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# A note on the closed-form solution for the longest head run problem of Abraham de Moivre

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## Abstract

The problem of the longest head run was introduced and solved by Abraham de Moivre in the second edition of his book *Doctrine of Chances* (de Moivre, 1738). The closed-form solution as a finite sum involving binomial coefficients was provided in Uspensky (1937). Since then, the problem and its variations and extensions have found broad interest and diverse applications. Surprisingly, a very simple closed form can be obtained, which we present in this note.

*Keywords:* longest run problem; generating functions; history of probability

## 1 Introduction

In a series of  $n$  independent trials, an event  $E$  has a probability  $p$  of occurrence for each trial. If, in these trials, event  $E$  occurs at least  $r$  times without interruption, then we have a *run* of size  $r$ . What is the probability  $y_n$  of having a run of size  $r$  in  $n$  trials? This problem was formulated and solved by Abraham de Moivre in the second edition of his book *The Doctrine of Chances: or, A Method of Calculating the Probabilities of Events in Play* (de Moivre (1738), Problem LXXXVIII, p. 243). Although more than 280 years have passed since then, de Moivre's problem and its variations remain of great interest in probability and statistics; see for example Novak (2017) and references therein.

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de Moivre did not provide a proof, but demonstrated a method of finding  $y_n$ . Reviewing that method, one can see that he used the method of generating functions. He demonstrated the method with an example of ten trials having  $p = 1/2$ , in which the probability of a run of size 3 equals  $65/128$ .

The closed-form solution was given by Uspensky (1937) as a polynomial with binomial coefficients, arrived at by first obtaining a difference equation and then using a method of generating functions to solve it. Surprisingly, a simple closed form of  $y_n$  can be obtained as a corollary from the difference equation given by Uspensky. This closed-form solution seems to have never been reported in the literature. In this note, we present it along with Uspensky's original derivations.

## 2 Uspensky's solution

We present Uspensky's solution (Uspensky (1937), pages 77-79) while keeping his original notations. This solution demonstrates the power of the use of ordinary linear difference equations along with the generating functions. Recall that we denoted by  $y_n$  the probability of a run of size  $r$  in  $n$  independent trials. Let's consider  $n + 1$  trials with the corresponding probability  $y_{n+1}$ . A run of size  $r$  in  $n + 1$  trials can happen in two mutually-exclusive ways:

(W1) : the run is obtained in the first  $n$  trials or

(W2) : the run is obtained as of trial  $n + 1$ .

(W2) means that among the first  $n - r$  trials, there is no run of size  $r$ ; event  $E^C$  occurred at trial  $n - r + 1$ ; and event  $E$  occurred in the trials  $n - r + 2, \dots, n + 1$ . Combining (W1) and (W2), we obtain a linear difference equation of order  $r + 1$ ,

$$y_{n+1} = y_n + (1 - y_{n-r})qp^r, \quad (1)$$

with initial conditions

$$y_0 = y_1 = \dots = y_{r-1} = 0, \quad y_r = p^r,$$

where  $q = 1 - p$ . Substituting  $y_n = 1 - z_n$ , we then have

$$z_{n+1} - z_n + qp^r z_{n-r} = 0, \quad (2)$$

with the corresponding initial conditions,

$$z_0 = z_1 = \dots = z_{r-1} = 1; \quad z_r = 1 - p^r. \quad (3)$$

The solution of (2) was obtained by the method of generating functions. The generating function of the sequence  $z_0, z_1, z_2, \dots$  is the power function of  $t$  defined as

$$\varphi(t) = z_0 + z_1 t + z_2 t^2 + \dots + z_n t^n + \dots$$

Using (2) and (3), one hopes to find a definite function  $\varphi(t)$ ; then the coefficient of  $t^n$  will be precisely  $z_n$ . In our case, this outcome is possible and can be obtained by multiplying  $\varphi(t)$  by  $1 - t + qp^r t^{r+1}$ , applying (2) and substituting (3),

$$\varphi(t) = \frac{1 - p^r t^r}{1 - t + qp^r t^{r+1}}. \quad (4)$$

Then, the generating function  $\varphi(t)$  can be developed into a power series of  $t$  with a coefficient  $z_n$  of  $t^n$  as

$$z_n = \beta_{n,r} - p^r \beta_{n-r,r}$$

$$\beta_{n,r} = \sum_{l=0}^{\frac{n}{r+1}} (-1)^l \binom{n-lr}{l} (qp^r)^l. \quad (5)$$

Going back to de Moivre's original example, where  $n = 10, r = 3$  and  $p = 1/2$ , and using (5), we obtain  $z_n = 63/128$  and  $y_n = 65/128$ , which coincides with de Moivre's answer on page 245 of his *Doctrine of Chances* (de Moivre, 1738).

### 3 Closed-form solution for the case $r \geq n/2$

Surprisingly, a simple closed-form solution follows from formula (1) of Uspensky (1937), arrived at by substituting  $n = r, r+1, r+2, \dots$  and considering an appropriate range of  $r$ .

Substituting  $n = r$ , we obtain

$$y_{r+1} = y_r + (1 - y_0)p^r q = \begin{cases} p^r + p^r q & \text{if } r \geq 1 \\ 1 & \text{if } r = 0. \end{cases} \quad (6)$$

Substituting  $n = r + 1$  and using (6), we obtain

$$y_{r+2} = y_{r+1} + (1 - y_1)p^r q = \begin{cases} p^r + 2p^r q & \text{if } r \geq 2 \\ p + pq + (1 - p)pq = 1 - q^3 & \text{if } r = 1 \\ 1 & \text{if } r = 0. \end{cases} \quad (7)$$

Substituting  $n = r + 1$  and using (6) and (7), we obtain

$$y_{r+3} = y_{r+2} + (1 - y_2)p^r q = \begin{cases} p^r + 3p^r q & \text{if } r \geq 3 \\ p^2 + 2p^2 q + (1 - p^2)p^2 q & \text{if } r = 2 \\ 1 - q^3 + (1 - p - pq)pq = 1 - q^4 & \text{if } r = 1 \\ 1 & \text{if } r = 0. \end{cases} \quad (8)$$

By continuing in similar manner, we obtain the following corollary:

**Corollary 1.** *If  $n/2 \leq r \leq n$ , where  $r$  is an integer, then*

$$y_n = p^r + (n - r)p^r q. \quad (9)$$

**Remark 1.** *Corollary 1 can be obtained directly from the difference equation  $y_n = y_{n-1} + (1 - y_{n-1-r})qp^r$ , which has initial conditions  $y_0 = \dots = y_{r-1} = 0, y_r = p^r$ . If  $n-1-r \leq r-1$  (i.e.  $n/2 \leq r$ ) then  $y_{n-1-r} = 0$ , and Corollary 1 follows.*

## 4 Comments

There are a number of interesting problems discussed in the Uspensky (1937) book, many of which have roots in the classics of probability, their origins tracing back to founders of modern-probability such as Pascal, Fermat, Huygens, Bernoulli, de Moivre, Laplace, Markoff, Bernstein, and others. For example, Uspensky (1937) considered another problem of de Moivre's that was latter discussed and extended by Diaconis and Zabel (1991). A large collection of classic problems in probability with historical comments and original citations are nicely presented in the book by Gorroochurn (2012).

There are many follow-ups on and extensions of de Moivre's longest head run problem. An interesting recursive solution of the problem in the case of the fair coin was given

by Székely and Tusnády (1979). That problem was then extended to the Markov chain setting, where Uspensky's generation function (4) was generalized to the case of dependent observations (Novak, 1989). The problem has also found applications in numerous fields. Among which are reliability (Derman *et al.*, 1982), computational biology (Schbath, 2000), and finance where time dependent-sequences naturally occur (see Novak (2011) and references therein).

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