)$ decays uniformly exponentially to zero in \mathbf{H} for $x_0 \in B_{\mathbf{H}}(R)$ (with rate depending on k and β).*

Remark 5.2. As in the analyses [37, 38, 39], we note that stabilizing the delay system (2.10) requires a damping coefficient scaled by the time of delay (itself depending on the intrinsic parameters of the problem).

LEMMA 5.3. *There exists a $k_Q(R)$ such that for any $k > k_Q$ all $\beta \geq 0$ the evolution $(w, w_t; w^t)$ on \mathbf{H} corresponding to (5.2) has that $(w, w_t, w_{tt}) \in C([t_0, \infty); H^r)$, where $H^r \equiv H^4(\Omega) \times H^2(\Omega) \times L_2(\Omega)$.*

Remark 5.3. The presence of β in Lemma 5.3 above is benign; $\beta > \beta_e$ is required for impacting the margin of stability in Lemma 5.2.

We will need to utilize the following estimates on q , coming from [16]. It is shown that they are valid for q as given in (2.10)—see also (2.13) above. These estimates indicate a mechanism for *compensated compactness* acquired by the flow and translated into delay part of the plate system:

$$(5.3) \quad ||q^u(t)|| \leq Ct^* \int_{t-t^*}^t ||\Delta u(\tau)|| d\tau, \quad ||q^u(t)||_{-1} \leq Ct^* \int_{t-t^*}^t ||u(\tau)||_1 d\tau,$$

$$(5.4) \quad |\langle q_t^u(t), \psi \rangle| \leq C \left\{ ||u(t)||_1 + ||u(t-t^*)||_1 + \int_{t-t^*}^t ||u(\tau)||_2 d\tau \right\} ||\psi||_1,$$

$$(5.5) \quad ||q_t^u(t)|| \leq Ct^* \left[||\Delta u(t)|| + ||\Delta u(t-t^*)|| + \int_{t-t^*}^t ||u(s)||_3 ds \right].$$

The last inequality follows from (2.13), after noting that M_θ^2 is the second order differential operator controlled by $||\Delta u||$ (due to clamped boundary conditions imposed on the structure). Additionally, we note the following elementary bound:

$$\int_0^{t^*} \int_{t-s}^t G(u(\tau)) d\tau ds \leq t^* \int_{t-t^*}^t G(u(\tau)) d\tau,$$

where G is any positive functional.

5.3. Stability of (5.1)—Proof of Lemma 5.2. We consider the Lyapunov approach which was used in [14, 16] (based upon [13]) in showing the existence of an absorbing set (and used in [33] to show Hadamard continuity of the full flow-plate dynamics). Define the Lyapunov-type function:

$$(5.6) \quad V(T_t^z(x_0)) \equiv E_\beta(z) - \langle q(z^t), z \rangle + \langle z_t, z \rangle + \frac{k}{2} ||z||^2 \\ + \mu \left(\int_{t-t^*}^t ||\Delta z||^2 ds + \int_0^{t^*} \int_{t-s}^t ||\Delta z(\tau)||^2 d\tau ds \right),$$

where $T_t^z(x_0) = (z(t), z_t(t); z^t)$ for $t \geq t_0$, and μ is some positive number to be specified below. Using Young's inequality and (5.3), we have immediately that

$$(5.7) \quad c(\mu)E_\beta(z(t)) \leq V(T_t^z(x_0)) \leq C(k)E_\beta(z(t)) + C(\mu, t^*) \int_{t-t^*}^t \|z(\tau)\|_2^2 d\tau.$$

The constant $c(\mu)$ does not depend on either damping parameter. We now continue with the Lyapunov calculus by computing the time derivative of $V(T_t^z(x_0))$. Since the dynamics has been shown sufficiently regular for regular data, we can consider such regular initial data and perform differential calculus on a priori smooth solutions. Density and passage with the limit on initial data will allow us to draw the same conclusion for finite energy initial data:

$$(5.8) \quad \begin{aligned} \frac{d}{dt} V(T_t^z(x_0)) &= \frac{d}{dt} E_\beta(z(t)) + \langle z_{tt}, z \rangle + \|z_t\|^2 + k \langle z_t, z \rangle \\ &\quad - \langle q, z_t \rangle - \langle q_t, z \rangle \\ &\quad + \mu \frac{d}{dt} \left\{ \int_{t-t^*}^t \|\Delta z\|^2 ds + \int_0^{t^*} \int_{t-s}^t \|\Delta z(\tau)\|^2 d\tau ds \right\} \\ &= -k \|z_t\|^2 + (\mu t^* + \mu - 1) \|\Delta z\|^2 - \beta \|z\|^2 \\ &\quad - \mu \|\Delta z(t-t^*)\|^2 - \mu \int_{t-t^*}^t \|\Delta z\|^2 ds \\ &\quad + \langle q, z \rangle - \langle q_t, z \rangle - U \langle z_x, z \rangle - U \langle z_x, z_t \rangle \\ &\quad - \langle \mathcal{F}(z, u), z_t \rangle - \langle \mathcal{F}(z, u), z \rangle. \end{aligned}$$

We twice made use of relation (5.1): $z_{tt} + \beta z + (k+1)z_t + \Delta^2 z + \mathcal{F}(z, u) = q^z - Uz_x$. We estimate from above; first we note that

$$(5.9) \quad \|\mathcal{F}(z, u)\| = \| [b - \|\nabla u\|^2] \Delta z \| \leq C(R, b) \|z\|_2.$$

Then, using (5.3), Young's inequality, and compactness of the Sobolev embeddings, we have

$$\begin{aligned} |\langle \mathcal{F}(z, u), z_t \rangle| + |\langle \mathcal{F}(z, u), z \rangle| &\leq \epsilon \|\Delta z\|^2 + C_\epsilon(R, b) [\|z_t\|^2 + \|z\|^2] \\ |\langle q^z(t), z \rangle| &\leq \epsilon \int_{t-t^*}^t \|\Delta z\|^2 + C_\epsilon(t^*) \|z\|^2 \\ |\langle q_t^z(t), z \rangle| &\leq \epsilon \left[\|\Delta z(t)\|^2 + \|\Delta z(t-t^*)\|^2 + \int_{t-t^*}^t \|\Delta z(\tau)\| d\tau \right] \\ &\quad + C_\epsilon(t^*) \|z\|^2 \\ U |\langle z_x, z \rangle| + U |\langle z_x, z_t \rangle| &\leq \epsilon [\|\Delta z\|^2 + \|z_t\|^2] + C_\epsilon(U) \|z\|^2. \end{aligned}$$

We may choose μ sufficiently small, then ϵ small, then β (depending on R and μ) sufficiently large, and finally k sufficiently large (depending on β and R), in order to obtain the following.

LEMMA 5.4. *There exists k_e and β_e so that for $k > k_e > 0$ and $\beta > \beta_e > 0$*

$$(5.10) \quad \frac{d}{dt}V(T_t^z(x)) \leq -c(k, \beta) \left\{ \|z_t\|^2 + \|\Delta z\|^2 + \|z\|^2 + \|\Delta z(t - t^*)\|^2 + \int_{t-t^*}^t \|\Delta z(\tau)\|^2 d\tau \right\},$$

where $c(k, \beta) \rightarrow \infty$ as $\min\{k, \beta\} \rightarrow \infty$.

From this lemma and the upper bound in (5.7), we have for some $\gamma(k, \beta) > 0$

$$E_\beta(z(t)) \leq CV(T_t^z(x_0)) \leq CV(x_0)e^{-\gamma t}, \quad t \geq t_0.$$

This concludes the proof of the uniform exponential stability of the z portion of the decomposed dynamics on \mathcal{B} .

5.4. Regularity of (5.2)—Proof of Lemma 5.3. Again, recall that we have chosen t_0 large enough so that $(u(t_0), u_t(t_0); u^{t_0}) \in \mathcal{B}$. Let $T_t^w(\mathbf{0}) = (w(t), w_t(t); w^t)$ correspond to the (5.2) dynamics. First, we note the following.

LEMMA 5.5. *The dynamics $T_t^w(\mathbf{0})$ (such that $T_t^w(\mathbf{0}) + T_t^z(x_0) = T_t(x_0)$, and $u = w + z$) are uniformly bounded on \mathbf{H} . This is to say that*

$$\|(w(t), w_t(t); w^t)\|_{\mathbf{H}} \leq C(R) \quad \forall \quad t \geq t_0.$$

Proof of Lemma 5.5. This follows immediately from the existence of a uniform absorbing set \mathcal{B} for the dynamics $T_t(x_0) = (u(t), u_t(t); u^t)$ on \mathbf{H} , and the uniform exponential stability of the $T_t^z(x_0)$ dynamics. \square

To continue with the proof of Lemma (5.3), recall $\mathcal{F}(w, u) = [b - \|\nabla u\|^2]\Delta w$. Consider the time-differentiated w dynamics; let $\overline{w} = w_t$:

$$(5.11) \quad \begin{cases} \overline{w}_{tt} + (k+1)\overline{w}_t + \Delta^2 \overline{w} = \frac{d}{dt} \{q(w^t)\} - \frac{d}{dt} \{\mathcal{F}(w, u)\} - U\overline{w}_x + \beta z_t & \text{in } \Omega, \\ \overline{w} = \partial_\nu \overline{w} = 0 & \text{on } \Gamma, \\ \overline{w}(t_0) = 0, \quad \overline{w}_t(t_0) = 0, \quad \overline{w}|_{(t_0-t^*, t_0)} = 0. \end{cases}$$

Recall that the term $\|z_t\|$ decays exponentially and is thus bounded. We will now obtain a global in time bound on $\|\overline{w}\|_2 + \|\overline{w}_t\|$ in what follows.

We invoke the well known *exponential decay of the linear, damped (static and viscous) plate equation* for

$$\overline{w}_{tt} + (k+1)\overline{w}_t + K\overline{w} + \Delta^2 \overline{w} = G.$$

(A proof of this decay is given in the argument section 5.3 leading up to Lemma 5.4.) It is critical to note that the margin of stability is controlled by k, K : $\omega(k, K) \rightarrow 0$ as $\min[k, K] \rightarrow \infty$ (see the analysis (5.10)). We consider the following:

$$(5.12) \quad \begin{aligned} \overline{w}_{tt} + (k+1)\overline{w}_t + K\overline{w} + \Delta^2 \overline{w} &= \frac{d}{dt} \{q^w\} - \frac{d}{dt} \{\mathcal{F}(w, u)\} - U\overline{w}_x + \beta z_t + K\overline{w} \\ &= G(u, z, w, \overline{w}, w^t). \end{aligned}$$

Let $\mathbf{w}(t) = (\overline{w}(t), \overline{w}_t(t))$. For t_0 (again, sufficiently large) we utilize the *variation of parameters formula*:

$$(5.13) \quad \left\| \mathbf{w}(t) \right\|_{Y_{pl}} \leq C \left\| \mathbf{w}(t_0) \right\|_{Y_{pl}} + \int_{t_0}^t e^{-\omega(k,K)(t-s)} \left[\left\| q_t^w \right\|_0 + \left\| \frac{d}{dt} \mathcal{F}(w, u) \right\|_0 + U \left\| \overline{w} \right\|_1 + K \left\| \overline{w} \right\|_0 + \beta \left\| z_t \right\|_0 \right] ds.$$

The critical terms involve q_t^w and $\frac{d}{dt} \mathcal{F}(w, u)$, as the other terms are bounded by Lemma 5.5.

For the term $\partial_t [q^w]$ we note the bounds in (5.5) above (see [16] for more details):

$$(5.14) \quad \|q_t^w(\tau)\| \leq Ct^* \left[\|\Delta w(\tau)\| + \|\Delta w(\tau - t^*)\| + \int_{\tau-t^*}^{\tau} \|w(s)\|_3 ds \right].$$

At this point, elliptic theory can be applied: a bound on $\|w\|_4$ is to be calculated from an elliptic equation in terms of $\|(\overline{w}, \overline{w}_t)\|_{Y_{pl}}$. Indeed, we consider the biharmonic problem with the clamped boundary conditions:

$$\Delta^2 w = -w_{tt} - (k+1)w_t - \mathcal{F}(w, u) - Uw_x - q^w + \beta z; \quad w = \partial_\nu w = 0 \text{ on } \partial\Omega.$$

This gives, via elliptic estimates (after accounting for the a priori bound in Lemma 5.5),

$$\begin{aligned} \|w\|_4 &\leq C \|w_{tt} - (k+1)w_t - \mathcal{F}(w, u) - w_x - q^w + \beta z\| \\ &\leq C \left[\|w_{tt}\| + (k+1)\|w_t\| + (1 + \|u\|_1^2)\|w\|_2 + U\|w\|_1 + (1+t^*) \sup_{s \in [t-t^*, t]} \|w(s)\|_2 + \beta C \right] \\ &\leq C [\|(w_t, w_{tt})\|_{Y_{pl}} + 1]. \end{aligned}$$

Thus we have

$$(5.15) \quad \|w(t)\|_4 \leq C(R) [\|(\overline{w}, \overline{w}_t)\|_{Y_{pl}} + 1].$$

We also have

$$(5.16) \quad \left\| \frac{d}{dt} \mathcal{F}(w, u) \right\|_0 = \left\| (\nabla u, \nabla u_t) \Delta w + \|\nabla u\|^2 \Delta w_t \right\|_0 \leq \|(\Delta u, u_t) \Delta w\| + \|\nabla u\|^2 \|\Delta \overline{w}\|.$$

Remark 5.4. Equation (5.16) is the key step for which no analog exists if performing this decomposition analysis on von Karman's plate.

Thus

$$(5.17) \quad \left\| \frac{d}{dt} \mathcal{F}(w, u) \right\|_0 \leq C(R) (1 + \|\Delta \overline{w}\|).$$

Returning to (5.13) and implementing the above bounds in (5.15) and (5.17), we have

$$\begin{aligned}
 & \left\| (\bar{w}(t), \bar{w}_t(t)) \right\|_{Y_{pl}} \\
 & \leq C \|(\bar{w}(t_0), \bar{w}_t(t_0))\|_{Y_{pl}} \\
 (5.18) \quad & + C(R, U) \int_{t_0}^t e^{-\omega(k, K)(t-s)} \left[1 + K + \|\bar{w}\|_2 + \int_{s-t^*}^s \|w(\tau)\|_3 d\tau \right] ds \\
 & \leq C \|(\bar{w}(t_0), \bar{w}_t(t_0))\|_{Y_{pl}} \\
 (5.19) \quad & + C(R, U) \int_{t_0}^t e^{-\omega(k, K)(t-s)} \left[1 + K + \|\bar{w}\|_2 + \int_{s-t^*}^s \|(\bar{w}, \bar{w}_t)\|_{Y_{pl}} d\tau \right] ds.
 \end{aligned}$$

At this point we note by our choice of decomposition—the w problem with null initial data—the initial condition $\bar{w}|_{(t_0-t^*, t^*)} = 0$ (and hence $\bar{w}_t = 0$ as well) on $(t_0 - t^*, t_0)$. Thus

$$\begin{aligned}
 (5.20) \quad & \left\| (\bar{w}(t), \bar{w}_t(t)) \right\|_{Y_{pl}} \leq C \|(\bar{w}(t_0), \bar{w}_t(t_0))\|_{Y_{pl}} \\
 & + C(R, U, t^*) \left[1 + K + \sup_{s \in [t_0, t]} \|(\bar{w}, \bar{w}_t)\|_{Y_{pl}} \right] \int_{t_0}^t e^{-\omega(k, K)(t-s)} ds \\
 (5.21) \quad & \leq C \left\| (\bar{w}(t_0), \bar{w}_t(t_0)) \right\|_{Y_{pl}} \\
 & + C(R, U, t^*) \left[1 + K + \sup_{s \in [t_0, t]} \|(\bar{w}, \bar{w}_t)\|_{Y_{pl}} \right] \frac{1 - e^{-\omega(k, K)(t-t_0)}}{\omega(k, K)}.
 \end{aligned}$$

A global-in-time bound on $(w_t, w_{tt}) = (\bar{w}, \bar{w}_t)$ in Y_{pl} will now follow: (i) we first take supremums in $t \in [t_0, \infty)$. Then (ii) we may choose $\min[k, K]$ sufficiently large (and thus up-scale $\omega(k, K)$) such that

$$\frac{C(R, U, t^*)}{\omega(k, K)} [1 - \exp(-\omega(k, K)(t - t_0))] < 1.$$

Absorbing this term in the LHS completes the proof of Lemma 5.3.

Remark 5.5. We note that, above, K appears on the RHS of the estimate and, thus, the global-in-time bounds for the (\bar{w}, \bar{w}_t) dynamics depend on the size of K , an auxiliary quantity which itself depends on the intrinsic parameters in the problem in the above argument.

5.5. Decomposition of flow dynamics. Consider data $y_0 \in Y_\rho$. At a time t_0 sufficiently large, we employ the reduction result Theorem 2.5 and utilize initial data $x_0 = (u(t_0), u_t(t_0); u^{t_0})$ coming from the full flow plate dynamics, and we denote the resulting semiflow from (2.10) by $T_t(x_0)$.

We consider the decomposition of the dynamics $T_t(x_0) = (u(t), u_t(t); u^t)$ into z and w components as in (5.1) and (5.2). We then consider the potential flow equation, with w given from (5.2) and initial time t_0 sufficiently large:

$$(5.22) \quad \begin{cases} (\partial_t + U \partial_x)^2 \phi^w = \Delta \phi^w & \text{in } \mathbb{R}_+^3, \\ \partial_\nu \phi^w = -[w_t + U w_x]_{ext} & \text{in } \mathbb{R}^2, \\ \phi^w(t_0) = 0; \quad \phi_t^w(t_0) = 0. \end{cases}$$

In this case, via the smoothing in Lemma 5.3, the Neumann data $[w_t + Uw_x]_{\text{ext}} \in H^1(\Omega)$ is sufficiently regular for Theorem 2.1 to apply. This yields a resulting potential flow $(\phi^w, \phi_t^w) \in C(0, \infty; H^1(\mathbb{R}_+^3) \times L_2(\mathbb{R}_+^3))$. Owing to the calculations in [14] and [7, p. 241] (due to the zero flow initial condition), we have that

$$(5.23) \quad r_{\Omega} \text{tr}[(\partial_t + U\partial_x)\phi^w] = -(\partial_t + U\partial_x)w - q^w(t).$$

With this identity, we can consider the nonautonomous semiflow $S_t^w(\cdot)$ on Y_ρ , in reference to the flow-plate problem:

$$(5.24) \quad \begin{cases} w_{tt} + kw_t + \Delta^2 w = p_0 + (\partial_t + U\partial_x)\phi^w|_{\Omega} - \mathcal{F}(w, u) + \beta z & \text{in } \Omega, \\ w = \partial_\nu w = 0 & \text{on } \Gamma, \\ w(t_0) = 0, \quad w_t(t_0) = 0, \\ (\partial_t + U\partial_x)^2 \phi^w = \Delta \phi^w & \text{in } \mathbb{R}_+^3 \times (0, T), \\ \phi^w(t_0) = 0; \quad \phi_t^w(t_0) = 0, \\ \partial_\nu \phi^w = -[(\partial_t + U\partial_x)w]_{\text{ext}} & \text{on } \mathbb{R}_{(x,y)}^2 \times (0, T). \end{cases}$$

Note that $\mathcal{F}(w, u)$ is linear in w , and we have the estimate

$$(5.25) \quad \|\mathcal{F}(w, u)\| \leq C\|\nabla u\|_{L^2(\Omega)}^2 \|w\|_2 \leq C\|u\|_2^2 \|w\|_2.$$

Owing to the well-posedness of the full flow-plate problem (1.1) [45] and the estimate in (5.25), one can assert well-posedness on Y of the (ϕ^w, w) —linear—dynamics by appealing to the results for the *linear* flow in [45, Theorem 2.1, pp. 3128–3128]. (See [15] as well.) We now consider the restriction to the flow ϕ from the full $S_t(y_0)$. By considering the difference $\phi - \phi^w$, we can construct the function ϕ^z , satisfying the corresponding potential flow equation with Neumann data

$$\partial_\nu \phi^z = -[z_t + Uz_x]_{\text{ext}}$$

on \mathbb{R}^2 . Combining this potential flow ϕ^z and the delayed z plate dynamics, as well as considering the difference between the flow-plate systems (1.1) and (5.24), we have constructed a *nonautonomous* flow $S_t^z(\cdot)$ on Y_ρ corresponding to the dynamics:

$$(5.26) \quad \begin{cases} z_{tt} + kz_t + \Delta^2 z = (\partial_t + U\partial_x)\phi^z|_{\Omega} - \mathcal{F}(z, u) - \beta z & \text{in } \Omega, \\ z = \partial_\nu z = 0 & \text{on } \Gamma, \\ z(t_0) = u(t_0), \quad z_t(t_0) = u_t(t_0), \\ (\partial_t + U\partial_x)^2 \phi^z = \Delta \phi^z & \text{in } \mathbb{R}_+^3 \times (0, T), \\ \phi^z(t_0) = \phi(t_0); \quad \phi_t^z(t_0) = \phi_t(t_0), \\ \partial_\nu \phi^z = -[(\partial_t + U\partial_x)z]_{\text{ext}} & \text{on } \mathbb{R}_{(x,y)}^2 \times (0, T). \end{cases}$$

Note that by considering k and β sufficiently large (as in Lemma 5.2), we have that $(z, z_t; z^t)$ decays exponentially in \mathbf{H} (uniformity with respect to R).

Recall that \mathcal{N} corresponds to the points (u, ϕ) which satisfy (1.22). Consider the stationary problem for some given function $u \in H_0^2(\Omega)$:

$$(5.27) \quad \begin{cases} \Delta^2 z + \beta z + \mathcal{F}(z, u) = Ur_{\Omega} \text{tr}[\partial_x \phi^z] & \mathbf{x} \in \Omega, \\ z = \partial_\nu z = 0 & \mathbf{x} \in \Gamma, \\ \Delta \phi^z - U^2 \partial_x^2 \phi^z = 0 & \mathbf{x} \in \mathbb{R}_+^3, \\ \partial_z \phi^z = U \partial_x z_{\text{ext}} & \mathbf{x} \in \partial \mathbb{R}_+^3. \end{cases}$$

Let

$$\mathcal{N}^z \equiv \left\{ (\widehat{z}, \widehat{\phi}^z) \in H_0^2(\Omega) \times W^1(\mathbb{R}_+^3) : (\widehat{z}, \widehat{\phi}^z) \text{ satisfy (5.27) variationally} \right\},$$

and let $\mathcal{N}^{z,*} \subset \mathcal{N}^z$ be the subset of points of the form $\{0; \widehat{\phi}^z\}$.

5.6. (z, ϕ^z) dynamics. In this section we will show that the desired convergence to equilibria result, as in Theorem 1.8, holds for the (z, ϕ^z) dynamics. This will be accomplished by critically using the uniform exponential decay of the (z, z_t) plate dynamics and running an approximation argument with smooth data as in [33]. The main result of this section is the following.

THEOREM 5.6. *Assume $k > k_e$ and $\beta > \beta_e$ with $y_0 \in Y_{\rho_0}$; then for any $t_n \rightarrow +\infty$ there is a subsequence t_{n_j} and a point $\widehat{y}^z = (\widehat{z}, 0; \widehat{\phi}^z, 0)$ with $(\widehat{z}, \widehat{\phi}^z) \in \mathcal{N}^{z,*}$ (i.e., $\widehat{z} = 0$) so that for any $\rho > 0$*

$$\lim_{j \rightarrow \infty} \|S_{t_{n_j}}^z(y_0) - \widehat{y}^z\|_{Y_\rho} = 0.$$

Proof of Theorem 5.6. The proof follows through several lemmas. Note that the linear part of (5.26) is equivalent to the system (1.13) (since both have static damping, $\beta > 0$). The decomposed nonlinear term $\mathcal{F}(z, u)$ enjoys the estimate,

$$(5.28) \quad \|\mathcal{F}(z, u)\| \leq C(R, b)\|z\|_2.$$

We now discuss the Hadamard continuity of the $S_t^z(\cdot)$ dynamics with respect to time. Finite time Hadamard continuity (shown as an intermediate step below) does not require assumptions on the damping. However, to obtain global-in-time Hadamard continuity we require sufficiently large damping parameters— k and β . The proof essentially follows the proof of Theorem 4.2, as originally given in [33].

LEMMA 5.7 (uniform-in-time Hadamard continuity for $S_t^z(\cdot)$). *Given initial datum $y_0^m = (u_0^m, u_1^m; \phi_0^m, \phi_1^m)$ and $y_0 = (u_0, u_1; \phi_0, \phi_1)$ with corresponding decompositions as in (5.26) and (5.24), we have that*

$$\sup_{t \geq 0} \|S_t^z(y_0^m) - S_t^z(y_0)\|_{Y_\rho} \rightarrow 0 \quad \text{as } y_0^m \rightarrow y_0 \text{ in } Y.$$

Proof of Lemma 5.7. We need only replace $f(z)$ by $\mathcal{F}(z, u)$ in the proof of Theorem 4.2, as given in [33]; we show this now.

Indeed, consider the differences $\mathbf{z} = z^m - z$ and $\Phi^z = \phi^{z^m} - \phi^z$ which satisfy the system

$$(5.29) \quad \begin{cases} \mathbf{z}_{tt} + k\mathbf{z}_t + \Delta^2 \mathbf{z} = (\partial_t + U\partial_x)\Phi^z|_\Omega - [\mathcal{F}(z^m, u^m) - \mathcal{F}(z, u)] - \beta\mathbf{z} & \text{in } \Omega, \\ z = \partial_\nu z = 0 & \text{on } \Gamma, \\ \mathbf{z}(t_0) = u^m(t_0) - u(t_0), \quad \mathbf{z}_t(t_0) = u_t^m(t_0) - u_t(t_0), \\ (\partial_t + U\partial_x)^2 \Phi^z = \Delta \Phi^z & \text{in } \mathbb{R}_+^3 \times (0, T), \\ \Phi^z(t_0) = \phi^m(t_0) - \phi(t_0); \quad \Phi_t^z(t_0) = \phi_t^m(t_0) - \phi_t(t_0), \\ \partial_\nu \phi^z = -[(\partial_t + U\partial_x)\mathbf{z}]_{\text{ext}} & \text{on } \mathbb{R}_{(x,y)}^2 \times (0, T). \end{cases}$$

We then utilize the velocity multipliers \mathbf{z}_t and Φ_t , respectively. The linear portion of the dynamics yields the standard energy identity for the differences:

$$(5.30) \quad \begin{aligned} E_z(t) + \int_0^t k\|\mathbf{z}_t\|^2 d\tau &= E_z(0) + 2U\langle \Phi(0), \mathbf{z}_x(0) \rangle - 2U\langle \Phi(t), \mathbf{z}_x(t) \rangle \\ &+ \int_0^t \langle \mathcal{F}(z^m, u^m) - \mathcal{F}(z, u), \mathbf{z}_t \rangle_\Omega d\tau, \end{aligned}$$

where we have utilized the notation

$$E_z = \frac{1}{2} (\|\Delta \mathbf{z}\|^2 + \|\mathbf{z}_t\|^2 + \beta \|\mathbf{z}\|^2 + \|\nabla \Phi\|^2 - U^2 \|\Phi_x\|^2 + \|\Phi_t\|^2).$$

By the bounds on E_{int} as in Lemma 1.2, as well as the global-in-time energy bounds (Lemma 1.5), one obtains

$$\begin{aligned} E_z(t) + \int_0^t k \|\mathbf{z}_t\|^2 d\tau \\ (5.31) \quad \leq C(U) E_z(0) + C \|\mathbf{z}_x(t)\|^2 + C \int_0^t |\langle \mathcal{F}(z^m, u^m) - \mathcal{F}(z, u), \mathbf{z}_t \rangle_\Omega| d\tau. \end{aligned}$$

Via compactness $\|\mathbf{z}_x\|^2 \leq \epsilon \|\Delta \mathbf{z}\|^2 + C(\epsilon) \|\mathbf{z}\|^2$, and hence

$$\begin{aligned} E_z(t) + \int_0^t k \|\mathbf{z}_t\|^2 d\tau \\ (5.32) \quad \leq C(U) E_z(0) + C \|\mathbf{z}(t)\|^2 + C \int_0^t |\langle \mathcal{F}(z^m, u^m) - \mathcal{F}(z, u), \mathbf{z}_t \rangle_\Omega| d\tau. \end{aligned}$$

Observing that

$$(5.33) \quad \|\mathbf{z}(t)\|^2 = \left\| \int_0^t \mathbf{z}_t(\tau) d\tau + \mathbf{z}(0) \right\|^2 \leq \int_0^t \|\mathbf{z}_t\|^2 d\tau + \|\mathbf{z}(0)\|^2 \leq \int_0^t E_z(\tau) d\tau + E_z(0),$$

we then have

$$\begin{aligned} (5.34) \quad E_z(t) + \int_0^t k \|\mathbf{z}_t\|^2 d\tau \leq C(U) E_z(0) + C(R) \int_0^t E_z(\tau) d\tau \\ + C \int_0^t |\langle \mathcal{F}(z^m, u^m) - \mathcal{F}(z, u), \mathbf{z}_t \rangle_\Omega| d\tau \end{aligned}$$

(where $C(R)$ denotes the dependence of the constant on the size of the ball containing the initial datum y^0 and y_m^0). To note *finite-in-time* Hadamard continuity, let $T > 0$ be fixed. From (5.30), we need only consider the nonlinear term

$$\int_0^T |\langle \mathcal{F}(z^m, u^m) - \mathcal{F}(z, u), \mathbf{z}_t \rangle| d\tau.$$

Note the identity

$$\begin{aligned} (b - \|\nabla u^m\|^2) \Delta z^m - (b - \|\nabla u\|^2) \Delta z \\ (5.35) \quad = b \Delta(z^m - z) - (\|\nabla u^m\|^2 - \|\nabla u\|^2) \Delta z^m - \|\nabla u\|^2 \Delta(z^m - z). \end{aligned}$$

And thus

$$(5.36) \quad \|\mathcal{F}(z^m, u^m) - \mathcal{F}(z, u)\| \leq C(b, R) [\|u^m - u\|_2 + \|z^m - z\|_2].$$

Implementing this bound, we see that

$$(5.37) \quad \left| \int_0^T \langle \mathcal{F}(z^m, u^m) - \mathcal{F}(z, u), \mathbf{z}_t \rangle d\tau \right| \leq C(b, R) \int_0^T [\|\mathbf{u}\|_2 + \|\mathbf{z}\|_2] \|\mathbf{z}_t\| d\tau$$

$$(5.38) \quad \leq C_\epsilon(b, R) \left[\int_0^T E_z(\tau) d\tau + \int_0^T \|u^m - u\|_2^2 d\tau \right]$$

We can now invoke the Hadamard continuity of $S_t(\cdot)$ for the u -dynamics on any finite $[0, T]$ (Theorem 4.1), as well as the initial conditions in (5.29), and thus

$$\|u^m(\tau) - u(\tau)\|_2^2 \leq C(T)E_z(0).$$

Gronwall's inequality applied in (5.34) with (5.38) then yields that

$$(5.39) \quad E_z(t) \leq C(b, U, R, T)E_z(0).$$

Hence, for any *fixed* T , we obtain a Hadamard continuity of the semiflow.

We now address this continuity on the infinite-time horizon. We return to (5.30):

$$(5.40) \quad \begin{aligned} & E_z(t) + \int_0^t k \|\mathbf{z}_t\|^2 d\tau \\ & \leq C \left\{ E_z(0) + \|\mathbf{z}(t)\|_1^2 + \left| \int_0^t \langle \mathcal{F}(z^m, u^m) - \mathcal{F}(z, u), \mathbf{z}_t \rangle d\tau \right| \right\} \\ & \leq C \left\{ E_z(0) + \epsilon E_z(t) + \|\mathbf{z}(t)\|^2 + \int_0^t \|\mathcal{F}(z^m, u^m) - \mathcal{F}(z, u)\|_0 \|\mathbf{u}_t\|_0 d\tau \right\}. \end{aligned}$$

Again using (5.33), we obtain

$$(5.41) \quad \begin{aligned} & E_z(t) + \int_0^t k \|\mathbf{z}_t\|^2 d\tau \\ & \leq C \left\{ E_z(0) + \int_0^t \|\mathbf{z}_t\|^2 d\tau + \int_0^t \|\mathcal{F}(z^m, u^m) - \mathcal{F}(z, u)\|_0 \|\mathbf{u}_t\|_0 d\tau \right\}. \end{aligned}$$

We invoke the bound (5.36) and note the global-in-time bounds on the (u^m, u_t^m) , (u, u_t) and (z^m, z_t^m) , and (z, z_t) dynamics in Y_{pl} :

$$\|\mathcal{F}(z^m, u^m) - \mathcal{F}(z, u)\| \leq C(b, R)[\|u^m - u\|_2 + \|z^m - z\|_2] \leq C(R),$$

and thus

$$E_z(t) + \int_0^t k \|\mathbf{z}_t\|^2 d\tau \leq C \left\{ E_z(0) + \int_0^t (\|\mathbf{z}_t\|^2 + C(R)\|\mathbf{z}_t\|) d\tau \right\}.$$

Scaling up β and k provides exponential decay of z_t and z_t^m to 0, and thus $\mathbf{z}_t \rightarrow 0$ exponentially—uniform with respect to $y_0^m, y_0 \in B_Y(R)$ (see above Lemma 5.2). Via this uniform exponential decay, for every $\epsilon > 0$ there exist a time $T^* > t^\#$ such that⁴

$$\int_{T^*}^{\infty} (\|\mathbf{z}_t\|^2 + C(R)\|\mathbf{z}_t\|) d\tau \leq \epsilon.$$

Thus, for any $t > T^*$ we may write

$$(5.42) \quad E_z(t) \leq C \left\{ E_z(0) + \int_0^{T^*} (\|\mathbf{z}_t\|^2 + C(R)\|\mathbf{z}_t\|) d\tau + \int_{T^*}^{\infty} (\|\mathbf{z}_t\|^2 + C(R)\|\mathbf{z}_t\|) d\tau \right\}.$$

⁴ T^* depends on the underlying parameters of the problem p_0, Ω, U , as well as the time of absorption for \mathcal{B} and the size of the support of the initial flow data ρ .

Utilizing (5.39), for any $\epsilon > 0$ we have

$$(5.43) \quad E_z(t) \leq C(R, T^*)(E_z(0) + E_z^{1/2}(0)) + \epsilon/2.$$

Taking y_m^0 sufficiently close to y^0 will yield that $C(R, T^*)(E_z(0) + E_z^{1/2}(0)) \leq \epsilon/2$. This concludes the proof of Lemma 5.7. \square

With uniform-in-time Hadamard continuity available, to continue with an *approximation by smooth data* approach for the (z, ϕ^z) dynamics, we must show that the desired convergence to equilibria holds for *smooth data* in the (z, ϕ^z) equation.

LEMMA 5.8. *Assume that*

$$x_0 = (u(t_0), u_t(t_0); u^{t_0}) \in (H^4 \cap H_0^2)(\Omega) \times H_0^2(\Omega) \times L_2(-t^*, 0; (H^4 \cap H_0^2)(\Omega)).$$

Then the dynamics $T_t^z(x_0)$ (such that $T_t^w(\mathbf{0}) + T_t^z(x_0) = T_t(x_0)$, and $u = w + z$) are uniformly bounded on $(H^4 \cap H_0^2)(\Omega) \times H_0^2(\Omega) \times L_2(-t^, 0; (H^4 \cap H_0^2)(\Omega))$. This is to say that*

$$\|(z(t), z_t(t); z^t)\|_{(H^4 \cap H_0^2)(\Omega) \times H_0^2(\Omega) \times L_2(-t^*, 0; (H^4 \cap H_0^2)(\Omega))} \leq C(R) \quad \forall \quad t \geq t_0.$$

Proof of Lemma 5.8. We do not repeat the argument in full, as it is directly follows the proof of Theorem 4.7, mutatis mutandis. Indeed, we consider smooth initial data (as above), differentiate (5.1), and perform the Lyapunov analysis in the proof of Theorem (4.7). The key fact (which greatly simplifies the proof in this case) is below:

$$(5.44) \quad \begin{aligned} \frac{d}{dt} \mathcal{F}(z, u) &= -2(\nabla u, \nabla u_t) \Delta z + [b - \|\nabla u\|^2] \Delta z_t \\ &= 2(\Delta u, u_t) \Delta z + [b - \|\nabla u\|^2] \Delta z_t, \end{aligned}$$

$$(5.45) \quad \left\| \frac{d}{dt} \mathcal{F}(z, u) \right\|_0 \leq C(b, R)[1 + \|z_t\|_2].$$

Remark 5.6. Note this type of estimate *fails* for von Karman dynamics because of the precise structure of the nonlocality; this is precisely the stage at which our *decomposition* approach will not obtain for the analogous von Karman problem.

At this point, we repeat the arguments in the proof of the propagation of regularity result Theorem 4.7. Specifically, the analysis of the term $B(t)$ (involving the decomposition involving terms P_1, Q_1) is not needed here. In particular, we take P_1 and Q_1 to be zero here. Indeed, utilizing (5.45), we have the estimate:

$$(5.46) \quad \left| \left\langle \frac{d}{dt} \mathcal{F}(z, u), z_{tt} \right\rangle \right| \leq \epsilon \|\Delta z_t\|^2 + C_\epsilon \|z_{tt}\|^2 + C_\epsilon(R).$$

At the junctures of (4.17), (4.28), and in the proof of Lemma 4.8, we implement (5.46) and up-scale k appropriately. \square

Remark 5.7. The subtlety needed in showing the propagation result in Theorem 4.7 (involving the decomposition into P_1 and Q_1) is needed to demonstrate that the *propagation of regularity for the original full flow-plate dynamics in (u, ϕ) can be obtained for any damping coefficient $k > 0$* . In proving Lemma 5.8 above, we already require k large to yield other supporting results, and thus scaling k up to simplify the argument for the reduced z dynamics (5.1) analogously is acceptable.

As a corollary, the global-in-time boundedness in Lemma 5.8 yield the desired convergence to equilibria result for smooth initial data to the full z problem (5.26).

COROLLARY 5.9. *Consider $y_0^M \in \mathcal{D}(\mathbb{T}) \cap Y$. Thus, for any sequence $t_n \rightarrow \infty$, there is a subsequence depending on M (also labeled t_n) and a point $(\hat{z}, \hat{\phi}) \in \mathcal{N}$ with $\hat{y} = (\hat{z}, 0); \hat{\phi}, 0$ (depending on the subsequence, and thus depending on M) such that*

$$\lim_{n \rightarrow \infty} \|S_{t_n}^z(y_0^M) - \hat{y}^z\|_{Y_\rho} = 0, \quad \forall \rho > 0.$$

Proof of Corollary 5.9. Due to the identical structure of the (z, ϕ^z) dynamics in (5.1) to those of (1.1) (with $f(u)$ replaced by $\mathcal{F}(z, u)$ and removing p_0), the proof of Theorem 4.3 can be repeated identically for the z dynamics (5.1), bearing in mind the bound in Lemma 5.8. (Recall that Theorem 4.3—which translates a global-in-time bound on plate dynamics into the desired strong convergence to equilibria for the full flow-plate dynamics—does not “care” about the structure of the specific plate equation, only the nature of the coupling. See Remark 4.4.) \square

We are now in a position to finish the proof of Theorem 5.6. Now, let $\epsilon > 0$ be given. We consider a sequence of smooth initial data $y_0^m \in \mathcal{D}(\mathbb{T})$ with the property that $y_0^m \rightarrow y_0$ in Y . We may choose M sufficiently large so that

$$\sup_{t > 0} \|S_t^z(y_0) - S_t^z(y_0^M)\|_{Y_\rho} < \epsilon/2.$$

Hence, there exists a $T(\epsilon, M)$ so that (perhaps passing to a subsequence) for all $t_n > T(\epsilon, M)$

$$d_{Y_\rho}(S_{t_n}^z(y_0^M), \hat{y}^z) < \frac{\epsilon}{2}.$$

Then

$$(5.47) \quad d_{Y_\rho}(S_{t_n}^z(y_0), \hat{y}^z) \leq d_{Y_\rho}(S_{t_n}^z(y_0), S_{t_n}^z(y_0^M)) + d_{Y_\rho}(S_{t_n}^z(y_0^M), \hat{y}^z) < \epsilon.$$

Owing to the decay of (u, u_t) in Lemma 5.2, we note that, in fact, $(\hat{z}, \hat{\phi}^z) \in \mathcal{N}^{z,*}$. This implies the desired convergence result and concludes the proof of Theorem 5.6. \square

5.7. (w, ϕ^w) dynamics. Now we consider the dynamics given by $S_t^w(\cdot)$ on Y_ρ and note the regularity of the dynamics (see Theorem 5.3).

THEOREM 5.10. *Assume $y_0 \in Y_{\rho_0}$. For $k > k_Q$, the dynamics $S_t^w(\cdot)$ have the property that for any sequence $t_n \rightarrow \infty$ there is a subsequence t_{n_j} and a point $\hat{y}^w \equiv (\hat{w}, 0; \hat{\phi}^w, 0)$ with $(\hat{w}, \hat{\phi}^w) \in \mathcal{N}$ so that*

$$\lim_{j \rightarrow \infty} \|S_{t_{n_j}}^w(y_0) - \hat{y}^w\|_{Y_\rho} = 0$$

for any $\rho > 0$. This implies the desired convergence result for $S_t^w(\cdot)$ to the set \mathcal{N} .

Proof of Theorem 5.10. Consider the plate dynamics $T_t^w(\cdot)$ corresponding to

$$(5.48) \quad \begin{cases} w_{tt} + (k+1)w_t + \Delta^2 w = p_0 + q^w(t) - Uw_x - [b - \|\nabla u\|^2]\Delta w + \beta z & \text{in } \Omega, \\ w = \partial_\nu w = 0 & \text{on } \Gamma, \\ w(t_0) = 0, \quad w_t(t_0) = 0, \quad w^{t_0} = 0. \end{cases}$$

Lemma 5.3 guarantees a bound in H^r for the $(w, w_t; w^t)$ dynamics, satisfying (4.1) in Theorem 4.3. The analysis of Theorem 4.3 obtains the equivalent of Theorem 4.9

for the w dynamics in (5.24). Here, we critically use the fact that the z terms above decay along any sequence of times $t_n \rightarrow \infty$, i.e., along any sequence of times $t_n \rightarrow \infty$ we have that $z(t_n) \rightarrow 0$ in $H^2(\Omega)$ and $z_t(t_n) \rightarrow 0$ in $L_2(\Omega)$. Considering the estimate

$$\|\beta z\|_0 \leq C\|z\|_2,$$

the identity

$$[b - \|\nabla u\|^2]\Delta w = [b - \|\nabla w\|^2]\Delta w - \|\nabla z\|^2\Delta w,$$

and the aforementioned exponential decay, we see that $S_{t_n}^w(y_0) \rightarrow (\widehat{w}, 0; \widehat{\phi}^w, 0)$ (with $(\widehat{w}, \widehat{\phi}^w) \in \mathcal{N}$) in Y_ρ for any $\rho > 0$. This concludes the proof of Theorem 5.10. \square

5.8. Final step in the proof of Theorem 1.8. By Theorems 5.6 and 5.10, for any sequence $t_n \rightarrow \infty$, there is a subsequence (also labeled) t_n and two points $\widehat{y}^z = (\widehat{z}, 0; \widehat{\phi}^z, 0)$ (with $(\widehat{z}, \widehat{\phi}^z) \in \mathcal{N}^{z,*}$) and $\widehat{y}^w = (\widehat{w}, 0; \widehat{\phi}^w, 0)$ (with $(\widehat{w}, \widehat{\phi}^w) \in \mathcal{N}$) so that $S_{t_n}^z(y_0) \rightarrow (\widehat{z}, 0; \widehat{\phi}^z, 0)$ and $S_{t_n}^w(y_0) \rightarrow (\widehat{w}, 0; \widehat{\phi}^w, 0)$ in Y_ρ for any $\rho > 0$. Note the fact that for $(\widehat{z}, \widehat{\phi}^z) \in \mathcal{N}^{z,*}$ and $(\widehat{w}, \widehat{\phi}^w) \in \mathcal{N}$, we have that

$$(5.49) \quad (\widehat{z}, \widehat{\phi}^z) + (\widehat{w}, \widehat{\phi}^w) \in \mathcal{N}$$

(due the linear structure of the stationary flow equation (1.22)). Thus

$$(5.50)$$

$$\begin{aligned} \|S_{t_n}(y_0) - (\widehat{y}^z + \widehat{y}^w)\|_{Y_\rho} &= \|(S_{t_n}^z(y_0) - \widehat{y}^z) + (S_{t_n}^w(y_0) - \widehat{y}^w)\|_{Y_\rho} \\ (5.51) \quad &\leq \|S_{t_n}^z(y_0) - \widehat{y}^z\|_{Y_\rho} + \|S_{t_n}^w(y_0) - \widehat{y}^w\|_{Y_\rho} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Since $(\widehat{z}, \widehat{\phi}^z) + (\widehat{w}, \widehat{\phi}^w) \in \mathcal{N}$, this implies the desired strong convergence result and completes the proof of Theorem 1.8.

6. Open problem: Eliminating damping size requirements. In experimental and numerical simulations with panels that have no imposed damping, flutter does not occur for subsonic flows (see section 1.1). The techniques herein (and in [12, 33, 40, 41]) certainly require *some* damping. However, for finite energy solutions we require a damping coefficient whose size depends on the *inherent parameters of the problem* in order to show convergence of full trajectories to the set of stationary states. We note that there is viscous damping in every physical plate system, but we do question whether the dependence of the results above on a minimal damping coefficient k_{\min} is necessary (especially since there is no size restriction for smooth initial data—Theorem 4.9—even for von Karman dynamics). It seems that: either (i) the requirement could be removed in order to yield an agreement with the experiment, or (ii) that there is some physical situation where the structural damping is sufficiently small that the subsonic panel may exhibit nonstationary end behavior. The authors could find no discussion of the latter situation in the engineering literature.

In our work above, the key issue necessitating large damping is the interaction of the terms Lu and $q(u^t)$ with the multiplier analysis in sections 5.3 and 5.4. The following intermediate assertion is presently clear: Taking $b = 0$, $p_0 \equiv 0$, $U \ll 1$ will not eliminate the need for a damping restriction to obtain strong stabilization of the full flow-plate dynamics due to the nonconservative, nondissipative contributions from q . An interesting question is this: under what restrictions (of the form $b \equiv 0$ and $p_0 \equiv 0$, or at least small in some sense) can we expect *stability* of the plate

dynamics—no nontrivial stationary states? From what is known physically, large damping, diminished loading, or small unperturbed flow velocities should lead to a simplification of the stationary set \mathcal{N} .

Acknowledgment. The authors sincerely thank professor E. H. Dowell of Duke University's Mechanical Engineering Department for his assistance, and for his insightful (and immensely helpful) commentary since 2010.

REFERENCES

- [1] J. M. BALL, *Global attractors for damped semilinear wave equations*, Discrete Contin. Dyn. Syst., 10 (2004), pp. 31–52.
- [2] H. M. BERGER, *A new approach to the analysis of large deflections of plates*, J. Appl. Mech., 22 (1955), pp. 465–472.
- [3] J. BERGH AND J. LÖFSTRÖM, *Interpolation Spaces. An Introduction*, Springer-Verlag, Berlin, 1976.
- [4] V. V. BOLOTIN, *Nonconservative Problems of Elastic Stability*, Pergamon Press, Oxford, 1963.
- [5] A. BOUTET DE MONVEL AND I. CHUESHOV, *The problem of interaction of von Karman plate with subsonic flow gas*, Math. Methods Appl. Sci., 22 (1999), pp. 801–810.
- [6] L. BOUTET DE MONVEL AND I. CHUESHOV, *Non-linear oscillations of a plate in a flow of gas*, C. R. Acad. Sci. Paris Ser. I, 322 (1996), pp. 1001–1006.
- [7] L. BOUTET DE MONVEL AND I. CHUESHOV, *Oscillation of von Karman's plate in a potential flow of gas*, Izv. RAN. Ser. Mat., 63 (1999), pp. 219–244.
- [8] L. BOUTET DE MONVEL, I. CHUESHOV, AND A. REZOUNENKO, *Long-time behaviour of strong solutions of retarded nonlinear PDEs*, Comm. Partial Differential Equations, 22 (1997), pp. 1453–1474.
- [9] I. CHUESHOV, *Remark on an Elastic Plate Interacting with a Gas in a Semi-Infinite Tube: Periodic Solutions*, Evol. Equ. Control Theory, accepted, <http://arxiv.org/abs/1601.04644>.
- [10] I. CHUESHOV, *Dynamics of Quasi-Stable Dissipative Systems*, Springer-Verlag, Berlin, 2015.
- [11] I. CHUESHOV, *Introduction to the Theory of Infinite-Dimensional Dissipative Systems*, Acta, Kharkov, 1999 (in Russian); also available online from <http://www.emis.de/monographs/Chueshov/>.
- [12] I. CHUESHOV, *Dynamics of von Karman plate in a potential flow of gas: Rigorous results and unsolved problems*, Proceedings of the 16th IMACS World Congress, Lausanne, Switzerland, 2000, pp. 1–6.
- [13] I. CHUESHOV AND I. LASIECKA, *Long-time Behavior of Second-order Evolutions with Nonlinear Damping*, Mem. Amer. Math. Soc. 195, AMS, Providence, RI, 2008.
- [14] I. CHUESHOV AND I. LASIECKA, *Von Karman Evolution Equations, Well-Posedness and Long-Time Behavior*, Springer Monogr. Math., Springer-Verlag, Berlin, 2010.
- [15] I. CHUESHOV AND I. LASIECKA, *Generation of a semigroup and hidden regularity in nonlinear subsonic flow-structure interactions with absorbing boundary conditions*, J. Abstr. Differ. Equ. Appl., 3 (2012), pp. 1–27.
- [16] I. CHUESHOV, I. LASIECKA, AND J. T. WEBSTER, *Attractors for delayed, non-rotational von Karman plates with applications to flow-structure interactions without any damping*, Comm. Partial Differential Equations, 39 (2014), pp. 1965–1997.
- [17] I. CHUESHOV, I. LASIECKA, AND J. T. WEBSTER, *Flow-plate interactions: Well-posedness and long-time behavior*, Discrete Contin. Dyn. Syst. Ser. S, 7 (2014), pp. 925–965.
- [18] I. CHUESHOV, I. LASIECKA, AND J. T. WEBSTER, *Evolution semigroups for supersonic flow-plate interactions*, J. Differential Equations, 254 (2013), pp. 1741–1773.
- [19] I. CHUESHOV AND A. REZOUNENKO, *Global attractors for a class of retarded quasilinear partial differential equations*, C. R. Acad. Sci. Paris Ser. 1, 321 (1995), pp. 607–612.
- [20] C. M. DA FERROS AND M. SLEMROD, *Asymptotic behavior of nonlinear contraction semigroups*, J. Funct. Anal., 13 (1973), pp. 97–106.
- [21] E. DOWELL, *Nonlinear Oscillations of a Fluttering Plate I*, AIAA J., 4 (1966), pp. 1267–1275.
- [22] E. DOWELL, *Nonlinear Oscillations of a Fluttering Plate II*, AIAA J., 5 (1967), pp. 1857–1862.
- [23] E. DOWELL, *Panel flutter—A review of the aeroelastic stability of plates and shells*, AIAA J., 8 (1970), pp. 385–399.
- [24] E. DOWELL, *A Modern Course in Aeroelasticity*, Kluwer Academic, Norwell, MA 2004.
- [25] E. H. DOWELL, *Some recent advances in nonlinear aeroelasticity: Fluid-structure interaction in the 21st century*, in Proceedings of the 51st AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference, AIAA, 2010.

- [26] P. FABRIE, C. GALUSINSKI, A. MIRANVILLE, AND S. ZELIK, *Uniform exponential attractors for a singularly perturbed damped wave equation*, Discrete Contin. Dyn. Syst., 10, pp. 211–238.
- [27] P. G. GEREDLI, I. LASIECKA, AND J. T. WEBSTER, *Smooth attractors of finite dimension for von Karman evolutions with nonlinear frictional damping localized in a boundary layer*, J. Differential Equations, 254 (2013), pp. 1193–1229.
- [28] P. G. GEREDLI AND J. T. WEBSTER, *Decay rates to equilibrium for nonlinear plate equations with geometrically constrained, degenerate dissipation*, Appl. Math. Optim., 68 (2013), pp. 361–390.
- [29] P. HOLMES AND J. MARSDEN, *Bifurcation to divergence and flutter in flow-induced oscillations: An infinite dimensional analysis*, Automatica J. IFAC, 14 (1978), pp. 367–384.
- [30] H. KOCH AND I. LASIECKA, *Hadamard well-posedness of weak solutions in nonlinear dynamic elasticity-full von Karman systems*, Progr. Nonlinear Differential Equations Appl., 50 (2002), pp. 197–216.
- [31] J. LAGNESE, *Boundary Stabilization of Thin Plates*, SIAM, Philadelphia, 1989.
- [32] I. LASIECKA AND R. TRIGGIANI, *Sharp regularity theory for second order hyperbolic equations of Neumann type*, Ann. Mat. Pura Appl. (4), 157 (1990), pp. 285–367.
- [33] I. LASIECKA AND J. T. WEBSTER, *Eliminating flutter for clamped von Karman plates immersed in subsonic flows*, Comm. Pure Appl. Math., 13 (2014), pp. 1935–1969; also available online from <http://arxiv.org/abs/1409.3308>.
- [34] J.-L. LIONS AND E. MAGENES, *Problèmes aux limites non homogènes et applications*, Vol. 1, Dunod, Paris, 1968.
- [35] S. MIYATAKE, *Mixed problem for hyperbolic equation of second order*, J. Math. Kyoto Univ., 13 (1973), pp. 435–487.
- [36] J. L. NOWINSKI AND H. OHNABE, *On certain inconsistencies in Berger equations for large deflections of plastic plates*, Int. J. Mech. Sci., 14 (1972), pp. 165–170.
- [37] S. NICAISE AND C. PIGNOTTI, *Exponential stability of abstract evolution equations with time delay*, J. Evol. Equ., 15 (2015), pp. 107–129.
- [38] S. NICAISE AND C. PIGNOTTI, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*, SIAM J. Control Optim., 45 (2006), pp. 1561–1585.
- [39] S. NICAISE, C. PIGNOTTI, AND J. VALEIN, *Exponential stability of the wave equation with boundary time-varying delay*, Discrete Contin. Dyn. Syst. Ser. S, 4 (2011), pp. 693–722.
- [40] I. RYZHKOVA, *Stabilization of a von Karman plate in the presence of thermal effects in a subsonic potential flow of gas*, J. Math. Anal. Appl., 294 (2004), pp. 462–481.
- [41] I. RYZHKOVA, *Dynamics of a thermoelastic von Karman plate in a subsonic gas flow*, Z. Ang. Math. Phys., 58 (2007), pp. 246–261.
- [42] M. SLEMROD, *Weak asymptotic decay via a “relaxed invariance principle” for a wave equation with nonlinear, nonmonotone damping*, Proc. Roy. Soc. Edinburgh Sect. A., 113 (1989), pp. 87–97.
- [43] D. TATARU, *On the regularity of boundary traces for the wave equation*, Ann. Sc. Norm. Super. Pisa. Cl. Sci., (5), 26 (1998), pp. 185–206.
- [44] C. P. VENDHAN, *A study of Berger equations applied to nonlinear vibrations of elastic plates*, Int. J. Mech. Sci., 17 (1975), pp. 461–468.
- [45] J. T. WEBSTER, *Weak and strong solutions of a nonlinear subsonic flow-structure interaction: Semigroup approach*, Nonlinear Anal., 74 (2011), pp. 3123–3136.