APPROVAL SHEET

Title of dissertation: Robust Value-at-Risk (VaR) Portfolio Selection Problem Under the Joint Ellipsoidal Uncertainty Set In the Presence of Transactions Costs

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ABSTRACT

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The robust portfolio selection problem considers the worst case of return under uncertainty sets of parameters, such as mean return and covariance of return. Goldfarb and Iyengar defined the return of assets by a factor model and provide the 'Separable' uncertainty sets for mean return and covariance of factor returns. However the sets are too conservative and construct a non-diversified portfolio. To overcome the drawbacks, Lu defined the 'Joint' ellipsoidal uncertainty set for mean return and covariance of factor returns.

In this research we derive a robust portfolio under the 'Joint' ellipsoidal uncertainty set. The problem is to maximize the expected return on a portfolio while restricting loss to exceed an investor's specific acceptable loss on a specified degree of confidence, called the robust Value-at-Risk (VaR) constraint problem. The constraint establishes an upper bound ϵ on the probability of losing a given percentage δ on the investment. The constraint under the uncertainty set is a non-convex function, so we use two reasonable estimations, which can be derived as semidefinite and

second order cone constraints, so that the problem with the estimations can be easily solved. The computational results on real market data show why the estimations are reasonable, and these results are compared to the problem under the 'Separable' uncertainty sets.

Additionally we extend the robust VaR constraint problem under the 'joint' uncertainty set to the problem in the presence of transactions costs, which are expenses incurred when buying or selling stocks. The idea is from the multi-period portfolio management problem and uses the same notations. The problem is to maximize transactions costs-adjusted return with the VaR constraint under the ellipsoidal uncertainty set. The real market simulation examines the impact of transactions costs consideration in the model.

Robust Value-at-Risk (VaR) Portfolio Selection Problem Under the Joint Ellipsoidal Uncertainty Set In the Presence of Transactions Costs

by

Hyekyung Park

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List of Abbreviations

CDF	Cumulative density function
CLT	Central limit theorem
CV	Coefficient of variation
CVaR	Conditional Value-at-Risk
IID	Independent and identically distributed
IOS	Investment opportunity set
KKT	Karush-Kuhn-Tucker
LP	Linear programming
MVO	Mean variance optimization
PD	Positive definite
PDF	Probability density function
PSD	Positive semidefinite
PVaR	Partitioned Value-at-Risk
QP	Quadratic programming
RCVaR	Relative robust conditional Value-at-Risk
RMRAR	Robust maximum risk-adjusted return
SD	Standard deviation
SDP	Semidefinite programming
SOCP	Second order cone programming
T-Bill	Treasury bill
VaR	Value-at-Risk
WVaR	Worst-case Value-at-Risk

Chapter 1

Introduction

1.1 Background and Motivation

One of the main problems in finance is how to allocate money into a given number of financial assets, e.g., options, stocks, and cash equivalents, in other words, how to construct a portfolio with available assets. Not all investors behave in the same manner. There are many factors that affect an investor's decisions, such as risk tolerance, available capital, and time horizon, so they have various different goals. Many researchers developed a variety of real market problems which depend on investors' goals and other factors. Risk and potential return are especially considered by researchers. Risk, or volatility, includes a possibility of losing investment, and typically variance or standard deviation are used as measures of the risk. The first mathematical portfolio model, called Mean-Variance Optimization (MVO), which maximizes the expected return of a portfolio while the variance of the return is less than a certain value, was introduced by Markowitz [30]. The classical optimal strategy is to assume parameters, such as mean return and covariance of return, to be known from historical data, but the solution is too sensitive to small perturbations in the parameters. As such, researchers in robust optimization further developed the Markowitz model by assuming the worst case of return or adopting a range of parameters instead of an estimating a single value and finding an optimal strategy.

There are a variety of robust portfolio models and many helpful overviews of robust portfolio optimization [5, 13, 25].

In this dissertation research, we are interested in the problem of maximizing the expected return on the portfolio while restricting the loss not to exceed an investor's specific acceptable loss on a given a specified degree of confidence, called the Value-at-Risk (VaR) constraint. The VaR measures the level of financial risk within an investment portfolio and defines the expected loss of the portfolio over a time period for a level of probability. The VaR constraint imposes a threshold on the expected loss of the portfolio.

When investors buy or sell stocks, transactions costs are incurred. For passive investors, who seldom rebalance their portfolios, transactions costs do not significantly affect their wealth. However, investors who rebalance their portfolio regularly, i.e. a few times a year, need to consider transactions costs because the profit they earn from the portfolio can be negated by the cost. Thus, we also extend the robust VaR constraint problem to the problem in the presence of transactions costs. The problem is to maximize the return of the portfolio adjusted for transactions costs under the VaR constraint of the return.

1.2 Outline of Thesis

Assume that there are n assets in a market. Let r be an n-dimensional column vector such that r_i is the return of asset i, $i = 1, \dots, n$. Our objective is to find an optimal weight that maximizes the return of a portfolio under a probability constraint, called the VaR constraint, which establishes an upper bound ϵ on the probability of losing a given percentage δ on an investment, i.e.,

$$\begin{aligned} \max_{\phi} \quad & r^T\phi \\ \text{s.t} \quad & \mathop{\mathbf{P}}_{r\sim\mathbf{P}}(r^T\phi \leq -\delta) \leq \epsilon, \\ & \phi \in \Phi, \end{aligned} \tag{1.2.1}$$

where ϕ is a weight vector so that the entry ϕ_i represents the fraction of total wealth invested in the i-th asset. In the portfolio model, Φ represents the set of acceptable weight vectors, which is typically defined as

$$\Phi = \{\phi \in \mathbb{R}^n : e^T\phi = 1,\ \phi_i \geq 0,\ i = 1, \dots, n\}. \tag{1.2.2}$$

Note that a nonnegative weight means that short selling is disallowed.

To solve the problem (1.2.1), we study various robust portfolio models that relate to this dissertation work in Section 1.4. In Section 1.4.1, we examine how the robust models have been developed in finance. Since the return of assets or the distribution of the return are unknown in practice, so we examine the distributional uncertainty models that can handle the VaR constraint in problem (1.2.1) in Section 1.4.2. The returns used in this dissertation are all defined using a factor model, the details of which are described in Section 1.4.3. Using the factor model, Goldfarb and Iyengar [16] provide the uncertainty sets for the mean return μ and the covariance matrix of factor return V, called separable uncertainty sets, which are described in

Section 1.4.4. Lu [29] develops a set called the joint ellipsoidal uncertainty set, which shares common properties with separable sets, but also overcomes some drawbacks of the separable sets. The definition of the joint ellipsoidal uncertainty set is given in Section 1.4.5 and is applied to solve the problem (1.2.1) in the next chapter. Section 1.4.6 introduces the multi-period robust portfolio selection problem, which considers the transactions cost in the model. As such, models in the previous sections only consider the return of the portfolio, whereas this model considers the actual return produced by deducting the cost. The notations used in this section are used to extend the model described in Chapter 3.

In Chapter 2, we discuss the real market data and performance measures of portfolio used to simulate all models in this dissertation. Several models examined in Section 1.4.1 are applied to the real market data so that the computational results can be compared to our model in the subsequent chapters.

In Chapter 3, we derive a model that solves the problem (1.2.1) under the joint ellipsoidal uncertainty set. The VaR constraint under this set is difficult to solve directly, so we use two estimations of the constraint and derive models using each estimation. In Chapter 4, the model introduced in Chapter 3 is extended to a model in the presence of transactions costs. Theses results are compared to the earlier model results that did not consider transactions costs. The models use the notation in [6], described in detail in Section 2.6. We have used MOSEK [1] in Matlab for simulations.

[1]Available at www.mosek.com [1]

1.3 Notation

The list of mathematical notations are defined in this section.

$\mathbb{R}$	the set of all real numbers
$\mathbb{R}^n$	the set of all n dimensional real column vectors
$\mathbb{R}^{m\times n}$	the set of all matrix with m rows and n columns
$\mathbb{S}^n$	the set of all symmetric matrix in $\mathbb{R}^{n\times n}$
$\mathbb{S}^n_+$	the set of all symmetric positive semidefinite matrix in $\mathbb{R}^{n\times n}$
$\mathbb{S}^n_{++}$	the set of all symmetric positive definite matrix in $\mathbb{R}^{n\times n}$
$\mathcal{L}^n$	n-dimensional second order cone, $\{z \in \mathbb{R}^n \;:\; z_1 \geq \sqrt{\sum_{i=2}^n z_i^2}\}$

Let $x \in \mathbb{R}^n$ be a vector.

$\|x\|_1$	1-norm of x, $\sum_{i=1}^n \|x_i\|$
$\|x\|_2$	2-norm of x, $\sqrt{\sum_{i=1}^n x_i^2}$
$\|x\|_\infty$	∞-norm of x, $\max\{\|x_i\| \;:\; i = 1,\dots,n\}$

Let S be a vector space over the real numbers and a set C in S is convex, i.e., for all $x, y \in C$, and for all $t \in [0,1]$, the point $(1-t)x + ty$ also belongs to C.

$\mathrm{ri}(C)$	the relative interior of C, $\{x \in C \;: \forall y \in C, \exists \lambda > 1 \;:\; \lambda x + (1-\lambda)y \in C\}$

Let $r \in \mathbb{R}^n$ be a random variable.

$\mathbf{E}(r)$	the mean(or expected value) of r
$\mathbf{Var}(r)$	the covariance of matrix of r
$\rho_{x,y}$	a correlation of two random variables $x \in \mathbb{R}$ and $y \in \mathbb{R}$
$\mathcal{N}(m, \sigma^2)$	a normal (or Gaussian) distribution with mean $m \in \mathbb{R}$, variance $\sigma^2 \in \mathbb{R}$
$\mathcal{N}(\mu, \Sigma)$	a multivariate normal distribution with mean $\mu \in \mathbb{R}^n$, covariance $\Sigma \in \mathbb{R}^{n\times n}$
$\mathscr{Z} \sim \mathcal{N}(0,1)$	the standard normal random variable
$\mathcal{F}_{\mathscr{Z}}(\cdot)$	the cumulative density function (CDF) of a random variable $\mathscr{Z}$

Let $M \in \mathbb{R}^{n\times n}$ be a square matrix.

$\sigma(M)$	the spectrum of M, set of its eigenvalues
$\sigma_{\max}(M)$	the maximum eigenvalue of matrix M
$M \succ 0$	M is a symmetric postive definite matrix, i.e., $M \in \mathbb{S}^n_{++}$
$M \succeq 0$	M is a symmetric positive semidefinite matrix, i.e., $M \in \mathbb{S}^n_+$
$\mathrm{rank}(M)$	the dimension of the vector space generated by its columns (or rows)
$\|M\|_2$	2-norm of M, $\max\limits_{\|x\|_2=1} \|Mx\|_2 = \sigma_{\max}(M)$

Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$ be matrices.

$\text{vec}(A)$	the vectorization of A, $[a_{1,1}, \dots, a_{m,1}, a_{1,2} \dots, a_{m,2}, \dots, a_{1,n}, \dots, a_{m,n}]^T \in \mathbb{R}^{mn}$
$\text{vech}(A)$	the half-vectorization of $A \in \mathbb{S}^n$, vectorization of the lower triangular part of A $[a_{1,1}, \dots, a_{n,1}, a_{2,2}, \dots, a_{n,2}, \dots, a_{n,n-1}, a_{n,n}]^T \in \mathbb{R}^{(n+1)n/2}$
$A \otimes B$	the Kronecker product, $\begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \in \mathbb{R}^{mp\times nq}$

Let $A, B \in \mathbb{R}^{m\times n.}$

$A \cdot B$	the Frobenius inner product, $\sum_{i=1}^m \sum_{j=1}^n A_{ij}B_{ij} = \text{vec}(A)^T\text{vec}(B)$
$A \circ B$	the Hadamard product, $(A \circ B)_{i,j} = A_{ij}B_{ij}$

1.4 Literature Review

1.4.1 Uncertainty Sets for Mean and Covariance of Return

The first mathematical portfolio selection model, called the mean-variance optimization (MVO) model, was introduced by Markowitz [30]. The model can be formed in two ways: one way is the minimum variance problem that minimizes the variance of the portfolio while it yields at least a given target return R, i.e.,

$$\min_{\phi}\{\phi^T\Sigma\phi \ : \ \mu^T\phi \geq R, \ \phi \in \Phi\}, \tag{1.4.1}$$

and the other way is the maximization return problem that maximizes the return on the portfolio while it satisfies a given upper limit σ^2 on the variance of the portfolio, i.e.,

$$\max_{\phi}\{\mu^T\phi \ : \ \phi^T\Sigma\phi \leq \sigma^2, \ \phi \in \Phi\}, \tag{1.4.2}$$

where $\mu \in \mathbb{R}^n$ denotes the mean return and $\Sigma \in \mathbb{S}_+^n$ denotes the covariance matrix of returns. Despite the model's theoretical success, practitioners do not use the basic model because parameters μ and Σ are unknown, and the solution is too sensitive to small perturbations in these parameters. To overcome this drawback, many researchers have proposed robust optimization models for the portfolio selection problems, specifically modeling them as linear programming problems. Consider a linear programming (LP) problem in the form

$$\begin{aligned} \min_{x} \quad & c^T x \\ \text{s.t} \quad & \tilde{A}x \leq b, \\ & l \leq x \leq u. \end{aligned} \tag{1.4.3}$$

Soyster [36] developed a stable model by introducing the notion of the uncertainty set of the parameter, which is unknown but bounded and symmetric, such as $\tilde{a}_{ij} \in [a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$, and solves the problem by assuming the worst case on the parameters. Then model (1.4.3) can be reformulated as

$$\begin{aligned} \max_{x,y} \quad & c^T x \\ \text{s.t} \quad & \sum_j a_{ij} x_j + \sum_{j \in J_i} \hat{a}_{ij} y_j \leq b_i, \forall i, \\ & - y_j \leq x_j \leq y_j, \forall j, \\ & l \leq x \leq u, \ y \geq 0, \end{aligned} \tag{1.4.4}$$

where J_i is the set of coefficients in row i that are subject to uncertainty.

Although the models reduce the sensitivity of the parameters, these models are too conservative, resulting in solutions with much worse expected return than the solution of the nominal problem. By using Soyster's scheme, Ben-Tal and Nemirovski [4] proposed a model that gives a reliable robust solution of the linear programming problem with a specified amount of uncertain data. They proposed the following robust problem:

$$\begin{aligned} \max_{x,y} \quad & c^T x \\ \text{s.t} \quad & \sum_j a_{ij}x_j + \sum_{j\in J_i} \hat{a}_{ij}y_{ij} + \Omega_i \sqrt{\sum_{j\in J_i} \hat{a}_{ij}^2 z_{ij}^2} \leq b_i, \forall i, \\ & -y_{ij} \leq x_j - z_{ij} \leq y_{ij}, \forall i, j \in J_i, \\ & l \leq x \leq u, \ y \geq 0. \end{aligned} \tag{1.4.5}$$

The authors have shown that the probability that the i-th constraint is violated is at most $\exp(-\Omega^2/2)$. This model applied to portfolio problem will be examined in Section 2.7.

Bertsimas and Sim [7] relaxed the worst case of the parameters by defining a new parameter called the budget of uncertainty. They defined J_i as the set of coefficients $a_{ij}, \ j \in J_i$ that are subject to parameter uncertainty, and the budget of uncertainty Γ_i takes values in the interval $[0, |J_i|]$. The parameter Γ_i adjusts the level of robustness and if the parameter is integer, then it is interpreted as the number of uncertainty parameters that take their worst case value, $a_{ij} - \hat{a}_{ij}$. They flexibly adjusted the level of conservatism of the robust solution in terms of probabilistic bounds on constraint violations. Then the LP problem in (1.4.3) can be derived as

$$\begin{aligned} \max_{x,y,p,z} \quad & c^T x \\ \text{s.t} \quad & \sum_j a_{ij} x_j + z_i \Gamma_i + \sum_{j \in J_i} p_{ij} \le b_i, \forall i \\ & z_i + p_{ij} \ge \hat{a}_{ij} y_j, \forall i, j \in J_i \\ & - y_j \le x_j \le y_j, \forall j \\ & l \le x \le u, \ p, y, z \ge 0. \end{aligned} \tag{1.4.6}$$

Many other researchers have proposed different uncertainty sets for μ and Σ in order to solve the Markowitz model (1.4.2) by assuming the worst case of the parameters, more details of which are in [5]. Lobo and Boyd [27] proposed box, ellipsoidal, and other uncertainty sets for μ and Σ. For example, the box uncertainty sets have the form $\mathcal{M} = \{\mu \in \mathbb{R}^n \mid \underline{\mu}_i \le \mu \le \overline{\mu}_i, i = 1, \ldots, n\}$, $\mathcal{S} = \{\Sigma \in \mathbb{S}^n_+ \mid \underline{\Sigma}_{ij} \le \Sigma \le \overline{\Sigma}_{ij}, i = 1, \ldots, n, j = 1, \ldots, n\}$. With these uncertainty sets, they solve robust variants of problems (1.4.1) and (1.4.2):

$$\min_{\phi} \left\{ \sup_{\Sigma \in \mathcal{S}} \phi^T \Sigma \phi : \inf_{\mu \in \mathcal{M}} \mu^T \phi \ge R, \phi \in \Phi \right\}. \tag{1.4.7}$$

Tütüncü and Koenig [37] focused on the case of box uncertainty sets for μ and

Σ, and show that problem (1.4.7) is equivalent to

$$\min_{\phi} \left\{ \inf_{\mu \in \mathcal{M}, \Sigma \in \mathcal{S}} \mu^T \phi - \theta \phi^T \Sigma \phi \ : \ \phi \in \Phi \right\}, \tag{1.4.8}$$

where $\theta \geq 0$ is an investor-specified risk factor.

Ben-Tal, El Ghaoui, and Nemirovski [2] solved the problem in (1.4.3) under three different uncertainty sets, namely Box, Ball-Box, and Budgeted uncertainty sets. The return r is defined as $r_i = \mu_i + \sigma_i \zeta_i$, where ζ is in the three different uncertainty sets $\mathcal{Z}$:

- Box uncertainty set: $\mathcal{Z} = \{\zeta \in \mathbb{R}^n : \|\zeta\|_\infty \leq 1\}$,
- Ball-Box uncertainty set: $\mathcal{Z} = \{\zeta \in \mathbb{R}^n : \|\zeta\|_\infty \leq 1, \|\zeta\|_2 \leq \Omega\}$,
- Budgeted uncertainty set: $\mathcal{Z} = \{\zeta \in \mathbb{R}^n : \|\zeta\|_\infty \leq 1, \|\zeta\|_1 \leq \gamma\}$.

The problem under the box uncertainty set can then be formulated as

$$\max_{\phi} \{(\mu - \sigma)^T \phi \ : \ \phi \in \Phi\}. \tag{1.4.9}$$

The Box uncertainty set guarantees 100% immunization against perturbations. The problem under the Ball-Box uncertainty set is

$$\max_{\phi, z, w} \left\{ \mu^T \phi - \|z\|_1 - \Omega \|w\|_2 \ : \ z + w = \sigma \phi, \ \phi \in \Phi \right\}, \tag{1.4.10}$$

where Ω is a safety parameter. The problem under the Budgeted uncertainty set is

$$\max_{\phi,z,w} \left\{ \mu^T \phi - \|z\|_1 - \gamma \|w\|_\infty \ : \ z + w = \sigma\phi, \ \phi \in \Phi \right\}. \tag{1.4.11}$$

Goldfarb and Iyengar [16] provided an uncertainty set by using the multi-factor model of return, which is discussed in the next section. They solve several different portfolio selection problems, specifically the mean-variance optimal portfolio selection problem, the maximum Sharpe ratio portfolio selection problem, and the value-at-risk portfolio selection problem, under uncertainty sets.

Lu [29] found a drawback of the set provided by Goldfarb and Iyengar, and introduced a new set, called a joint ellipsoidal uncertainty set, for the model parameters. They showed that it can be constructed as a confidence region, and applied it to the mean-variance optimal portfolio selection problem. This set will be used to solve the robust VaR problem in the subsequent chapters. Ling and Xu [12] proposed a robust portfolio selection model involving options under marginal and joint ellipsoidal uncertainty sets.

1.4.2 Distributional Uncertainty Models

Recall the portfolio selection problem in (1.2.1) that maximizes return and establishes an upper bound ϵ on the probability of losing a given percentage δ on the portfolio. In other words, δ is a risk level that investors can accept. However, the distribution is unknown, so in this section, we construct an uncertainty set of the distribution and take the worst case of the set, which makes the probability constraint conservative and easily solvable. First, we assume that the mean return μ and the covariance matrix of return Σ of the distribution of returns are known exactly, and that the probability can have any distribution $\mathbf{P}$ in $\mathcal{P}$, which is the set of all distributions that have mean μ and covariance Σ. Then, we take the supremum over the probabilities instead of having a random distribution. Value-at-Risk is one of the most widely accepted risk measures in the financial industry. El Ghaoui, Oks, and Oustry [14] examined the problem of worst-case Value-at-Risk over portfolios with risky returns belonging to a restricted class of probability distributions. The worst-case VaR with level ϵ is less than δ, that is, the authors derive an equivalent second order cone programming (SOCP). Let $\mathcal{P}$ be the set of probability distributions with mean μ and covariance matrix $\Sigma \succ 0$. Let $\epsilon \in (0, 1]$ and $\delta \in \mathbb{R}$ be given. The following propositions are equivalent. The worst-case VaR with level ϵ is less than δ, that is,

$$\sup_{\mathbf{P}\in\mathcal{P}} \mathop{\mathbf{P}}_{r\sim\mathbf{P}}\{\delta \leq -r^T\phi\} \leq \epsilon,$$

where the **sup** is taken with respect to all probability distributions in $\mathcal{P}$. This is equivalent to the SOCP

$$\kappa(\epsilon)\|\Sigma^{1/2}\phi\|_2 - \mu^T\phi \leq \delta,$$

where $\kappa(\epsilon) = \sqrt{(1-\epsilon)/\epsilon}$. By using the proposition, the problem in (1.2.1) with the worst-case VaR (WVaR) constraint can be written as

$$\begin{aligned}
\max_{\phi,t,y} \quad & \mu^T \phi, \\
\text{s.t} \quad & \kappa(\epsilon)t - \mu^T \phi \leq \delta, \\
& y = \Sigma^{1/2}\phi, \\
& \|y\|_2 \leq t, \\
& \phi \in \Phi.
\end{aligned} \tag{1.4.12}$$

The problem in (1.4.12) is easy to solve if the future return, r, is known. However, r cannot be known before it happens, so the problem is not deterministic. Pinar and Tütüncü [34] proposed an investment concept related to arbitrage using partial probabilistic information. Popescu [35] considered the problem that maximizes the worst case expected return on the portfolio over all possible distributions when the mean and covariance of the distribution are known. Huang, Fabozzi, and Fukushima [22] extended the worst-case VaR approach and formulated the corresponding problems as semi-definite programs to deal with uncertain exit times in robust portfolio selection. Natarajan, Pachamanova, and Sim [31] proposed an approximation method for minimizing the VaR of a portfolio based on robust optimization techniques, which results in the optimization of a modified VaR measure, Asymmetry-Robust VaR (ARVaR) that considers asymmetries in the distributions of returns and is coherent. Goh, Lim, Sim, and Zhang [15] presented an approach that minimizes a Partitioned VaR (PVaR) measure by separating asset return distributions into positive and negative half-spaces. As an alternative risk measure, Conditional Value-at-Risk (CVaR) is defined to be the mean of the tail distribution exceeding VaR. Natarajan, Pachamanova, and Sim [32] presented a model for the worst-case CVaR based on partial moment information when the exact distributions of random variables are unknown. Zhu and Fukushima [40] studied the minimization of worst-case CVaR under mixture distribution uncertainty, box uncertainty, and ellipsoidal uncertainty when only partial information on the underlying probability distribution is available. Huang, Zhu, Fabozzi, and Fukushima [23] considered a relative robust CVaR (RCVaR) portfolio selection problem where the underlying probability distribution of the portfolio return is only known to belong to a certain set.

1.4.3 Robust Factor Model

Goldfarb and Iyengar [16] used a factor model to define asset returns. They assumed that the market opens for trading at discrete instants in time and has n trading assets. The vector of n random asset returns over a single period $r \in \mathbb{R}^n$ is given by

$$r = \mu + V^T f + \epsilon, \tag{1.4.13}$$

where $\mu \in \mathbb{R}^n$ is the vector of mean returns, $f \sim \mathcal{N}(0, F) \in \mathbb{R}^m$ is the vector of m factor returns that drive the market, $V \in \mathbb{R}^{m\times n}$ is the factor loading matrix of the n assets, and $\epsilon \sim \mathcal{N}(0, D)$ is the vector of residual returns. Let $x \sim \mathcal{N}(\mu, \Sigma) \in \mathbb{R}^n$ be a multivariate normal random variable with mean vector μ and covariance matrix Σ. In addition, it is assumed that the vector of residual returns ϵ is independent of the vector of factor returns f, and that the covariance matrix $F \succ 0$ and the covariance matrix D are typically much smaller than the covariance matrix $V^T FV$. Thus, the vector of asset returns is normally distributed as $r \sim \mathcal{N}(\mu, V^T FV + D)$. Then the return on the portfolio $r^T\phi$ is normally distributed as

$$r^T\phi = \mu^T\phi + f^T V\phi + \epsilon^T\phi \sim \mathcal{N}(\phi^T\mu, \phi^T(V^T FV + D)\phi). \tag{1.4.14}$$

Goldfarb and Iyengar defined the uncertainty sets S_d, S_m, and S_v for the residual covariance matrix D, the mean return vector μ and the factor loadings matrix V. These are

$$\begin{aligned} S_d &= \{D : D = \mathrm{diag}(d), d_i \in [\underline{\mathrm{d}}_i, \bar{d}_i], i = 1, \ldots, n\}, \\ S_m &= \{\mu : \mu = \mu_0 + \xi, |\xi_i| \le \gamma_i,\ i = 1, \ldots, n\}, \\ S_v &= \{V : V = V_0 + W, \|W_i\|_g \le \rho_i,\ i = 1, \ldots, n\}, \end{aligned}$$

where W_i is the ith column of W and $\|w\|_g = \sqrt{w^T G w}$ for $G \succ 0$. Suppose the market data consists of asset returns $\{r^t : t = 1, \ldots, p\}$ and factor returns $\{f^t : t = 1, \ldots, p\}$ for p periods. Then the linear model (1.4.13) implies that

$$r_i^t = \mu_i + \sum_{j=1}^{m} V_{ji} f_j^t + \epsilon_i^t, \quad i = 1, ..., n, \quad t = 1, ..., p. \tag{1.4.15}$$

As in typical linear regression analysis, it is assumed that $\{\epsilon_i^t : i = 1, \ldots, n, t = 1, \ldots, p\}$ are all independent normal random variables and $\epsilon_i^t \sim \mathcal{N}(0, \sigma_i^2)$ for all $t = 1, \ldots, p$. Now, let $B = (f^1, f^2, \ldots, f^p) \in \mathbb{R}^{m \times p}$ be the matrix of the factor returns and $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$ be the column vector of all ones.

1.4.4 Separable Uncertainty Sets

Goldfarb and Iyengar [16] defined uncertainty sets for the mean return of assets μ and factor loading matrix V, called separable uncertainty sets, using a factor model. Define

$$y_i = (r_i^1, \ldots, r_i^p)^T,\ A = (e\ B^T),\ x_i = (\mu_i, V_{1i}, \ldots, V_{mi})^T,\ \epsilon_i = (\epsilon_i^1, \ldots, \epsilon_i^p)^T \quad (1.4.16)$$

for $i = 1, \ldots, n$. Then (1.4.15) can be rewritten as

$$y_i = Ax_i + \epsilon_i,\ \forall i = 1, \ldots, n. \quad (1.4.17)$$

Suppose that matrix A has full column rank, that is $\text{rank}(A) = m + 1$. Then the least squares estimate $\bar{x}_i$ of the true parameter x_i is given by $\bar{x}_i = (A^TA)^{-1}A^Ty_i$. The following proposition is used to define the uncertainty sets in [16].

Proposition 1.4.1. *Let s_i^2 be the unbiased estimate of σ_i^2 given by*

$$s_i^2 = \frac{\|y_i - A\bar{x}_i\|^2}{p - m - 1}$$

for $i = 1, 2, \ldots, n$. Then the random variables

$$\mathscr{Y}^i = \frac{1}{(m+1)s_i^2}(\bar{x}_i - x_i)^T A^T A(\bar{x}_i - x_i),\ i = 1, \ldots, n,$$

are distributed according to the F-distribution with $m+1$ degrees of freedom in the numerator and $p-m-1$ degrees of freedom in the denominator. Moreover, $\{\mathscr{Y}^i\}_{i=1}^n$ are independent.

Let $\tilde{w} \in (0,1)$ be given, $\mathscr{F}_J$ denotes cumulative distribution function (CDF) of the F-distribution with J degrees of freedom in the numerator and $p - m - 1$ degrees of freedom in the denominator, and $c_J(\tilde{w})$ be the $\tilde{w}$-critical value, i.e., the solution of the equation $\mathcal{F}_J(c_J(\tilde{w})) = \tilde{w}$. Then the probability $\mathscr{Y}^i \leq c_{m+1}(\tilde{w})$ is $\tilde{w}$. Define

$$S_i(\tilde{w}) = \{x_i : (\bar{x}_i - x_i)^T A^T A(\bar{x}_i - x_i) \leq (m+1)c_{m+1}(\tilde{w})s_i^2\}, \quad i = 1, \ldots, n.$$

Then the set $S_i(\tilde{w})$ is a $\tilde{w}$-confidence set, and it follows that

$$S(\tilde{w}) = S_1(\tilde{w}) \times S_2(\tilde{w}) \times \cdots \times S_n(\tilde{w}) \tag{1.4.18}$$

is also a $\tilde{w}^n$-confidence set for (μ, V) since the residual errors $\{\epsilon_i : i = 1, \ldots, n\}$ are assumed to be independent. Let $S_m(\tilde{w})$ denote the projection of $S(\tilde{w})$ along the vector μ; i.e.,

$$S_m(\tilde{w}) = \{\mu \ : \ \mu = \mu_0 + \nu, |\nu_i| \leq \gamma_i, i = 1, \ldots, n\}, \tag{1.4.19}$$

where

$$\mu_{0,i} = \bar{\mu}_i,\ \ \gamma_i = \sqrt{(m+1)(A^TA)^{-1}_{11}c_{m+1}(\tilde{w})s_i^2},\ \ i = 1, \dots, n.$$

Let $Q = [e_2, \dots, e_{m+1}]^T \in \mathbb{R}^{m\times(m+1)}$ be a projection matrix that projects x_i along V_i. Define the projection $S_v(\tilde{w})$ of $S(\tilde{w})$ along V as follows:

$$S_v(\tilde{w}) = \{V\ :\ V = V_0 + W, \|W_i\|_g \leq \rho_i,\ i = 1, \dots, n\}, \tag{1.4.20}$$

where

$$\begin{aligned} V_0 &= [\bar{V}_1 \cdots \bar{V}_n], \\ G &= (Q(A^TA)^{-1}Q^T)^{-1} = BB^T - \frac{1}{p}(Be)(Be)^T, \\ \rho_i &= \sqrt{(m+1)c_{m+1}(\tilde{w})s_i^2},\ i = 1, \dots, n. \end{aligned}$$

Then (1.4.18) implies that $S_m(w)$ and $S_v(\tilde{w})$ are an $\tilde{w}^n$-confidence set for the mean asset return vector μ and the covariance matrix of factor return V, respectively. Let $\phi \in \mathbb{R}^n$ be the allocation vector in typical set, defined in (1.2.2). Then the rate of return of the portfolio is

$$r_\phi = r^T\phi = \mu^T\phi + f^TV\phi + \epsilon^T\phi \sim \mathcal{N}(\phi^T\mu, \phi^T(V^TFV + D)\phi).$$

Goldfarb and Iyengar proposed four different portfolio selection problems, (1.4.21)–(1.4.24), under the separable uncertainty sets S_m and S_v, defined in (1.4.19) and (1.4.20). Then the results from the theory of multivariate linear regression are used to justify the uncertainty sets S_v and S_m. The robust analog of the Markowitz mean-variance optimization problem in (1.4.2) is given by

$$\begin{aligned} &\min_{\phi \in \Phi} \ \max_{V \in S_v} \mathbf{Var}[r_\phi] \\ &\text{s.t} \quad \min_{\mu \in S_m} \mathbf{E}[r_\phi] \geq \alpha. \end{aligned} \tag{1.4.21}$$

The *robust minimum variance portfolio selection problem* in (1.4.21) is to minimize the worst case variance of the portfolio subject to the constraint that the worst case expected return on the portfolio is at least α.

A closely related problem shown in (1.4.22), the *robust maximum return problem*, is the dual of (1.4.21). The problem is to maximize the worst case of expected return subject to a constraint on the worst case variance:

$$\begin{aligned} &\max_{\phi \in \Phi} \ \min_{\mu \in S_m} \ \mathbf{E}[r_\phi] \\ &\text{s.t} \quad \max_{V \in S_v} \mathbf{Var}[r_\phi] \leq \lambda. \end{aligned} \tag{1.4.22}$$

Another problem is also provided, which is called the *robust maximum Sharpe ratio problem*. The objective of the problem is to maximize the worst case of the Sharpe ratio of the expected return on the portfolio, i.e., the return in excess of the risk-free rate r_f to the standard deviation of the return:

$$\max_{\phi \in \Phi} \ \min_{(\mu, V) \in S_m \times S_v} \left\{ \frac{\mathbf{E}[r_\phi] - r_f}{\sqrt{\mathbf{Var}[r_\phi]}} \right\}. \tag{1.4.23}$$

Assume the optimal value of this problem is strictly positive, that is, there exists a portfolio whose worst-case return is strictly greater than r_f. Then there is at least one asset with worst-case return greater than r_f.

The *robust portfolio selection problem with VaR constraint* is also discussed under the separable uncertainty set:

$$\begin{aligned} \max_{\phi \in \Phi} \quad & \min_{\mu \in S_m} \mathbf{E}[r_\phi] \\ \text{s.t} \quad & \max_{(\mu, V) \in S_m \times S_v} \mathbf{P}(r_\phi \leq \alpha) \leq \beta. \end{aligned} \tag{1.4.24}$$

They assumed that the probability $\mathbf{P}$ is normally distributed. The last portfolio selection problem in (1.4.24) is the main problem we focus in this dissertation. The models Goldfarb and Iyengar provided under the separable uncertainty sets are robust but too conservative. Moreover they constructed highly non-diversified portfolios in computational results. To overcome these drawbacks of the sets, Lu defined a new uncertainty set discussed in the next section.

1.4.5 Joint Ellipsoidal Uncertainty Set

Lu [28, 29] provided a joint uncertainty set to overcome the drawback of the separable uncertainty sets, provided by Goldfarb and Iyengar. Let $S_m(\tilde{w})$ and $S_v(\tilde{w})$

denote the projection of $S(\tilde{w})$ along μ and V. Then

$$\mathbf{P}(\mu \in S_m(\tilde{w})) \geq \mathbf{P}((\mu, V) \in S(\tilde{w})) = \tilde{w}^n, \text{ and}$$

$$\mathbf{P}(V \in S_v(\tilde{w})) \geq \mathbf{P}((\mu, V) \in S(\tilde{w})) = \tilde{w}^n.$$

Hence, $S_m(\tilde{w})$ and $S_v(\tilde{w})$ have at least $\tilde{w}^n$-confidence levels, but their actual confidence levels are unknown and can be much higher than $\tilde{w}^n$. Since $S(\tilde{w}) \subseteq S_m(\tilde{w}) \times S_v(\tilde{w})$,

$$\mathbf{P}((\mu, V) \in S_m(\tilde{w}) \times S_v(\tilde{w})) \geq \mathbf{P}((\mu, V) \in S(\tilde{w})) = \tilde{w}^n.$$

Thus $S_m(\tilde{w}) \times S_v(\tilde{w})$, as a joint uncertainty set of (μ, V), has at least $\tilde{w}^n$-confidence level. But, its actual confidence level is unknown and can be much higher than $\tilde{w}^n$. Thus, the robust portfolio selection models based on such uncertainty sets $S_m(\tilde{w})$ and $S_v(\tilde{w})$ can be too conservative. As such, Lu [29] introduced a 'joint' ellipsoidal uncertainty set for the Goldfarb and Iyengar factor model to overcome the drawbacks.

$$S_{m,v}(w) = \left\{ (\mu, V) \in \mathbb{R}^n \times \mathbb{R}^{m \times n} : \sum_{i=1}^{n} \frac{(x_i - \bar{x}_i)^T (A^T A)(x_i - \bar{x}_i)}{s_i^2 (m+1)} \leq c(w) \right\} \tag{1.4.25}$$

for some $c(w)$, where $x_i = (\mu_i, V_{1i}, V_{2i}, \ldots, V_{mi})^T$ for $i = 1, \ldots, n$. We know that $\{\mathscr{Y}^i\}_{i=1}^n$ are i.i.d.. Using this fact and the central limit theorem in [24], we can conclude that the distribution of the random variable

$$\mathscr{L}_n = \frac{\sum_{i=1}^n \mathscr{Y}^i - n\mu_F}{\sigma_F \sqrt{n}}$$

converges towards the standard normal distribution $\mathcal{N}(0,1)$ as $n \to \infty$. For a relatively large n, $\mathbf{P}(\mathscr{L}_n \le \tilde{c}(w)) \approx w$, hence, $\mathbf{P}(\sum_{i=1}^{n} \mathscr{Y}^i \le c(w)) \approx w$, where $c(w) = \tilde{c}(w)\sigma_F\sqrt{n} + n\mu_F$.

Proposition 1.4.2 (Proposition 3.1 of [29])**.** *$S_{m,v}$ is an w-confidence uncertainty set of (μ, V) for some $\tilde{c}(w)$ if and only if $P(\sum_{i=1}^{n} \mathscr{Y}^i \le c(w)) = w$. That is, $c(w)$ is the w-critical value of $\sum_{i=1}^{n} \mathscr{Y}^i$. Moreover, $c(w) = \tilde{c}(w)\sigma_F\sqrt{n} + n\mu_F$, where $\tilde{c}(w)$ is the w-critical value for a standard normal distribution,*

$$\mu_F = \frac{p-m-1}{p-m-3}, \ \textit{and}\ \sigma_F = \sqrt{\frac{2(p-m-1)^2(p-2)}{(m+1)(p-m-3)^2(p-m-5)}}.$$

Lu solved the robust maximum return problem in (1.4.22) under the joint uncertainty set (1.4.25):

$$\begin{aligned} \max_{\phi\in\Phi} \ & \min_{(\mu,V)\in S_{m,v}} \ \mathbf{E}[r_\phi] \\ \text{s.t} \ & \max_{(\mu,V)\in S_{m,v}} \mathbf{Var}[r_\phi] \le \lambda. \end{aligned} \tag{1.4.26}$$

However, the uncertainty set of (μ, V) used in the problem is essentially $S_m \times S_v$, where $S_m = \{(\mu, V) \in S_{m,v}$ for some V$\}$ and $S_v = \{(\mu, V) \in S_{m,v}$ for some $\mu\}$. Then problem (1.4.26) is under the separable uncertainty sets. Since $S_{m,v}$ has a w-confidence level,

$$\mathbf{P}((\mu, V) \in S_m \times S_v) \ge \mathbf{P}((\mu, V) \in S_{m,v}) = w^n.$$

So, $S_m \times S_v$ has at least a w-confidence level, but its actual confidence level is unknown and can be much higher than w^n. To resolve the issue, Lu derived the robust maximum risk-adjusted return problem (RMRAR) under the uncertainty set $S_{m,v}$:

$$\max_{\phi \in \Phi} \min_{(\mu, V) \in S_{m,v}} \mathbf{E}[r_\phi] - \theta \mathbf{Var}[r_\phi], \tag{1.4.27}$$

where $\theta \geq 0$ represents the risk-aversion parameter. The RMRAR problem will be extended to the problem in the presence of transactions costs in Chapter 4.

1.4.6 Multi-period Robust Models

Ben-Tal, Margalit, and Nemirovski [3] formulated the L-stage portfolio selection problem that maximizes the investor's wealth on L-stage. Glpinal and Rustem [19] extended the multi-period mean-variance portfolio optimization to the robust worst-case design with multiple rival return and risk scenarios. Bertsimas and Pachamanova [6] suggested robust optimization formulations of the multi-period. Bertsimas and Pachamanova proposed the following multi-period portfolio management problem [6]. There are n risky assets, one riskless asset (asset 0), and N trading periods. At time period N, an investor collects final wealth W_N. The objective is to manage the portfolio of assets to maximize expected final wealth. The following notation is used:

- x_t^n is the investor's dollar holdings at the beginning of time period.

- u_t^n, v_t^n are the amount for the investor to sell and buy the stock n at time t.

- $c_{sell} u_t^n$, and $c_{buy} v_t^n$ are the transactions cost.

- $\tilde{r}_t^n$ is the uncertainty returns.

Then the investor's dollar holdings are given by equations,

$$
\begin{aligned}
x_t^n &= (1+\tilde{r}_{t-1}^n)(x_{t-1}^n - u_{t-1}^n + v_{t-1}^n),\ t=1,\ldots,T,\ n=1,\ldots,N, \\
x_t^0 &= (1+r_{t-1}^0)\left(x_{t-1}^0 + \sum_{n=1}^{N}(1-c_{sell})u_{t-1}^n - \sum_{n=1}^{N}(1+c_{buy})v_{t-1}^n\right),\ t=1,\ldots,T.
\end{aligned}
$$

The maximum final wealth problem is the following:

$$
\begin{aligned}
\max \quad & \sum_{n=0}^{N} x_T^n \\
\text{s.t} \quad & x_t^n = (1+\tilde{r}_{t-1}^n)(x_{t-1}^n - u_{t-1}^n + v_{t-1}^n), t=1,\ldots,T,\ n=1,\ldots,N, \qquad (1.4.28) \\
& x_t^0 = (1+r_{t-1}^0)\left(x_{t-1}^0 + \sum_{n=1}^{N}(1-c_{sell})u_{t-1}^n - \sum_{n=1}^{N}(1+c_{buy})v_{t-1}^n\right),\ t=1,\ldots,T, \\
& x_t^n \geq 0,\ t=1,\ldots,T,\ n=0,\ldots,N, \\
& u_t^n \geq 0,\ v_t^n \geq 0,\ t=1,\ldots,T,\ n=1,\ldots,N.
\end{aligned}
$$

The maximum final wealth problem is considered with these notations in Chapter 4.

Chapter 2

Real Market Data Simulation

In this chapter, we discuss the real market data and performance measures that will be used on all models in this dissertation, and simulations of several LP models using the data and measures are provided.

In Section 2.1, the real market data are provided and performance measures are defined. In Section 2.2, we investigate the appropriate threshold on return δ with a certain confidence level $(1-\epsilon)$ on the VaR constraint using the market data. In Section 2.3, we apply a few robust LP models introduced in the previous chapter to solve the robust maximum return problem, then provide computational results on real market data. The results will be used to draw comparisons with the model we propose in the next chapter.

2.1 Real Market Data and Performance Measures

All models use the factor model to define the daily return of stocks as in Section 1.4.3, and are simulated on the same real market data. Note that the portfolio return is assumed to be normally distributed as

$$r^T\phi \sim \mathcal{N}(\phi^T\mu, \phi^T(V^TFV + D)\phi).$$

The assets that are chosen for investment are those which are currently ranked at the top of each sector by Fortune 500[1] in 2016. In total, there are $n = 36$ assets in this set (see Table A.1.3). The set of factors are 6 major market indices (see Table A.1.2). We use four years of historical daily returns[2] from January 1, 2013 through December 31, 2016. Rebalancing of the portfolio is done every four months. Each year has 252 trading days, hence the four years data contains 12 periods of length $p = 84$ trading days. For each period, the portfolio is rebalanced using the previous period data. The first investment starts on May 1, 2013, so there are 11 investment periods. For each investment period t, the factor covariance matrix F is computed on the factor returns of the previous trading period, and the upper bound of the variance $\bar{d}_i$ of the residual return is computed to be $\bar{d}_i = s_i^2$, where s_i^2 is given in Proposition 1.4.1.

Performance of each model on the same market data is measured by overall wealth growth rate, diversification number, transactions costs, and the Sharpe ratio. The *wealth growth rate (wgr)* of period t, $t = 1, \dots, 11$, is defined as

$$\mathrm{wgr}_t = \left[\prod_{p_t \le k \le p_{t+1}} (e + r_k) \right]^T \phi_t - 1,$$

where r_k is a daily return vector of k-th day, and ϕ_t is the weight vector of the portfolio on period t. Then the *overall wealth growth rate (owgr)* on period t is

[1] http://beta.fortune.com/fortune500/

[2] The daily stock return is calculated by $(P_1 - P_0)/P_0$, where P_0 and P_1 are adjusted closing prices, which reflect all of the dividends and splits, on the previous day and the current day. The adjusted closing prices are obtained from Yahoo! Finance.

defined as

$$\text{owgr}_t = \prod_{1 \le k \le t} (1 + \text{wgr}_k) - 1.$$

The *diversification number* of a portfolio is defined as the number of its components that are above 1%. The *transaction cost* is defined as expenses incurred when buying or selling a stock, i.e., broker's commissions. In the real market, most brokerage companies charge a flat trade commission fee, but this can be converted to rate. For instance, if an investor purchases 40 units of a stock priced at $25 ($1,000 investment) with commission of $5, then the transaction cost rate is 0.5%. Another investor purchases 400 units of the stock at the same price ($10,000 investment) with the same commission amount, then the cost rate is 0.05%. As total investment increases, the cost rate can be adjusted to a smaller rate. If an investor has $\$10,000$ and buys more than one stock, for example 20 different stocks, with the same commission, then the transaction cost rate is $20 \times \$5/\$10,000 = 1\%$. As the investor's portfolio increases in diversification, the cost rate will increase.

Another performance measure of the portfolio is *Sharpe ratio*, a measure for calculating risk-adjusted return, developed by William F. Sharpe. The Sharpe ratio is defined as excess return divided by risk.

$$\textit{Sharpe ratio} = \frac{\mathbf{E}(r_p) - r_f}{\sqrt{\mathbf{Var}(r_p)}},$$

where excess return is the expected return on the investment r_p less risk-free return r_f, and the risk is the standard deviation of the portfolio returns. The interest rate on the U.S. Treasury bill is commonly used as a risk-free rate. The Sharpe ratio is negative when the expected portfolio return is lower than the risk-free rate. Let the 4-month risk free rate during period t be calculated by

$$r_f^t = (1 + r_{f_1}^t/12)(1 + r_{f_2}^t/12)(1 + r_{f_3}^t/12)(1 + r_{f_4}^t/12) - 1$$

where $r_{f_k}^t$ is the four-week U.S. Treasury bill rate per year of the k-th month on period t. The 4-week U.S. Treasury bill rates per year for the 4-year investment period are given in Table A.1.1[3]. From the equation and 4-week U.S. Treasury bill rates, the 4-month risk-free rate is given as in Table 2.1.1. The factor model is

Table 2.1.1: 4-month U.S. Treasury bill rate per year

2013	2014	2015	2016
0.0183	0.0092	0.0058	0.0692
0.0075	0.0083	0.0042	0.0658
0.0150	0.0058	0.0167	0.0875

applied on daily return, not 4-month return. In order to get the expected 4-month return and standard deviation of 4-month return of the portfolio, we multiply the expected daily return and variance of daily return by 84 trading days. Thus the Sharpe ratio of a 4-month portfolio is defined as

$$\textit{Sharpe ratio} = \frac{84 \times \mu^T \phi - r_f}{\sqrt{84 \times \phi^T (V^T F V + D)\phi}}.$$

The mean stock return μ and covariance of factor return V are not single values,

[3]The 4-week U.S. T- bill rate is the 4-week bank discount rate of the first trading day of each month quoted from https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=billrates

they are in $S_{m,v}$. We want to calculate the worst case Sharpe ratio under the set, but the maximum Sharpe ratio problem under the set is not easy to calculate. So we substitute $\bar{\mu}$ and $\bar{V}$, defined as the least squares estimate $\bar{x}_i$ of problem (1.4.17) for μ and V in the Sharpe ratio.

2.2 Minimum Value-at-Risk

Value-at-Risk (VaR) is one of the risk measure, defined as the possible loss on a portfolio with $(1-\epsilon)$ confidence level during the next holding period. Typical values for the threshold on probability ϵ are 1%, 2.5%, and 5% in [26]. An investor wants to minimize the risk of losing 1% of investment on a 99% confidence level. But there might not exist such a portfolio that satisfies this specification. In this section, we demonstrate the smallest possible threshold on return under different confidence levels to help investors select the threshold on return. For a given weight vector $\phi \in \Phi$, where $\Phi = \{\phi \in \mathbb{R}^n : e^T\phi = 1,\ \phi_i \geq 0,\ i = 1, \ldots, n\}$, the return of the portfolio is defined as in (2.3.2). The VaR can be found by solving

$$\begin{aligned} &\min\ \delta && (2.2.1) \\ &\text{s.t}\ \mathop{\mathbf{P}}_{r\sim\mathbf{P}}(\delta \leq -r^T\phi) \leq \epsilon. \end{aligned}$$

Assume that the returns of assets are defined as in Section 1.4.3 using factor model. So the return is normally distributed with mean μ and covariance $V^TFV + D$. Then the probability constraint in (2.2.1) is changed to

$$
\begin{aligned}
\mathbf{P}(r^T\phi \le -\delta) \le \epsilon \Leftrightarrow\ & \mathbf{P}(\mu^T\phi + \mathscr{Z}\sqrt{\phi^T(V^TFV+D)\phi} \le -\delta) \le \epsilon,\\
\Leftrightarrow\ & \mathbf{P}\left(\mathscr{Z} \le \frac{-\mu^T\phi - \delta}{\sqrt{\phi^T V^T FV + D\phi}}\right) \le \epsilon,\\
\Leftrightarrow\ & \frac{-\mu^T\phi - \delta}{\sqrt{\phi^T(V^TFV+D)\phi}} \le \mathcal{F}^{-1}(\epsilon),\\
\Leftrightarrow\ & -\mathcal{F}^{-1}(\epsilon)\sqrt{\phi^T(V^TFV+D)\phi} - \mu^T\phi \le \delta,
\end{aligned}
$$

where $\mathscr{Z} \sim \mathcal{N}(0,1)$ is the standard normal random variable and $\mathcal{F}^{-1}(\cdot)$ is its cumulative density function(CDF). The optimal VaR on problem (2.2.1) can be expressed as function of ϕ,

$$\mathrm{VaR}(\phi) = \epsilon_0\sqrt{\phi^T(V^TFV+D)\phi} - \mu^T\phi,$$

where $\epsilon_0 = -\mathcal{F}^{-1}(\epsilon)$. The smallest Worst-case Value-at-Risk (WVaR) under the joint ellipsoidal uncertainty set, defined in 1.4.5, over the possible weight vector ϕ in typical set Φ is found by solving the following problem

$$\min_{\phi\in\Phi}\ \max_{(\mu,V)\in S_{m,v}} \mathrm{VaR}(\phi).$$

However, the objective function under the joint uncertainty set is hard to solve, so separable uncertainty set, defined in 1.4.4 is considered instead of the joint set. Note that the joint uncertainty set is smaller than the separable uncertainty set, so the minimum of WVaR under the joint set is smaller than the separable sets.

$$\min_{\phi\in\Phi} \max_{(\mu,V)\in S_{m,v}} \mathrm{VaR}(\phi) \;\le\; \min_{\phi\in\Phi} \max_{(\mu,V)\in S_m\times S_v} \mathrm{VaR}(\phi),$$

where the set S_m and S_v are defined as in (1.4.19) and (1.4.20). Then the problem (2.2.1) can be derived as second order cone programming (SOCP):

$$\begin{aligned}
&\min_{\phi,\alpha,\beta,t} \; (\epsilon_0 t - \min_{\mu\in S_m} \mu^T\phi) \\
&\text{s.t} \; \left\| \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\| \le t, \\
&\quad \|\bar{D}^{1/2}\phi\| \le \beta, \\
&\quad \max_{V\in S_v} \|F^{1/2}V\phi\| \le \alpha, \\
&\quad \phi \in \Phi.
\end{aligned} \tag{2.2.2}$$

The SOCP under the separable uncertainty sets was solved by Goldfarb and Iyengar [16].

Table 2.2.1 shows the smallest WVaR on the portfolio with different confidence levels on probability. The confidence levels are $99\%, 97.5\%, 95\%, 90\%$ when $\epsilon = 0.01, 0.025, 0.05, 0.1$. Let the confidence level of the joint uncertainty set w be 97.5%. As ϵ gets larger, the VaR becomes smaller. Assume a 99% confidence level, i.e., $\epsilon = 0.01$. Then the robust VaR constraint problem does not have a solution if the investor chooses a return threshold δ below 1%. However, since ϵ is 0.01, the problem can have a solution for all periods. Thus, an adequate pair of ϵ and δ needs

Table 2.2.1: The smallest VaR with different confidence levels

t \ ϵ	0.01	0.025	0.05	0.1
1	0.0128	0.0113	0.0100	0.0085
2	0.0125	0.0110	0.0097	0.0082
3	0.0111	0.0098	0.0086	0.0073
4	0.0130	0.0115	0.0103	0.0088
5	0.0123	0.0109	0.0097	0.0083
6	0.0131	0.0115	0.0102	0.0087
7	0.0131	0.0115	0.0102	0.0086
8	0.0118	0.0104	0.0092	0.0078
9	0.0129	0.0113	0.0100	0.0085
10	0.0141	0.0124	0.0109	0.0093
11	0.0105	0.0092	0.0081	0.0068
Mean	0.0125	0.0110	0.0097	0.0082

to be chosen. Under the joint ellipsoidal uncertainty set, the robust VaR problem can be solved using the smallest VaR in Table 2.2.1.

2.3 Simulation of LP Models

Consider the problem (1.2.1) without the probability constraint that maximizes return of the portfolio:

$$\max_{\phi} r^T\phi \qquad \text{s.t } \phi \in \Phi. \tag{2.3.1}$$

Using the multivariate factor model to define the return of stocks in Section 1.4.3, the least squares estimate $\bar{x}_i$ of the true parameter x_i is given by

$$\bar{x}_i = (A^TA)^{-1}A^Ty_i = (\bar{\mu}_i, \bar{V}_{1i}, \bar{V}_{2i}, \ldots, \bar{V}_{mi}).$$

Assume that the return of stocks have mean stock return $\bar{\mu}$ and covariance of factor return $\bar{V}$, and the return of stocks are assumed to have a normal distribution from the factor model as

$$r \sim \mathcal{N}(\bar{\mu}, \bar{V}^T F \bar{V} + D). \tag{2.3.2}$$

Let the return of each stock $r_i \sim \mathcal{N}(\mu_i, \sigma_i)$, then the random variable is standardized to a z-score by subtracting the mean and then dividing by its standard deviation:

$$z_i = \frac{r_i - \mu_i}{\sigma_i} \sim \mathcal{N}(0, 1).$$

The probabilities that the absolute value of the z-score is less than 1, 2, and 3 are, respectively,

$$\begin{aligned} \mathbf{P}\left(\mu_i - 3\sigma_i \leq r_i \leq \mu_i + 3\sigma_i\right) &= \mathbf{P}\left(|z_i| \leq 3\right) = 99.74\%, \\ \mathbf{P}\left(\mu_i - 2\sigma_i \leq r_i \leq \mu_i + 2\sigma_i\right) &= \mathbf{P}\left(|z_i| \leq 2\right) = 95.44\%, \\ \mathbf{P}\left(\mu_i - \sigma_i \leq r_i \leq \mu_i + \sigma_i\right) &= \mathbf{P}\left(|z_i| \leq 1\right) = 68.26\%. \end{aligned}$$

The probabilities determine the interval of the random return. From the return defined in (2.3.2), define the mean return of stock i by $\mu_i = \bar{\mu}_i$ and the standard derivation of the return $\sigma_i = 3\sqrt{(\bar{V}^T F \bar{V} + D)_{ii}}$. Then the i-th stock return r_i will lie within $[\mu_i - \sigma_i, \mu_i + \sigma_i]$ with 99.74% confidence level for $i = 1, \ldots, n$. Since the number of asset $n = 36$, the probability that return r is in $[\mu - \sigma, \mu + \sigma]$ is at least $91\% (= 0.9974^{36} = 0.9105)$.

2.3.1 Ben-Tal and Nemirovski

Ben-Tal and Nemirovski [4] assumed that the return on asset i is given by $r_i = \mu_i + \xi_i\sigma_i$, where the perturbations $\{\xi_i\}$ are independent random variables symmetrically distributed in $[-1, 1]$. By using an auxiliary variable c, problem (2.3.1) can be written as

$$\begin{aligned} \max_{\phi} \quad & c \\ \text{s.t} \quad & -r^T\phi \le -c, \\ & \phi \in \Phi. \end{aligned} \tag{2.3.3}$$

Since $r_i = \mu_i + \xi_i\sigma_i$ for every $i = 1, \ldots, n$, the negative return is $-r_i = -\mu_i + \xi_i\sigma_i$. The feasible solution of (2.3.3) can be extended to the solution of the following optimization problem with a positive parameter Ω:

$$\begin{aligned} \max_{\phi,y,z} \quad & c, \\ \text{s.t} \quad & -\mu^T\phi + \sigma^T y + \Omega\sqrt{\sum \sigma_i^2 z_i^2} \le -c, \\ & -y \le \phi - z \le y, \\ & e^T\phi = 1, \\ & y \ge 0, \ \phi \ge 0, \end{aligned} \tag{2.3.4}$$

where Ω is a positive parameter with a reliability level $\kappa = \exp\{-\Omega^2/2\}$, which means the first constraint in (2.3.3) is violated with probability at most κ. By bringing the constraint back to the objective function, the problem can be rewritten as:

$$
\begin{aligned}
\max_{\phi,y,z} \quad & \mu^T\phi - \sigma^T y - \Omega\sqrt{\sum \sigma_i^2 z_i^2} \\
\text{s.t} \quad & -y \leq \phi - z \leq y, \\
& \sum_{i=1}^{n} \phi_i = 1, \\
& y \geq 0,\ \phi \geq 0.
\end{aligned}
\tag{2.3.5}
$$

Real market data can be used to simulate model (2.3.5). If a reliability level k is 5%, then parameter $\Omega = -2\ln(k) = 6$. Figure 2.3.1(a) shows that the portfolio has steady growth return over the periods except for the period between 6 and 7, and the portfolio has reached 53% of overall growth return. According to Figure 2.3.1(b), the portfolio is highly diversified and contains at least 30 stocks over the entire period. In Figure 2.3.1(c), the transaction cost is 1 on the first period since we start with no investment in each stock, so that the portfolio changed 100%. Later on, it decreases and keeps the cost below 0.4. The Sharpe ratio in Figure 2.3.1(d) is positive for the entire period. A positive Shape ratio occurs if the return of the portfolio is greater than the risk free rate during the period. The maximum Shape ratio 0.75 occurs in period 3, the minimum Sharpe ratio 0.12 occurs in period 4, and the average Sharpe ratio for the 11 periods is 0.44.

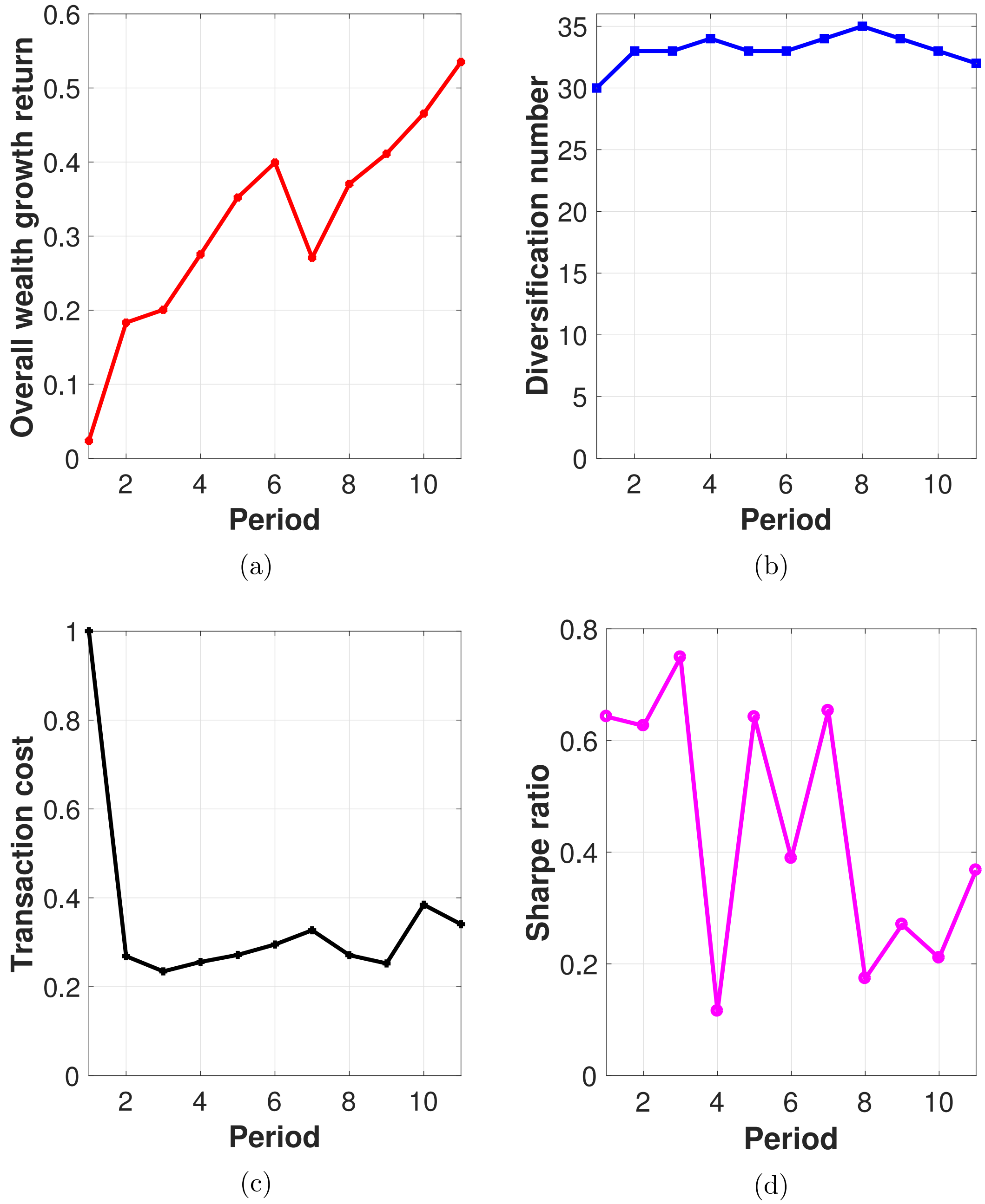


Figure 2.3.1: Performance of Ben-Tal and Nemirovski's model with 5% reliability level: (a) overall wealth growth return (b) diversification number (c) transaction cost (d) Sharpe ratio over the 11 investment periods

2.3.2 Bertsimas and Sim

Bertsimas and Sim [7] proposed a new approach to solve the linear programming problem in (1.4.3). Since the model is already applied to the robust maximum return problem in (2.3.1) by Gregory, Darby-Dowman, and Mitra [17], the details behind the construction of the model are omitted. A new stochastic variable η_i measures the deviation of parameter r_i from μ_i and takes values in [-1,1], where $\eta_i = (r_i - \mu_i)/\sigma_i$. That is, rearrange the rate of return as $r_i = \mu_i + \sigma_i \eta_i$, and let $|J|$ be the number of parameters r_i that are uncertain. Then from Soyster's and Ben-Tal and Nemirovski's models,

$$\sum_i \frac{| r_i - \mu_i |}{\sigma_i} = |J|.$$

Bertsimas and Sim relaxed this condition by defining a new parameter Γ, the budget of uncertainty, as the number of uncertain parameters that take their worst case value $\mu_i - \sigma_i$. Therefore, $\sum_i | \eta_i | \leq \Gamma$, such that $\Gamma \in [0, |J|]$. Then problem (2.3.3) can be derived as

$$\begin{aligned} &\underset{\phi,p,q}{\text{Max}}\ c \\ &\text{s.t}\quad c \leq \mu^T \phi - \Gamma p - \sum_i q_i, \\ &\qquad p + q_i \geq \sigma_i \phi_i,\ \forall i, \\ &\qquad e^T \phi = 1, \\ &\qquad p,\ q,\ \phi \geq 0. \end{aligned}$$

Bertsimas and Sim provided three different bounds on the probability of violation under the model. If ϕ^* is an optimal solution of problem (2.3.6), the bounds are:

$$\text{(i) } \Pr(r^T\phi^* < c) \leq \exp(-\frac{\Gamma^2}{2|J|}) \tag{2.3.6}$$

$$\text{(ii) } \Pr(r^T\phi^* < c) \leq B(n,\Gamma) \tag{2.3.7}$$

$$\text{(iii) } B(n,\Gamma) \leq (1-\mu)C(n,\lfloor v \rfloor) + \sum_{l=\lfloor v \rfloor+1}^{n} C(n,l) \tag{2.3.8}$$

where $n = |J_i|$, $v = (\Gamma_i + n)/2$, $\mu = v - \lfloor v \rfloor$, and

$$\begin{aligned} B(n,\Gamma) &= \frac{1}{2^n}\left\{(1-\mu)\sum_{l=\lfloor v \rfloor}^{n}\binom{n}{k} + \mu\sum_{l=\lfloor v \rfloor+1}^{n}\binom{n}{l}\right\} \\ &= \frac{1}{2^n}\left\{(1-\mu)\binom{n}{\lfloor v \rfloor} + \sum_{l=\lfloor v \rfloor+1}^{n}\binom{n}{l}\right\} \\ C(n,l) &= \begin{cases} \frac{1}{2^n}, & \text{if } l=0 \text{ or } l=n, \\ \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{(n-l)l}}\exp(n\log(\frac{n}{2(n-l)}) + l\log(\frac{n-l}{l})), & \text{otherwise.} \end{cases} \end{aligned}$$

By using Bertsimas and Sim's model, (2.3.1) can be reformulated as

$$\begin{aligned} \max_{\phi} \quad & \mu^T\phi - \Gamma p - e^T q \\ \text{s.t} \quad & p + q_i \geq \sigma_i\phi_i, \ \forall i \\ & e^T\phi = 1 \\ & y, \phi, q, p \geq 0. \end{aligned}$$

Figure 2.3.2 presents the bound's estimation of the probability of violation for the

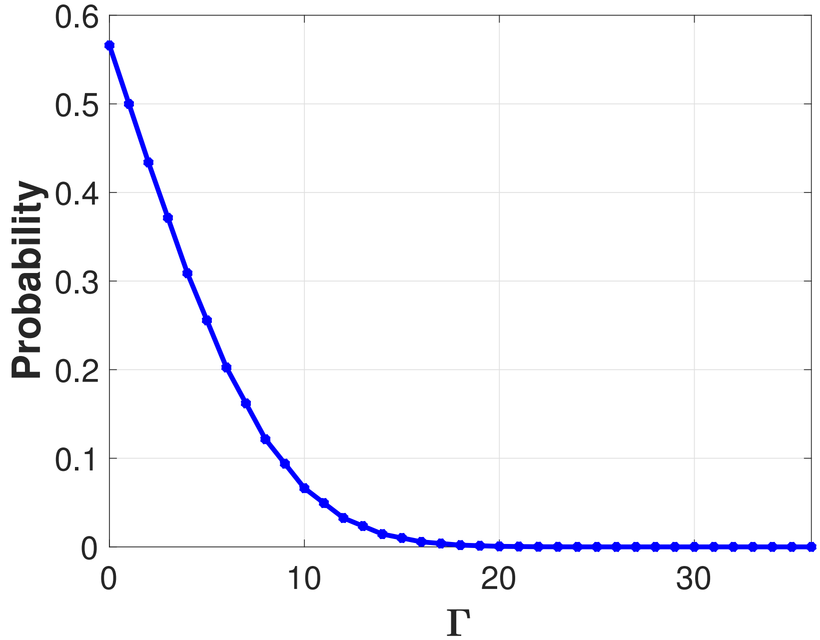


Figure 2.3.2: Probability bounds with respect to Γ.

budget of uncertainty $\Gamma = 0, 1, \ldots n$ by using estimation (2.3.7) when 36 assets are available (n = 36). The probability of violation is decreasing as Γ increases since the model with bigger Γ is more conservative by taking the worst case on more assets. Table 2.3.1 shows probability bounds for some Γ values.

Table 2.3.1: Probability bounds for some Γ values

Γ	0	1	4	6	9	11	14	19	23
bounds(%)	57	50	31	20	9	5	1	0.1	0.01

Our choice of $\Gamma = 11$ gives less than 5% probability of violation, in other words, the confidence level is 95%. Figure 2.3.3(a) overall growth similar to that of the BS model. It has steady growth and a big gain between period 1 and 2, and big loss between period 6 and 7. But the owgr of BN model is slightly higher than the owgr of BS models. Figure 2.3.3(b) shows the model is well-diversified and contains

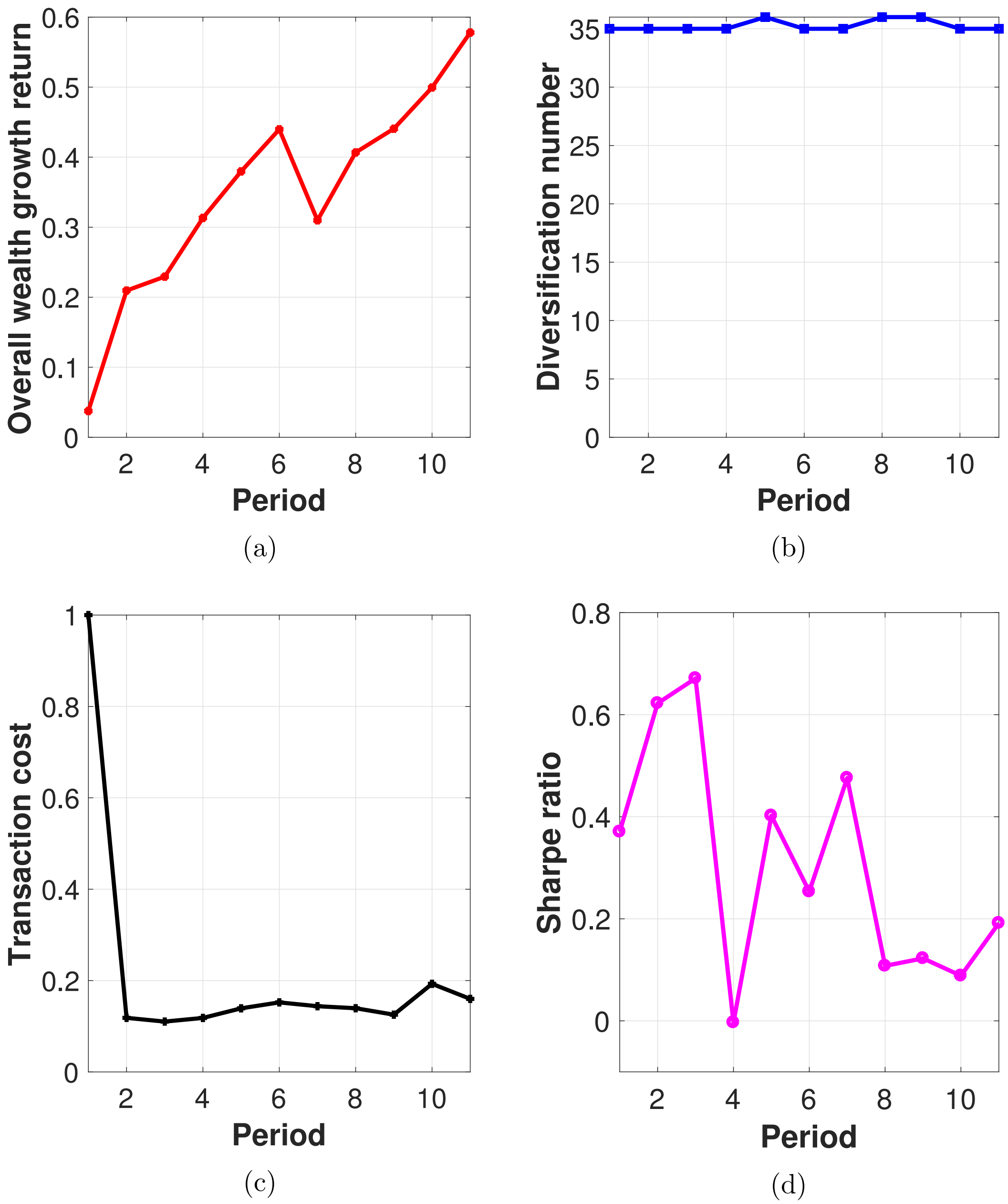


Figure 2.3.3: Performance of Bertsimas and Sim's model with 5% reliability level: (a) overall wealth growth return (b) diversification number (c) transaction cost (d) Sharpe ratio over the 11 investment periods

35 or 36 of the 36 stocks. Figure 2.3.3(c) starts with cost 1 on the first period and all cost stay less than 0.2. Compared to the BS model, the cost is half, which means the portfolio is changed less. For each period, the Sharpe ratio of Bertsimas and Sim's model in Figure 2.3.3(d) is less than the Sharpe ratio of Ben-tal and Nemirovski's model in Figure 2.3.1(d). The mean Sharpe ratio over 11 investment periods of Bertsimas and Sim's model is 0.3.

2.3.3 Box, Ball-Box, Budgeted Uncertainty Sets

Ben-Tal, El Ghaoui, and Nemirovski [2] provided three different uncertainty sets, namely Box, BallBox, and Budgeted, and produced the following propositions to solve problem (2.3.3). Let the unknown return $r_i = \mu_i + \sigma_i \zeta_i$, where ζ is a random perturbation vector.

Proposition 2.3.1 (proposition 2.3.3 in [2]). *The robust linear inequality constraint (2.3.3) under the Ballbox uncertainty set* $\mathcal{Z} = \{\zeta \in \mathbb{R}^L : \|\zeta\|_\infty \leq 1, \|\zeta\|_2 \leq \Omega\}$ *is equivalent to the system of conic quadratic constraints:*

$$
\begin{aligned}
&(a)\ z_l + w_l = \sigma_l \phi_l,\ \ l = 1, \dots, L,\\
&(b)\ \sum_l |z_l| + \Omega \sqrt{\sum_l w_l^2} \leq -c + \mu^T \phi.
\end{aligned}
$$

Also, the feasible solution violates the inequality condition (2.3.3) at most $\exp\{-\Omega^2/2\}$.

Proposition 2.3.2 (proposition 2.3.4 in [2]). *The constraint (2.3.3) under the Budgeted uncertainty set* $\mathcal{Z} = \{\zeta \in \mathbb{R}^L : \|\zeta\|_\infty \leq 1, \|\zeta\|_1 \leq \gamma\}$ *is equivalent to the system* *of conic quadratic constraints:*

$$(a)\ z_l + w_l = \sigma_l \phi_l,\ \ l = 1, \ldots, L,$$

$$(b)\ \sum_l |z_l| + \gamma \underset{l}{max}\ |w_l| \leq -c + \mu^T \phi.$$

Also, the feasible solution violates the inequality condition (2.3.3) at most $\exp\{\frac{-\gamma^2}{2L}\}$.

By using the above propositions, problem (2.3.3) under the three different uncertainty sets can be derived as quadratic cone programmings. The problem (2.3.1) under the Box uncertainty set, $\mathcal{Z} = \{\zeta : \|\zeta\|_\infty \leq 1\}$, is similar to Soyster's model:

$$\begin{aligned} &\max_{\phi}\ (\mu - \sigma)^T \phi \\ &\text{s.t}\ \ \phi^T e = 1, \\ &\qquad \phi \geq 0. \end{aligned}$$

To deal with the absolute value in constraint (b) of Proposition 2.3.1, introduce two positive variables p and q such that $z_l = p_l - q_l$ and $|z_l| = p_l + q_l$. By Proposition 2.3.1, the problem in (2.3.1) under the Ballbox uncertainty set, $\mathcal{Z} = \{\zeta : \|\zeta\|_\infty \leq$

$1, \|\zeta\|_2 \leq \Omega\}$ can be derived as

$$\begin{aligned}
\max_{\phi,w,p,q,s} \quad & \mu^T\phi - (p+q)^T e - \Omega s \\
\text{s.t} \quad & p_i - q_i + w_i = \sigma_i\phi_i, \; i = 1, \ldots, n, \\
& \|w\|_2 \leq s, \\
& \phi^T e = 1, \\
& \phi, p, q \geq 0.
\end{aligned}$$

Similarly, introduce four positive variables p, q, u, and v such that $z_l = p_l - q_l$, $|z_l| = p_l + q_l$, $w_l = u_l - v_l$, and $|w_l| = u_l + v_l$. By Proposition 2.3.2, the problem in (2.3.1) under the Budgeted uncertainty set, $\mathcal{Z} = \{\zeta : \|\zeta\|_\infty \leq 1, \|\zeta\|_1 \leq \gamma\}$ can be derived as

$$\begin{aligned}
\max_{\phi,p,q,u,v,s} \quad & \mu^T\phi - (p+q)^T e - \gamma s \\
\text{s.t} \quad & p_i - q_i + u_i - v_i = \sigma_i\phi_i, \; i = 1, \ldots, n, \\
& u_i + v_i \leq s, \; l = 1, \ldots, n, \\
& \phi^T e = 1, \\
& \phi, p, q, u, v, s \geq 0.
\end{aligned}$$

Figure 2.3.3(a) shows that the problem under the Ballbox and Budgeted uncertainty sets have similar performance but the return under the Box uncertainty

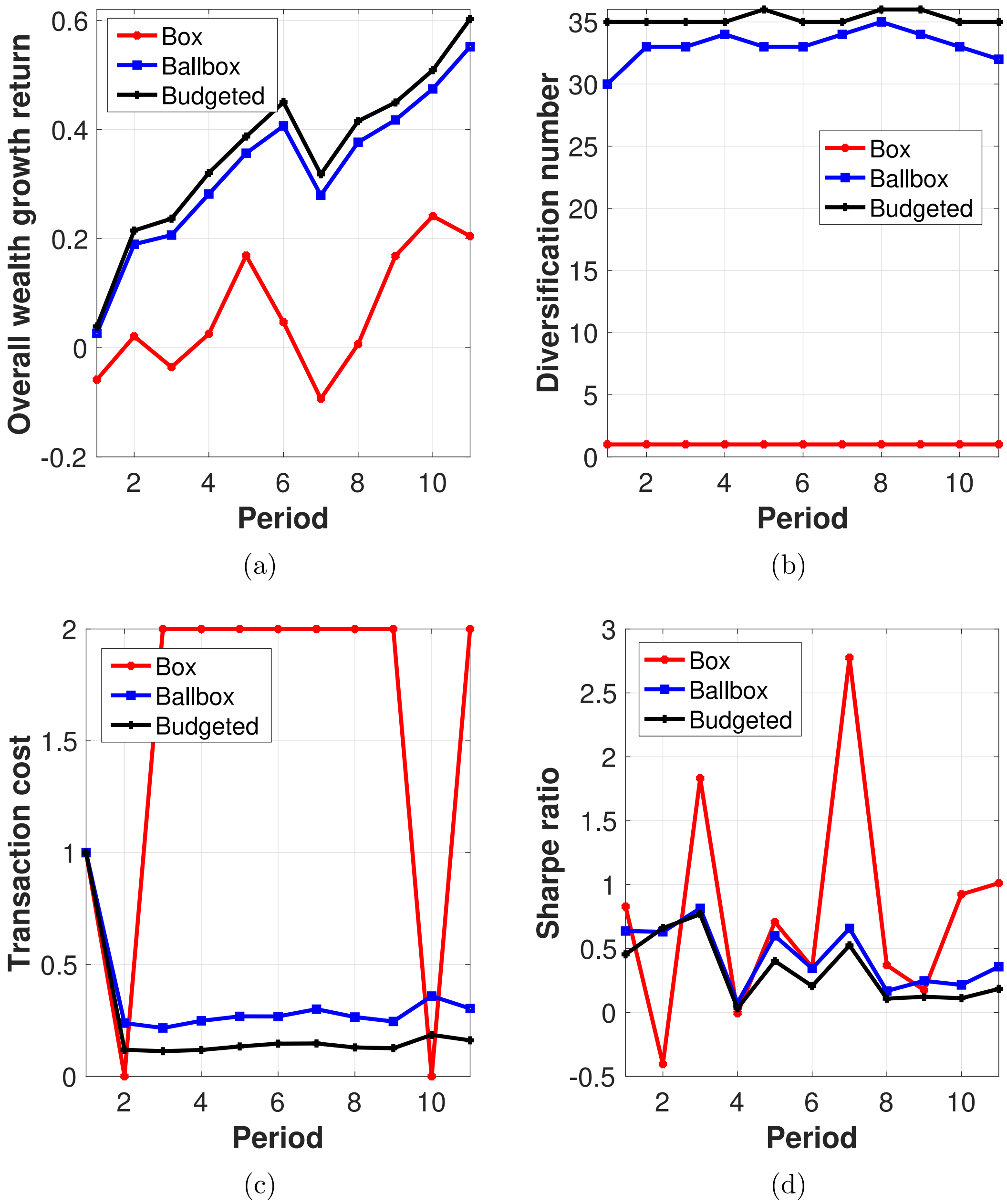


Figure 2.3.4: Performance under three different uncertainty sets: (a) overall wealth growth return over the periods under three different sets, (b) diversification number, (c) transaction cost, and (d) Sharpe ratio. The reliability level is 5% for the ballbox and budgeted sets.

set performs worse than these two sets. Figure 2.3.3(b) shows that both the Ballbox and Budgeted uncertainty sets construct well-diversified portfolios, which contain at least 30 stocks, so that the risk of portfolio return is low. However, the portfolio return under the box uncertainty set solely depends on one stock that had a good performance in the previous period. The stock that had a good performance in a previous period does not guarantee the similar performance in the next period. Hence, constructing the portfolio with one stock is too risky. Also Figure 2.3.3(c) shows that the Ballbox and Budgeted uncertainty sets have less transaction cost than the Box uncertainty set. Figure 2.3.3(d) shows that the Sharpe ratios of the Ballbox and Budgeted uncertainty sets are close to the models in the previous two sections, but the Box uncertainty set provides a completely different Sharpe ratio. The Sharpe ratio of the Box uncertainty set is much higher than two uncertainty sets for the most periods by putting all money into one stock, which has a high return and low risk. This shows that higher Sharpe ratio doesn't always guarantee a better portfolio if the portfolio is not diversified.

Chapter 3

Robust Portfolio Problem with VaR under Joint Uncertainty Set

3.1 Overview

Recall that the main goal is to find the optimal weight from the robust portfolio selection problem (1.2.1) under the joint ellipsoidal uncertainty set, defined in Section 1.4.5. The problem was solved by Goldfard and Iyengar [16] under the separable uncertainty sets for the mean return μ and the covariance matrix of factor returns V. Also, they solved three other problems, namely the robust minimum variance portfolio selection problem (1.4.21), the robust maximum return problem (1.4.22), and the robust maximum Sharpe ratio problem (1.4.23). However, Lu [29] addressed the drawbacks of the uncertainty sets, such as high conservativeness and the construction of non-diversified portfolios. As such, he defined a new uncertainty set, called the joint ellipsoidal uncertainty set (1.4.25), which is described in Section 1.4.5. Lu applied this new set to the robust maximum return and robust minimum variance problems, but did not do so with the robust maximum Sharpe ratio and robust VaR constraint problems.

In Section 3.2, the robust VaR constraint portfolio selection problem under the joint ellipsoidal uncertainty set is solved. Consider the same factor model for asset returns in Section 1.4.3. It is difficult to solve with VaR constraint directly since it is a non-convex constraint. So estimations of the constraint, written as a semidefinite and a second order cone constraints are used. In Section 3.3, the portfolio selection problem with VaR constraint in Section 3.3 is examined using historical real market data. This is compared to the performance of the problem under the separable uncertainty set.

The benefit of the joint ellipsoidal uncertainty set is that it constructs a well-diversified portfolio. This simple example explains why the diversification number of a portfolio matters. A well-diversified portfolio is safer because it reduces risk while maintaining strong return potential. There are two assets A and B, which have expected returns $E(r)$ and risks $\sigma(r)$ as in Table 3.1.1. Figure 3.1.1 presents the investment opportunity set (IOS) with these two stocks, which is all the returns and risks of portfolios consisting of these two assets. The IOS will depend on the correlation of returns of A and B.

Table 3.1.1: Two risky assets

Assets	$E(r)$	$\sigma(r)$
A	6%	10%
B	10%	20%

When investors put all their money into one asset, say A, then the expected return is 6% and the risk is given as 10%. By adding one more asset B in the portfolio, which is negatively correlated to the return of asset A, then we can construct a portfolio that has the same risk but higher expected return than asset A.

Similarly, if the portfolio is constructed by 100% of asset B, its expected return is 10% and its risk is 20%. By adding asset A in the portfolio, investors can reduce risk substantially with a small decrease of the expected return. Theoretically, if

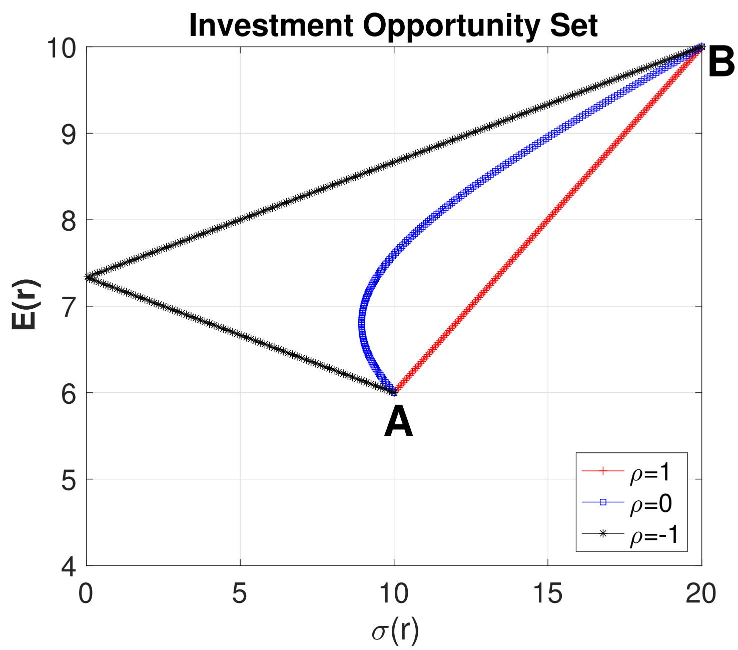


Figure 3.1.1: Investment opportunity set (IOS) for two risky assets when the correlation of the two assets in Table 3.1.1 is perfectly positively correlated $\rho = 1$, uncorrelated $\rho = 0$, and perfectly negatively correlated $\rho = -1$.

two assets are perfectly positively correlated, the constructed portfolio with these two assets has no risk reduction benefits. If two are perfectly negatively correlated, investors can reduce risk to zero.

In general, as the number of assets increases, the variance of the portfolio will decrease. Suppose there are n available assets. Bodie, Kane, and Marcus [8] provided the general formula for the variance of a portfolio as

$$\sigma_p^2 = \sum_{i,j=1}^{n} w_i w_j \mathbf{Cov}(r_i, r_j).$$

Consider the case of an equally weighted portfolio so that $w_i = 1/n$. For simplicity, assume that all assets have common standard deviation σ, and common correlation coefficient ρ of any pair of assets. Bodie, Kane, and Marcus showed the variance of the portfolio as

$$\sigma_p^2 = \frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2 = \sigma^2\left(\frac{1}{n} + \left(1 - \frac{1}{n}\right)\rho\right).$$

When correlation coefficient ρ is 0, as n increases, the variance of the portfolio decreases and eventually approaches 0. Suppose $\rho = 1$, then $\sigma_p^2 = \sigma^2$, and there is no benefit on the number of assets. For any positive ρ less than 1, the variance of the portfolio approaches $\rho\sigma^2$ as n increases. Let standard deviation σ be 1. If

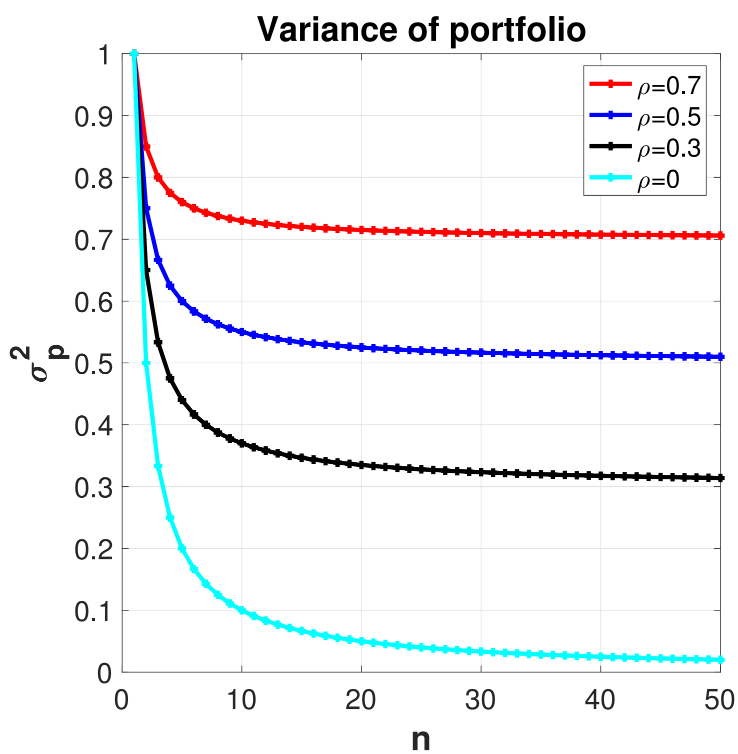


Figure 3.1.2: Variance of equally weighted portfolio of n assets. The variance of all assets $\sigma_i = \sigma$ and the correlation coefficient of any pair of assets $\rho_{ij} = \rho$.

more assets are added to the portfolio, the risk of the portfolio will eventually reach the level of the market portfolio. Unlike the separable sets, the joint uncertainty set will include many assets in the portfolio. In other words, it constructs a diversified portfolio, which can be seen from the computational results in next section.

3.2 The Robust VaR Problem under Joint Ellipsoidal Set

Recall the joint ellipsoidal uncertainty set for mean return μ and the covariance matrix for factor return V, provided by Lu [29], for some $c(w)$,

$$S_{m,v}(w) = \left\{ (\mu, V) \in \mathbb{R}^n \times \mathbb{R}^{m \times n} : \sum_{i=1}^{n} \frac{(x_i - \bar{x}_i)^T (A^T A)(x_i - \bar{x}_i)}{s_i^2 (m+1)} \leq c(w) \right\}, \quad (3.2.1)$$

where $x_i = (\mu_i, V_{1i}, V_{2i}, \ldots, V_{mi})^T$ for $i = 1, \ldots, n$, and $A = (e \ B^T)$ are defined as in (1.4.16). Consider the VaR constraint problem under the above uncertainty set.

$$\begin{aligned} &\max_{\phi \in \Phi} \min_{(\mu,V) \in S_{m,v}} \mathbf{E}[r_\phi] \\ &\text{s.t} \quad \max_{(\mu,V) \in S_{m,v}} \mathbf{P}(r_\phi \leq -\delta) \leq \epsilon, \end{aligned} \quad (3.2.2)$$

Since the return on the portfolio is $r_\phi \sim \mathcal{N}(\mu^T \phi, \phi^T (V^T F V + D)\phi)$, the VaR constraint can be written as below inequality

$$\begin{aligned} \mathbf{P}(r_\phi \leq -\delta) \leq \epsilon &\Leftrightarrow \mathbf{P}(\mu^T \phi + \mathscr{Z} \sqrt{\phi^T (V^T F V + D)\phi} \leq -\delta) \leq \epsilon \\ &\Leftrightarrow \mathbf{P}\left(\mathscr{Z} \leq \frac{-\mu^T \phi - \delta}{\sqrt{\phi^T (V^T F V + D)\phi}} \right) \leq \epsilon \\ &\Leftrightarrow \frac{-\mu^T \phi - \delta}{\sqrt{\phi^T (V^T F V + D)\phi}} \leq \mathcal{F}^{-1}(\epsilon) \\ &\Leftrightarrow -\mathcal{F}^{-1}(\epsilon) \sqrt{\phi^T (V^T F V + D)\phi} \leq \mu^T \phi + \delta \end{aligned}$$

where $\mathscr{Z} \sim \mathcal{N}(0,1)$ is the standard normal random variable and $\mathcal{F}^{-1}(\cdot)$ is its cumulative density function (CDF). Typically, we set the possibility of losing certain percentage on a portfolio small $\epsilon \ll 0.5$, and so $\mathcal{F}^{-1}(\epsilon) < 0$. Then both sides of the equation are nonnegative. Squaring both sides of the inequality will be valid, so the VaR constraint can be changed to a quadratic inequality constraint:

$$
\begin{aligned}
P(r_\phi \leq -\delta) \leq \epsilon \Leftrightarrow \; & \epsilon_0^2 \phi^T (V^T F V + D)\phi \leq (\mu^T \phi + \delta)^2 \\
\Leftrightarrow \; & \epsilon_0^2 \phi^T (V^T F V + D)\phi - (\mu^T \phi + \delta)^2 \leq 0,
\end{aligned}
$$

where ϵ_0 = -$\mathcal{F}^{-1}(\epsilon)$. Then problem (3.2.2) is equivalent to

$$
\begin{aligned}
& \max_{\phi \in \Phi} \min_{(\mu,V) \in S_{m,v}} \mu^T \phi \\
& \text{s.t} \quad \max_{(\mu,V) \in S_{m,v}} \{\epsilon_0^2 \phi^T V^T F V \phi - (\mu^T \phi)^2 - 2\delta(\mu^T \phi)\} + \epsilon_0^2 \phi^T D \phi - \delta^2 \leq 0.
\end{aligned}
$$

By introducing auxiliary variables ν and t, the objective function can be moved to the constraint, and the quadratic function is changed to a quadratic cone constraint as

$$
\begin{aligned}
\max_{\nu,t,\phi} \quad & \nu \\
\text{s.t} \quad & \min_{(\mu,V)\in S_{m,v}} \mu^T\phi \geq \nu, \\
& \max_{(\mu,V)\in S_{m,v}} \{\epsilon_0^2\phi^T V^T F V\phi - (\mu^T\phi)^2 - 2\delta(\mu^T\phi)\} + \epsilon_0^2 t - \delta^2 \leq 0, \qquad (3.2.3) \\
& \phi^T D\phi \leq t, \\
& \phi \in \Phi.
\end{aligned}
$$

By using the equivalent conditions of the first two constraints, problem (3.2.3) can be rewritten as

$$
\begin{aligned}
\max_{\nu,t,\phi} \quad & \nu \\
\text{s.t} \quad & \nu - \mu^T\phi \leq 0, && \forall(\mu,V)\in S_{m,v}, \quad (3.2.4) \\
& \epsilon_0^2\phi^T V^T F V\phi - (\mu^T\phi)^2 - 2\delta(\mu^T\phi) + \epsilon_0^2 t - \delta^2 \leq 0, && \forall(\mu,V)\in S_{m,v}, \quad (3.2.5) \\
& \phi^T D\phi \leq t, \\
& \phi \in \Phi.
\end{aligned}
$$

The third constraint can be reformulated as second-order cone constraint by using below Lemma provided by Nesterov and Nemirovski in [33].

Lemma 3.2.1. *The restricted hyperbolic constraints, $z^Tz \leq xy,\ x, y \geq 0,$ can be* *reformulated as second-order cone constraints as follows*

$$z^T z \leq xy \;\Leftrightarrow\; 4z^T z \leq (x+y)^2 - (x-y)^2 \;\Leftrightarrow\; \left\| \begin{pmatrix} 2x \\ x-y \end{pmatrix} \right\| \leq x+y.$$

Let $\mathcal{L}^n$ be the n-dimensional second-order cone given by

$$\mathcal{L}^n = \left\{ z \in \mathbb{R}^n : z_1 \geq \sqrt{\sum_{i=2}^{n} z_i^2} \right\}.$$

Then the third quadratic constraint of the problem can be written as

$$\begin{aligned} \phi^T D\phi \leq t \;\Leftrightarrow\; \|D^{1/2}\phi\|^2 \leq t \;\Leftrightarrow\; & \|2D^{1/2}\phi\|^2 \leq (1+t)^2 - (1-t)^2 \\ \Leftrightarrow\; & \sqrt{(1-t)^2 + \|2D^{1/2}\phi\|^2} \leq (1+t) \\ \Leftrightarrow\; & \begin{pmatrix} 1+t \\ 1-t \\ 2D^{1/2}\phi \end{pmatrix} \in \mathcal{L}^{n+2}. \end{aligned}$$

The following proposition is called the $\mathscr{S}$-procedure lemma and the proof is given in [9].

Proposition 3.2.2. *Let $F_i(x) = x^T A_i x + 2b_i^T x + c_i, i = 0, \ldots, p$ be quadratic functions of $x \in \mathbb{R}^n$. Then $F_0(x) \leq 0$ for all x such that $F_i(x) \leq 0, i = 1, \ldots, p$, if there* *exists* $\tau_i \geq 0$ *such that*

$$\sum_{i=1}^{p} \tau_i \begin{pmatrix} c_i & b_i^T \\ b_i & A_i \end{pmatrix} - \begin{pmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{pmatrix} \succeq 0.$$

Moreover, if $p = 1$ *then the converse holds if there is* $x_0 \in \mathbb{R}^n$ *such that* $F_1(x_0) < 0$.

The next proposition states a property of the Kronecker product of positive semidefinite matrices. For the proof, see [21].

Proposition 3.2.3. *If* $H \succeq 0$ *and* $K \succeq 0$, *then* $H \otimes K \succeq 0$.

By the above propositions, two constraints (3.2.4) and (3.2.5) can be derived to equivalent semidefinite constraints.

Lemma 3.2.4. *Let* $S_{m,v}$ *be an* w*-confidence uncertainty set given in (3.2.1) for* $w \in (0,1)$. *Then two inequality constraints (3.2.4) and (3.2.5) are equivalent to*

$$\begin{pmatrix} \tau_1 R - (\phi\phi^T) \otimes \begin{pmatrix} -1 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} & \tau_1 h + \delta q \\ \tau_1 h^T + \delta q^T & \tau_1 \eta - \epsilon_0^2 t + \delta^2 \end{pmatrix} \succeq 0,$$

$$\begin{pmatrix} \tau_2 R & \tau_2 h + q \\ \tau_2 h^T + q^T & \tau_2 \eta - 2\nu \end{pmatrix} \succeq 0,$$

$$\tau_1, \tau_2 \geq 0,$$

where

$$R = \begin{pmatrix} \frac{A^TA}{s_1^2(m+1)} & & \\ & \ddots & \\ & & \frac{A^TA}{s_n^2(m+1)} \end{pmatrix} \in \mathbb{R}^{[(m+1)n\times(m+1)n]}, \qquad \eta = \sum_{i=1}^{n} \bar{x}_i^T \left(\frac{A^TA}{s_i^2(m+1)} \right) \bar{x}_i - c(w),$$

$$h = \begin{pmatrix} -\frac{A^TA\bar{x}_1}{s_1^2(m+1)} \\ \vdots \\ -\frac{A^TA\bar{x}_n}{s_n^2(m+1)} \end{pmatrix} \in \mathbb{R}^{(m+1)n}, \qquad q = (\phi_1, 0 \ldots, \phi_n, 0)^T \in \mathbb{R}^{(m+1)n}.$$

Proof. Given any $(t, \nu, \phi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$, we define

$$H(\mu, V) = \epsilon_0^2 \phi^T V^T F V \phi - (\mu^T \phi)^2 - 2\delta(\mu^T\phi) + \epsilon_0^2 t - \delta^2,$$

$$L(\mu, V) = -\mu^T\phi + \nu.$$

For $x_i = (\mu_i, V_{1i}, V_{2i}, \ldots, V_{mi})^T$ for $i = 1, \ldots, n$,

$$\frac{\partial H}{\partial x_i} = \begin{pmatrix} -2(\mu^T\phi)\phi_i - 2\delta\phi_i \\ 2\epsilon_0^2 \phi^T V^T F \phi_i \end{pmatrix}, \quad \frac{\partial^2 H}{\partial x_i \partial x_j} = \begin{pmatrix} -2\phi_i\phi_j & 0 \\ 0 & 2\epsilon_0^2 \phi_i\phi_j F \end{pmatrix},$$

$$\frac{\partial H}{\partial x_i}(0,0) = \begin{pmatrix} -2\delta\phi_i \\ 0 \end{pmatrix}, \quad \frac{\partial^2 H}{\partial x_i \partial x_j}(0,0) = \begin{pmatrix} -2\phi_i\phi_j & 0 \\ 0 & 2\epsilon_0^2 \phi_i\phi_j F \end{pmatrix}.$$

By the Taylor series expansion for $H(\mu, V)$ at $x = 0$, we get

$$
\begin{aligned}
H(\mu, V) &= \frac{1}{2}\sum_{i,j=1}^{n} x_i^T \begin{pmatrix} -2\phi_i\phi_j & 0 \\ 0 & 2\epsilon_0^2\phi_i\phi_j F \end{pmatrix} x_j + \sum_{i=1}^{n} \begin{pmatrix} -2\delta\phi_i \\ 0 \end{pmatrix} x_i + \epsilon_0^2 t - \delta^2 \\
&= \sum_{i,j=1}^{n} x_i^T \begin{pmatrix} -\phi_i\phi_j & 0 \\ 0 & \epsilon_0^2\phi_i\phi_j F \end{pmatrix} x_j + 2\sum_{i=1}^{n} \begin{pmatrix} -\delta\phi_i \\ 0 \end{pmatrix} x_i + \epsilon_0^2 t - \delta^2.
\end{aligned}
$$

Similarly, the Taylor series expansion for $L(\mu, V)$ at $x = 0$, we obtain

$$
\frac{\partial L}{\partial x_i}(0,0) = \begin{pmatrix} -\phi_i \\ 0 \end{pmatrix}, \; L(\mu, V) = \sum_{i=1}^{n} \begin{pmatrix} -\phi_i \\ 0 \end{pmatrix} x_i + \nu.
$$

The joint uncertainty set $S_{m,v}$ in (3.2.1) can be written as

$$
\begin{aligned}
S_{m,v} = \Bigg\{ (\mu, V) \in & \mathbb{R}^n \times \mathbb{R}^{m\times n} : \sum_{i=1}^{n} x_i^T \left(\frac{A^T A}{(m+1)s_i^2} \right) x_i \\
& + 2\sum_{i=1}^{n} \left(\frac{-A^T A \bar{x}_i}{(m+1)s_i^2} \right)^T x_i + \sum_{i=1}^{n} \bar{x}_i^T \left(\frac{A^T A}{(m+1)s_i^2} \right) \bar{x}_i - c(w) \leq 0 \Bigg\}.
\end{aligned}
\tag{3.2.6}
$$

We see that $x = \bar{x}$ strictly satisfies the inequality given in (3.2.6). By using proposition 3.2.2, we can conclude that $H(\mu, V) \leq 0$ and $L(\mu, V) \leq 0$ for all $(\mu, V) \in S_{m,v}$

if and only if there exist $\tau_1, \tau_2 \in \mathbb{R}$ such that

$$\tau_1 \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} - \begin{pmatrix} E & -\delta q \\ -\delta q^T & \epsilon_0^2 t - \delta^2 \end{pmatrix} \succeq 0, \quad \tau_1 \geq 0,$$
$$\tau_2 \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} - \begin{pmatrix} 0 & -q \\ -q^T & 2\nu \end{pmatrix} \succeq 0, \quad \tau_2 \geq 0,$$

where $R, q, h,$ and η are the same as those defined in proposition 3.2.3, and E is given by

$$E_{ij} = \begin{pmatrix} -\phi_i\phi_j & 0 \\ 0 & \epsilon_0^2 \phi_i \phi_j F \end{pmatrix}, \; E = (\phi\phi^T) \otimes \begin{pmatrix} -1 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix}.$$

Using proposition (3.2.3) and the fact that $F \succeq 0$ implies that

$$\begin{pmatrix} \tau_1 R - (\phi\phi^T) \otimes \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} & \tau_1 h + \delta q \\ \tau_1 h^T + \delta q^T & \tau_1 \eta - \epsilon_0^2 t + \delta^2 \end{pmatrix} \succeq 0,$$
$$\begin{pmatrix} \tau_2 R & \tau_2 h + q \\ \tau_2 h^T + q^T & \tau_2 \eta - 2\nu \end{pmatrix} \succeq 0,$$
$$\tau_1, \tau_2 \geq 0.$$

$\square$

From the lemma, the constraint (3.2.5) is equivalent to the semidefinite constraint.

For simplicity, replace the quadratic variable $\phi\phi^T$ by semidefinite matrix $S \in \mathbb{R}^{n\times n}$. Then above constraint becomes

$$\begin{pmatrix} \tau_1 R - S \otimes \begin{pmatrix} -1 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} & \tau_1 h + \delta q \\ \tau_1 h^T + \delta q^T & \tau_1 \eta - \epsilon_0^2 t + \delta^2 \end{pmatrix} \succeq 0, \ \tau_1 \geq 0, \ S = \phi\phi^T. \tag{3.2.7}$$

The constraint $S \succeq \phi\phi^T$ can be changed to the semidefinite constraint by Schur's complements

$$\tilde{S} = \begin{pmatrix} 1 & \phi \\ \phi & S \end{pmatrix} \succeq 0.$$

However, the reverse inequality constrain $\phi\phi^T \succeq S$ cannot be changed to semidefinite constraint. So, we try to find the equivalent condition to $S = \phi\phi^T$. One we found was a constraint that includes the rank of the matrix. Vandenberche and Boyd [38] showed that the nonconvex constraint $S = \phi\phi^T$ is equivalent to

$$\tilde{S} \succeq 0, \ \text{and} \ \operatorname{rank}(\tilde{S}) = 1.$$

But the rank-constrained semidefinite programming is also hard to compute. So we use the estimated matrix to remove the nonconvex constraint. First, we use the below estimate function of the constraint (3.2.5):

$$\epsilon_0^2 \phi^T V^T F V \phi - 2\delta(\mu^T \phi) + \epsilon_0^2 t - \delta^2 \leq 0 \quad \forall (\mu, V) \in S_{m,v}. \tag{3.2.8}$$

The mean of the daily return μ is small, and $\phi^T e = 1$, so the quadratic term of the product, $(\mu^T \phi)^2$, can be ignored. Moreover, the term is nonnegative, estimation (3.2.8) implies VaR constraint (3.2.5). Thus, the estimation is a stronger constraint than (3.2.5). By Lemma 3.2.3 with the fact that $F \succeq 0$, the estimated constraint (3.2.8) holds if and only if for some $\tau_1 \geq 0$,

$$\begin{pmatrix} \tau_1 R - S \otimes \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} & \tau_1 h + \delta q \\ \tau_1 h^T + \delta q^T & \tau_1 \eta - \epsilon_0^2 t + \delta^2 \end{pmatrix} \succeq 0, \ S \succeq \phi\phi^T. \tag{3.2.9}$$

We also find another estimation of constraint (3.2.5), and the equivalent semidefinite constraint (3.2.7) is hard to compute because of the nonconvex constraint $S = \phi\phi^T$. So we relaxed the condition by $S \succeq \phi\phi^T$ and add more conditions to reduce the gap between S and $\phi\phi^T$. Assume that $S = \phi\phi^T$. Then S is doubly non-negative matrix that is both non-negative, $S_{ij} \geq 0 \ \forall i, j$, and positive semidefinite, $S \succeq 0$. By the fact that the sum of the weight vector is $\phi^T e = 1$,

$$\phi = \phi(\phi^T e) = (\phi\phi^T)e = Se, \ 1 = (e^T\phi)(\phi^T e) = e^T(\phi\phi^T)e = e^T Se.$$

The estimated constraint of (3.2.7) including all of the properties above is

$$\begin{pmatrix} \tau_1 R - S \otimes \begin{pmatrix} -1 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} & \tau_1 h + \delta q \\ \tau_1 h^T + \delta q^T & \tau_1 \eta - \epsilon_0^2 t + \delta^2 \end{pmatrix} \succeq 0, \ S \succeq \phi\phi^T, \ \phi = Se, \ S_{ij} \geq 0. \tag{3.2.10}$$

VaR constraint (3.2.7) implies estimation (3.2.10), thus the estimation is a weaker constraint than (3.2.7). The following two theorems show that problem (3.2.2) with two estimations of the VaR constraint can be reformulated as cone programming problems.

Theorem 3.2.5. *Let $S_{m,v}$ be an w-confidence uncertainty set given in (3.2.1) for $w \in (0,1)$. Then, problem (3.2.2) with estimation (3.2.9) on VaR constraint is equivalent to*

$$
\begin{aligned}
\max_{\phi,S,\tau_1,\tau_2,\nu,t,} \quad & \nu \\
s.t \quad & \begin{pmatrix} \tau_1 R - S \otimes \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} & \tau_1 h + \delta q \\ \tau_1 h^T + \delta q^T & \tau_1 \eta - \epsilon_0^2 t + \delta^2 \end{pmatrix} \succeq 0, \\
& \begin{pmatrix} \tau_2 R & \tau_2 h + q \\ \tau_2 h^T + q^T & \tau_2 \eta - 2\nu \end{pmatrix} \succeq 0, \\
& \begin{pmatrix} 1 & \phi^T \\ \phi & S \end{pmatrix} \succeq 0, \\
& \begin{pmatrix} 1+t \\ 1-t \\ 2D^{1/2}\phi \end{pmatrix} \in \mathcal{L}^{n+2}, \\
& S_{ij} \geq 0, \; \tau_1, \tau_2 \geq 0, \; \phi \in \Phi,
\end{aligned}
\tag{3.2.11}
$$

where R, h, q and η are the same as those defined in Lemma 3.2.4.

Theorem 3.2.6. *Let $S_{m,v}$ be an w-confidence uncertainty set given in (3.2.1) for $w \in (0,1)$. Then, problem (3.2.2) with estimation (3.2.10) on VaR constraint is equivalent to*

$$
\begin{aligned}
\max_{\phi,S,\tau_1,\tau_2,\nu,t,} \quad & \nu \\
s.t \quad & \begin{pmatrix} \tau_1 R - S \otimes \begin{pmatrix} -1 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} & \tau_1 h + \delta q \\ \tau_1 h^T + \delta q^T & \tau_1 \eta - \epsilon_0^2 t + \delta^2 \end{pmatrix} \succeq 0, \\
& \begin{pmatrix} \tau_2 R & \tau_2 h + q \\ \tau_2 h^T + q^T & \tau_2 \eta - 2\nu \end{pmatrix} \succeq 0, \\
& \begin{pmatrix} 1 & \phi^T \\ \phi & S \end{pmatrix} \succeq 0, \\
& \phi = Se, \\
& S_{ij} \geq 0, \\
& \begin{pmatrix} 1+t \\ 1-t \\ 2D^{1/2}\phi \end{pmatrix} \in \mathcal{L}^{n+2}, \\
& \tau_1, \tau_2 \geq 0, \;\; \phi \in \Phi,
\end{aligned}
\tag{3.2.12}
$$

where R, h, q and η are the same as those defined in lemma 3.2.4.

The following proposition, called the strict $\mathscr{S}$-procedure, is obtained from [39].

Proposition 3.2.7 (Theorem 1.2 in [39])**.** *Let $F_i(x) = x^T A_i x + 2b_i^T x + c_i, i = 0, 1$ be* *quadratic functions of* $x \in \mathbb{R}^n$*, and assume* $F_1(x) > 0$ *for some* x*. Then* $F_0(x) < 0$ *for all nonzero* x *such that* $F_1(x) \leq 0$*, if and only if there exists* $\tau \geq 0$ *such that*

$$\tau \begin{pmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{pmatrix} - \begin{pmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{pmatrix} \succ 0. \tag{3.2.13}$$

Lemma 3.2.8. *Let* δ^* *to be the minimum of the WVaR problem over* ϕ*, i.e.,*

$$\delta^* := \underset{\phi \in \Phi}{min} \underset{(\mu,V) \in S_{m,v}}{max} \epsilon_0 \sqrt{(\phi)^T(V^T FV + D)\phi} - \mu^T \phi.$$

Let $ri(\cdot)$ *be the relative interior of the associated set. Then there exists* δ^0 *such that for any* $\delta > \delta^0$*, there exist a solution* $\phi \in ri(\Phi)$ *such that*

$$\begin{aligned} &\underset{(\mu,V) \in S_{m,v}}{max} \phi^T V^T FV\phi - 2\delta(\mu^T\phi) + \epsilon_0^2 t - \delta^2 < 0 \text{ and} \\ &\phi^T D\phi < t. \end{aligned}$$

Proof. By the definition of δ^*, there exists solution $\phi^* \in \Phi$ such that

$$\begin{aligned} \delta^* &= \max_{(\mu,V) \in S_{m,v}} \epsilon_0 \sqrt{(\phi^*)^T(V^T FV + D)\phi^*} - \mu^T \phi^*, \\ 0 &= \max_{(\mu,V) \in S_{m,v}} \epsilon_0 \sqrt{(\phi^*)^T(V^T FV + D)\phi^*} - (\mu^T \phi^* + \delta^*). \end{aligned}$$

Since both terms are positive, it can be stated that

$$\max_{(\mu,V) \in S_{m,v}} \epsilon_0^2 (\phi^*)^T(V^T FV + D)\phi^* - (\mu^T \phi^* + \delta^*)^2 = 0.$$

Then for any $\delta > \delta^*$, there exists $\phi \in \Phi$ such that

$$\max_{(\mu,V)\in S_{m,v}} \phi^T(V^TFV + D)\phi - (\mu^T\phi + \delta)^2 < 0.$$

Since the mean daily return μ is small, there is δ^0 slightly greater than δ^* such that for any $\delta > \delta^0$, it can be stated that

$$\max_{(\mu,V)\in S_{m,v}} \phi^T(V^TFV + D)\phi - 2\delta(\mu^T\phi) - \delta^2 < 0,$$

and there is $t \in \mathbb{R}$,

$$\max_{(\mu,V)\in S_{m,v}} \phi^T V^T FV\phi - 2\delta(\mu^T\phi) + \epsilon_0^2 t - \delta^2 < 0,$$
$$\phi^T D\phi < t.$$

$\square$

Theorem 3.2.9. *Assume that* $0 \neq F \succ 0$, *and* $w \in (0, 1)$. *Problem (3.2.11) and its dual are strictly feasible, both problem are solvable, and duality gap is zero.*

Proof. We first show that problem (3.2.11) is strictly feasible. By Lemma 3.2.8, there exists $\phi^0 \in \mathrm{ri}(\Phi)$ and $t^0 \in \mathbb{R}$ such that

$$\max_{(\mu,V)\in S_{m,v}} (\phi^0)^T V^T FV\phi^0 - 2\delta(\mu^T\phi^0) + \epsilon_0^2 t^0 - \delta^2 < 0,$$
$$(\phi^0)^T D\phi^0 < t^0.$$

Then we observe that

$$\begin{pmatrix} 1+t^0 \\ \\ 1-t^0 \\ \\ 2D^{1/2}\phi^0 \end{pmatrix} \in \mathrm{ri}(\mathcal{L}^{n+2}).$$

By Proposition 3.2.7, there exists $\tau_1 > 0$ such that

$$\begin{pmatrix} \tau_1^0 R - S^0 \otimes \begin{pmatrix} 0 & 0 \\ \\ 0 & \epsilon_0^2 F \end{pmatrix} & \tau_1^0 h + \delta q^0 \\ \\ (\tau_1^0 h + \delta q^0)^T & \tau_1^0 \eta - \epsilon_0^2 t^0 + \delta^2 \end{pmatrix} \succ 0,$$

$$S^0 \succ \phi^0(\phi^0)^T,$$

for some matrix S^0 that has all positive entries. Then by the Schur Complement Lemma, the following holds:

$$\begin{pmatrix} 1 & (\phi^0)^T \\ \\ \phi^0 & S^0 \end{pmatrix} \succ 0.$$

Since A has full column rank, $R \succ 0$. Let τ_2^0 be any given positive number and ν^0 be sufficiently small such that

$$\tau_2^0 \eta - 2\nu^0 - (\tau_2^0 h + q^0)^T (\tau_2^0 R)^{-1} (\tau_2^0 h + q^0) > 0,$$

where $q^0 = (\phi_1^0, 0, \ldots, \phi_n^0, 0)^T \in \mathbb{R}^{(m+1)n}$. By the Schur's Complement Lemma, this is equivalent to

$$\begin{pmatrix} \tau_2^0 R & \tau_2^0 h + q^0 \\ (\tau_2^0 h + q^0)^T & \tau_2^0 \eta - 2\nu^0 \end{pmatrix} \succ 0.$$

Thus, $(\phi^0, S^0, \tau_1^0, \tau_2^0, \nu^0, t^0)$ is a strictly feasible point of problem (3.2.11). Next, we show that the dual of problem (3.2.11) is also strictly feasible. Let

$$X^1 = \begin{pmatrix} X_{11}^1 & X_{12}^1 \\ X_{21}^1 & X_{22}^1 \end{pmatrix}, \; X^2 = \begin{pmatrix} X_{11}^2 & X_{12}^2 \\ X_{21}^2 & X_{22}^2 \end{pmatrix}, \; X^3 = \begin{pmatrix} X_{11}^3 & X_{12}^3 \\ X_{21}^3 & X_{22}^3 \end{pmatrix}, \; x^4 = \begin{pmatrix} x_1^4 \\ x_2^4 \\ x_3^4 \end{pmatrix}$$

be the dual variables corresponding to the first five constraints of the problem (3.2.11), respectively, where $X_{11}^1, X_{11}^2 \in \mathbb{R}^{[(m+1)n]\times[(m+1)n]}$, $X_{12}^1, X_{12}^2 \in \mathbb{R}^{(m+1)n}$, $X_{22}^3 \in \mathbb{R}^{n\times n}$, $X_{21}^3, x_3^4 \in \mathbb{R}^n$, $X_{22}^1, X_{22}^2, X_{11}^3, x_1^4, x_2^4 \in \mathbb{R}$. Let x^5 be the dual variable corresponding to the condition $e^T\phi = 1$. Then the Lagrange function of problem (3.2.11) is defined as

$$\begin{aligned} &\mathscr{L}(\phi, S, \tau_1, \tau_2, \nu, t, X^1, X^2, X^3, x^4, x^5) = -\nu \\ &\quad - X^1 \cdot \begin{pmatrix} \tau_1 R - S \otimes \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} & \tau_1 h + \delta q \\ \tau_1 h^T + \delta q^T & \tau_1 \eta - \epsilon_0^2 t + \delta^2 \end{pmatrix} \\ &\quad - X^2 \cdot \begin{pmatrix} \tau_2 R & \tau_2 h + q \\ \tau_2 h^T + q^T & \tau_2 \eta - 2\nu \end{pmatrix} - X^3 \cdot \begin{pmatrix} 1 & \phi^T \\ \phi & S \end{pmatrix} - x^4 \cdot \begin{pmatrix} 1+t \\ 1-t \\ 2D^{1/2}\phi \end{pmatrix} + x^5(e^T\phi - 1). \end{aligned}$$

$$
\begin{aligned}
\frac{\partial \mathscr{L}}{\partial \phi} &= -2\delta\Psi(X^1_{12}) - 2\Psi(X^2_{12}) - 2X^3_{21} - 2D^{\frac{1}{2}}x^4_3 + x^5 e \geq 0,\\
\frac{\partial \mathscr{L}}{\partial S} &= \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} \odot X^1_{11} - X^3_{22} \succeq 0,\\
\frac{\partial \mathscr{L}}{\partial \tau_1} &= -\begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \cdot X^1 \geq 0,\\
\frac{\partial \mathscr{L}}{\partial \tau_2} &= -\begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \cdot X^2 \geq 0,\\
\frac{\partial \mathscr{L}}{\partial t} &= \epsilon_0^2 X^1_{22} - x^4_1 + x^4_2 = 0,\\
\frac{\partial \mathscr{L}}{\partial \nu} &= -1 + 2X^2_{22} = 0.
\end{aligned}
$$

Then dual of the problem (3.2.11) is

$$
\begin{aligned}
\min_{X^1,X^2,X^3,x^4,x^5} \quad & \delta^2 X^1_{22} + X^3_{11} + x^4_1 + x^4_2 + x^5,\\
\text{s.t} \quad & 2\delta\Psi(X^1_{12}) + 2\Psi(X^2_{12}) + 2X^3_{21} + 2D^{\frac{1}{2}}x^4_3 - x^5 e \leq 0,\\
& \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} \odot X^1_{11} - X^3_{22} \geq 0,\\
& \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \cdot X^k \leq 0, k = 1, 2,\\
& -\epsilon_0^2 X^1_{22} + x^4_1 - x^4_2 = 0,\\
& 2X^2_{22} = 1,\\
& X^1 \succeq 0, X^2 \succeq 0, X^3 \succeq 0, x^4 \in \mathcal{L}^{n+2},
\end{aligned}
$$

where $\Psi : \mathbb{R}^{(m+1)n} \to \mathbb{R}^n$ is defined as $\Psi(x) = (x_1, x_{m+2}, \ldots, x_{(n-2)(m+1)+1}, x_{(n-1)(m+1)+1})^T$ for any $x \in \mathbb{R}^{(m+1)n}$, and

$$\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot X \equiv \left[\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \cdot X_{ij} \right] \in \mathbb{R}^{n \times n}$$

for any $X = (X_{ij}) \in R^{[(m+1)n] \times [(m+1)n]}$ with $X_{ij} \in R^{(m+1)\times(m+1)}$ for $i, j = 1, \cdots, n$.

Let $X^2 = \frac{1}{2(1+\gamma)} \left[\begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \\ 1 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \\ 1 \end{pmatrix}^T + \gamma I \right]$ for some γ and $X^1 = tX^2$ for some t. Then it follows that $2X^2_{22} = 1$ and $X^1_{22} = t/2$. Let $x^4 = (\epsilon_0^2 t/2, 0, \ldots, 0) \in \mathrm{ri}(\mathcal{L}^{n+2})$ so that $-\epsilon_0^2 X^1_{22} + x^4_1 - x^4_2 = 0$. Lu [29] showed that

$$-\begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \cdot X^1 \succ 0$$

and $X^1 \succ 0$ for sufficiently small positive γ. Thus it implies that $X^2 \succ 0$ and

$$-\begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \cdot X^2 = -\frac{1}{2} \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \cdot X^1 \succ 0.$$

Now, let

$$X^3_{22} = t \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} \odot X^1_{11}$$

for some small positive number $t < 1$. Using the fact that

$$\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot X_{11}^1 \succ 0,$$

which is proven by Lu [29], we get

$$\begin{pmatrix} 0 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} \odot X_{11}^1 - X_{22}^3 = (1-t) \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} \odot X_{11}^1 \succ 0.$$

By letting $X_{12}^3 = 0$ and $X_{11}^3 = 1$, the matrix $X^3 \succ 0$. Thus for a sufficiently large number x^5, we can get

$$2\delta\Psi(X_{21}^1) + 2\Psi(X_{12}^2) + 2X_{21}^3 + 2D^{\frac{1}{2}} x_3^4 - x^5 e < 0.$$

Hence, $(X^1, X^2, X^3, x^4, x^5)$ is a strictly feasible solution of the dual problem. $\square$

3.3 Real Market Data Simulation

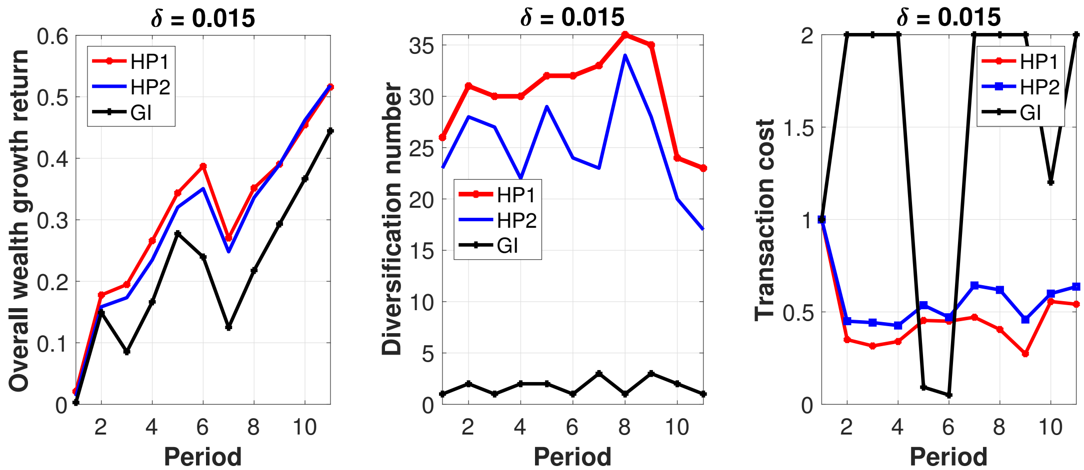
We compute the robust portfolio selection problem with the VaR constraint (3.2.2) on the same real market data in the previous section with the 'joint' and 'separable' uncertainty sets. In total, there are $n = 36$ assets in this set (see Table A.1.3). The set of factors are 6 major market indices (see Table A.1.2). Note that there are 252 trading days per year. The time period is from January 1, 2013 through December 31, 2016, containing 12 periods of length p = 84 trading days, 4 years,

Table 3.3.1: Maximum weight

Period	HP1	HP2	GI	HP1	HP2	GI	HP1	HP2	GI
1	0.0933	0.1114	1	0.1010	0.1010	1	0.1010	0.1010	1
2	0.0881	0.1359	0.9052	0.0687	0.0687	1	0.0687	0.0687	1
3	0.1062	0.1196	1	0.1020	0.1019	1	0.1020	0.1019	1
4	0.0825	0.1105	0.9536	0.0803	0.0803	1	0.0803	0.0803	1
5	0.0550	0.0849	0.9749	0.0550	0.0550	1	0.0550	0.0550	1
6	0.0971	0.1310	1	0.0723	0.0723	1	0.0730	0.0730	1
7	0.0867	0.1082	0.6062	0.1018	0.0982	1	0.0757	0.0758	1
8	0.0719	0.0757	1	0.0812	0.0856	1	0.0844	0.0844	1
9	0.0893	0.1342	0.5418	0.1030	0.1188	1	0.0948	0.0948	1
10	0.1428	0.1909	0.7668	0.1382	0.1405	1	0.0914	0.0914	1
11	0.1015	0.1242	1	0.1035	0.1035	1	0.1035	0.1035	1
Mean	0.0922	0.1206	0.8862	0.0915	0.0933	1	0.0845	0.0845	1
	(a) $\delta = 0.015$			(b) $\delta = 0.02$			(c) $\delta = 0.03$		

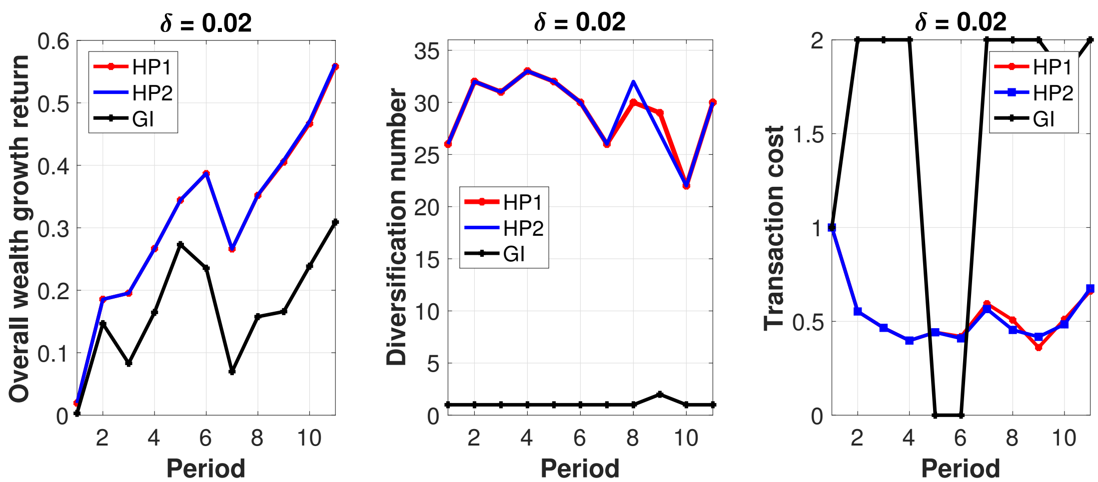
and so there are 11 investment periods. For a given desired confidence level $w > 0$, the confidence level for Goldfarb and Iyengar's separable uncertainty set in [16] is defined as $\tilde{w} = w^{1/n}$. Let the confidence on the set w be 97.5%, thus $\tilde{w}$ is 99.93%. Typical values for the probability ϵ are 1, 2.5, and 5 percent in [26], so we assume the confidence level on the probability constraint to be 97.5%, i.e., $\epsilon = 0.025$. Then the confidence level on the set and the constraint is at least 95%($= 0.975^2$). The symbol 'HP1' represents the VaR constraint problem with estimation (3.2.9) and the symbol 'HP2' represents the problem with estimation (3.2.10) under the joint ellipsoidal uncertainty set. The symbol 'GI' represents the VaR constraint problem under the separable uncertainty sets.

Figure 3.3.1 is the performance of the portfolio when the threshold on the investor's loss δ is 1.5% on the investment, Figure 3.3.2 is the performance when δ is 2%, and Figure 3.3.3 is the performance when δ is 3%. The performance does not



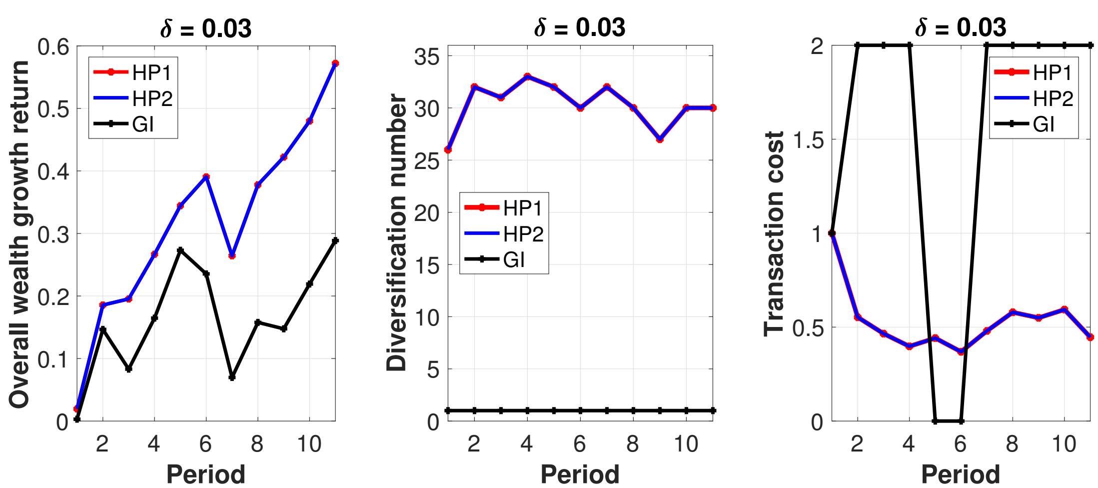
(a) (b) (c)

Figure 3.3.1: Performance of portfolio with $\delta = 0.015$



(a) (b) (c)

Figure 3.3.2: Performance of portfolio with $\delta = 0.02$

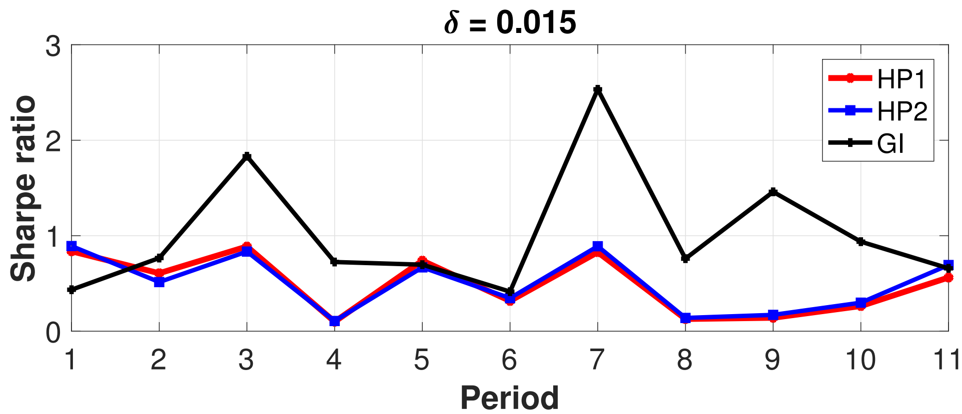


(a) (b) (c)

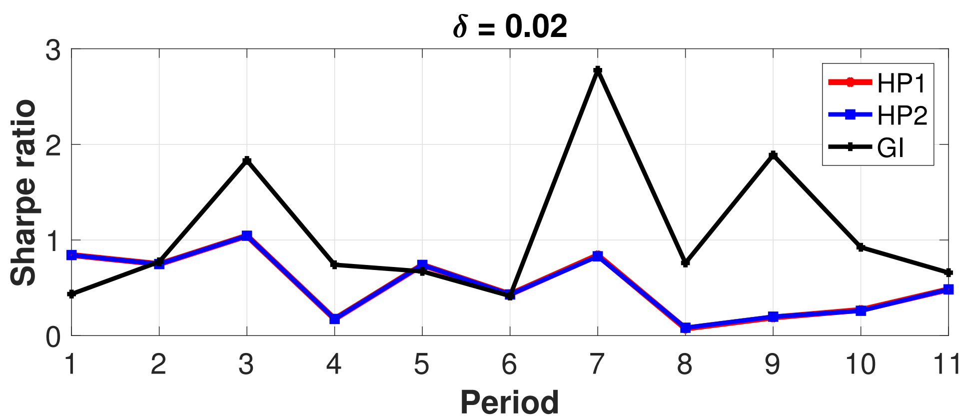
Figure 3.3.3: Performance of portfolio with $\delta = 0.03$

significantly change beyond $\delta = 3\%$ for each model. The 'HP1' and 'HP2' models construct similar portfolios for any δ and have the exact same portfolio when δ is greater than 3%. The 'HP1' and 'HP2' models have better overall wealth growth returns than the 'GI' model regardless δ. The figures of diversification number (3.3.1(b), 3.3.2(b), and 3.3.3(b)) and Table 3.3.1 show that the 'GI' model has a highly non-diversified portfolio. The diversification number of 1 means the investor puts all the portfolio into one stock, so the risk of the portfolio depends entirely on that stock. When $\delta = 0.015$, the 'GI' model includes a small number of stocks, so it is not that well diversified. In Table 3.3.1, the model has still about a 90% weight on one stock when $\delta = 0.015$. So it is not well diversified. On the other hand, the 'HP1' and 'HP2' models construct the portfolios with more than 26 stocks, and the maximum weight of the portfolio is around 10%, so the portfolio is well diversified. All three models start with transaction cost 1 since the initial amount invested in each stock is 0. Transaction cost 0 means no change in the portfolio. Transaction cost 2 means the portfolio is changed completely, which doesn't included any stocks in the previous portfolio, thus cost 1 is from selling all the stocks and the other 1 is to buy all new stocks. Since the 'GI' model includes only one stock in the portfolio, the transaction cost is close to 2 for the most period by selling one stock and buying new stock, while the 'HP1' and 'HP2' models are around 0.5 in Figure 3.3.3(c). Similarly, the 'GI' model has a huge weight on a single stock, the transaction cost is close to 2 in Figure 3.3.1(c) and Figure 3.3.2(c).

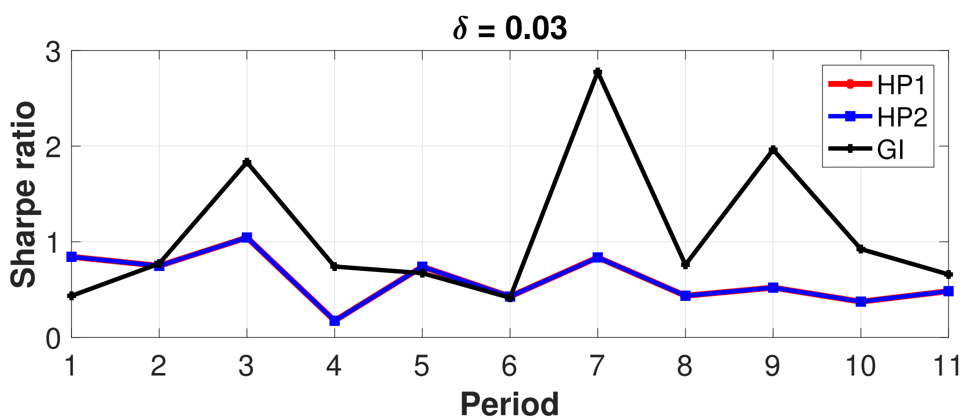
In Table 3.3.2 and Figure 3.3.4, all three models have positive Sharpe ratios. For any δ, the 'GI' model has a higher Sharpe ratio than the other two models since



(a) Sharpe ratio when $\delta = 0.015$



(b) Sharpe ratio when $\delta = 0.02$



(c) Sharpe ratio when $\delta = 0.03$

Figure 3.3.4: Sharpe ratio over the investment period of three different models with $\delta = 0.015, 0.02, 0.03$. The transaction cost is not considered in the return of the portfolio, and the risk-free rate r_f is chosen from the U.S. Treasury bill rate.

Table 3.3.2: Mean Sharpe ratio

δ	HP1	HP2	GI
0.015	0.4917	0.5049	1.0197
0.02	0.5306	0.5302	1.0792
0.03	0.6023	0.6023	1.0860

the 'GI' model constructs a conservative portfolio by putting most of the portfolio into one stock that has high return and low variance.

From the computational results, we see the 'HP1' and 'HP2' models have similar owgr, dn, and tc. Table 3.3.3 shows the difference of weights between the 'HP1' and 'HP2' models, which is calculated by the absolute sum of the weights on each period:

$$\|\phi^t_{\mathrm{HP1}} - \phi^t_{\mathrm{HP2}}\|_1,\ t = 1, \ldots, 11.$$

Two models have different estimations on the VaR constraint, but they have similar

Table 3.3.3: Difference of weights between the HP1 and HP2 models

Period	0.015	0.02	0.03
1	0.1816	0.0001	0.0000
2	0.2536	0.0002	0.0001
3	0.2136	0.0002	0.0001
4	0.2336	0.0001	0.0001
5	0.3710	0.0000	0.0001
6	0.2465	0.0160	0.0001
7	0.2604	0.0570	0.0001
8	0.2202	0.0918	0.0002
9	0.3021	0.0558	0.0001
10	0.2561	0.0397	0.0003
11	0.2519	0.0002	0.0002
Mean	0.2537	0.0237	0.0001

weights on each period. As δ gets bigger, the two models construct closer portfolios.

From the result, the problem with the actual VaR constraint is also expected to have similar weights as these two models.

3.4 Discussion

In this chapter, we used the factor model to define the random asset returns and solved the robust VaR problem under the joint ellipsoidal uncertainty set with estimations on the VaR constraint. This set is less conservative and produces a diversified portfolio compared to the separable uncertainty set. The difficult part in solving the robust VaR problem is that the VaR constraint is non-convex, so we estimate the constraint to be semidefinite constraint, which is easier to handle.

For the VaR constraint, the 'HP1' model uses estimation (3.2.9), which implies the VaR constraint, so the estimation is a strong condition. The 'HP2' model uses estimation (3.2.10), in which the VaR constraint implies the estimation, so the estimation is a weak condition. From the simulation, the 'HP1' and 'HP2' models construct very close optimal portfolios. Thus we can conclude that the VaR problem constructs the close portfolio and so the 'HP1' and 'HP2' models are reasonable estimations.

Chapter 4

Robust Portfolio Problem in the Presence of Transactions Costs under Joint Uncertainty Set

4.1 Overview

When investors rebalance their portfolios, they need to consider the transactions costs-adjusted return of the portfolio, which is defined as

$$(P_1 - P_0 - \mathrm{Tc})/P_0,$$

where P_1 is the value of portfolio in the next period, P_0 is the current portfolio value, and Tc is transactions costs. The next simple example will show the effect of the transactions costs on return. Consider two assets, A and B, where asset A has

Table 4.1.1: No transactions costs

Assets	Return	Initial amount	Rebalance
A	5%	\$100	\$0
B	5.1%	\$0	\$100
Expected Profit		\$5	\$5.1

expected return 5% and asset B has expected return 5.1%. The initial amount of money in asset A is \$100 and no money is in asset B. In the case of no transactions costs considered (see Table 4.1.1), the investor will rebalance the portfolio by moving all of money into asset B to maximize the expected profit. Consider transactions

Table 4.1.2: 0.5% transactions costs

Assets	Return	Initial amount	Rebalance
A	5%	\$100	\$0
B	5.1%	\$0	\$100
Expected Profit		\$5	\$4.1

costs rate of 0.5% to buy or sell assets. The expected profit without rebalancing is \$5. The expected profit putting all of the money into asset B gives \$4.1 since the transactions costs of \$0.5 to sell asset A and \$0.5 to buy asset B are deducted from the profit (see Table 4.1.2). In the presence of the 0.5% transactions costs, it is better to keep the same portfolio to get more profit. This simple example shows that the small transactions costs affects portfolio so we construct totally different portfolio from the case without the transactions costs.

In this chapter, the robust VaR constraint portfolio model under the joint ellipsoidal uncertainty set in Chapter 3 will be extended to a model which considers transactions costs. The models in Chapter 3 find the optimal weight of assets to construct the portfolio, but the models did not consider the initial portfolio. However, the model in this chapter will find the optimal weights that rebalance the initial portfolio to maximize the transactions costs-adjusted return. To involve the costs, we use similar notation from the multi-period portfolio problem (1.4.28) that Bertisimas and Pachamanova provided in [6]. The next section derives the robust VaR constraint problem, which maximizes transactions costs-adjusted return while having a threshold on the return as a constraint. The computational results on the real market data of this model are presented in Section 4.3 and they are compared to the model in Chapter 3. In Section 4.4, we extend the RMRAR model introduced in Section 1.4.5 to consider transactions costs. The computational results of the model are provided in Section 4.5 using the same data used in Section 4.3 to compare the results.

4.2 The Robust VaR Constraint in the Presence of Transactions Costs

To introduce transactions costs, we use similar notations from the multi-period maximum final wealth problem (1.4.28) that Bertsimas and Pachamanova [6] provided. For a single period, the following notation will be used:

- x_i^0 and x_i^1 are the investor's initial and final dollar holdings on stock i.
- x_0^0 and x_0^1 are the investor's initial and final cash holdings.
- y_i and z_i are the amount for the investor to sell and buy the stock i.
- $c_{sell}y_i$, and $c_{buy}z_i$ are the transactions costs to sell and buy.
- $\tilde{r}_i$ and r_0 are the uncertainty returns on stock i and the certain cash return.

Using the notations, the dollar holding on stock i and cash holding after one period are defined as

$$
\begin{aligned}
&x_i^1 = (1+\tilde{r}_i)(x_i^0 - y_i + z_i),\ i = 1, 2, \ldots, n, \\
&x_0^1 = (1+r_0)\left(x_0^0 + \sum_{i=1}^{n}(1-c_{sell})y_i - \sum_{i=1}^{n}(1+c_{buy})z_i\right).
\end{aligned}
$$

Then $\sum_{i=0}^{n} x_i^1$ represents the final wealth, and $\sum_{i=0}^{n} x_i^0$ represents the initial wealth. The transactions costs-adjusted return that we want to maximize is defined by profit over initial wealth, i.e.,

$$
\left(\sum_{i=0}^{n} x_i^1 - \sum_{i=0}^{n} x_i^0\right) \Big/ \sum_{i=0}^{n} x_i^0 \ = \ \sum_{i=0}^{n} x_i^1 \Big/ \sum_{i=0}^{n} x_i^0 - 1.
$$

Using the notations, the robust VaR portfolio problem under the joint uncertainty set in the presence of transactions costs can be formulated as

$$
\begin{aligned}
\max_{x^1,y,z}\ \ & \min_{(\mu,V)\in S_{m,v}} \mathbf{E}\left(\frac{\sum_{i=0}^{n} x_i^1}{\sum_{i=0}^{n} x_i^0} - 1\right) \\
\text{s.t}\ \ & \max_{(\mu,V)\in S_{m,v}} \mathbf{P}\left(\frac{\sum_{i=0}^{n} x_i^1}{\sum_{i=0}^{n} x_i^0} - 1 \le -\delta\right) \le \epsilon, \\
& x_i^1 = (1+\tilde{r}_i)(x_i^0 - y_i + z_i),\ i = 1, 2, \ldots, n, \\
& x_0^1 = (1+r_0)\left(x_0^0 + \sum_{i=1}^{n}(1-c_{sell})y_i - \sum_{i=1}^{n}(1+c_{buy})z_i\right), \\
& y \ge 0, z \ge 0.
\end{aligned}
\tag{4.2.1}
$$

In the objective function, the constant 1 can be dropped. For given initial amount in stock i, x_i^0, let $\bar{c}_i$ be the dollar amount after reconstructing portfolio on stock i. Then $\bar{c} \in \mathbb{R}^n$ is a column vector that represents a dollar holding on stocks after reconstructing, i.e., $\bar{c}_i = (x_i^0 - y_i + z_i),\ i = 1, \ldots, n$. The transaction-adjusted return can be written as

$$\frac{\sum_{i=1}^n x_i^1}{\sum_{i=0}^n x_i^0} - 1 \ = \ \frac{\sum_{i=1}^n (1+\tilde{r}_i)\bar{c}_i}{\sum_{i=0}^n x_i^0} - 1 \ = \ \sum_{i=1}^n (1+\tilde{r}_i)c_i - 1, \tag{4.2.2}$$

where $c = \bar{c}/(\sum_{i=0}^n x_i^0)$. By using the new variable c, variable x^1 can be removed in the objective function and probability constraint. The new variable c is an non-negative vector, which means no short selling is allowed. The sum of its entries is

$$\begin{aligned}
e^T c = \frac{e^T(x^0 - y + z)}{\bar{x}} &= \frac{e^T x^0 - e^T y + e^T z}{e^T x^0 + x_0^0} \\
&= \frac{e^T x^0 - (1 - c_{sell})e^T y + (1 + c_{buy})e^T z - c_{sell}e^T y - c_{buy}e^T z}{e^T x^0 + x_0^0} \\
&= \frac{x_0^0 + e^T x^0 - c_{sell}e^T y - c_{buy}e^T z}{e^T x^0 + x_0^0} \\
&= 1 - \frac{c_{sell}e^T y + c_{buy}e^T z}{e^T x^0 + x_0^0}
\end{aligned}$$

where $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$ is the column vector with all entries equal to one. Let $tcr_i = (c_{sell}e^T y_i + c_{buy}e^T z_i)/(e^T x^0 + x_0^0)$ be the portion of the transactions costs incurred from stock i on total investment. Then the variable of sum $c+tcr$ is defined as the allocation vector $\phi \in \Phi = \{\phi \in \mathbb{R}^n : \phi_i \geq 0, e^T\phi = 1\}$. To allocate all money into stocks, set the cash holding after reconstructing portfolio equal to zero,

$$x_0^0 + \sum_{i=1}^{n}(1 - c_{sell})y_i - \sum_{i=1}^{n}(1 + c_{buy})z_i = 0.$$

Consequently, the final cash holding x_0^1 also becomes zero. By using new variable c and the fact no cash holding after rebalancing is allowed, the problem (4.2.1) becomes

$$\begin{aligned}
&\max_{c,y,z} \min_{(\mu,V)\in S_{m,v}} \mathbf{E}(c^T(e+\tilde{r})) \\
&\text{s.t} \quad \max_{(\mu,V)\in S_{m,v}} \mathbf{P}(c^T(e+\tilde{r}) \leq (1-\delta)) \leq \epsilon, \\
&\qquad x_0^0 = -(1-c_{sell})e^Ty + (1+c_{buy})e^Tz, \\
&\qquad \bar{x}c = x^0 - y + z, \\
&\qquad c \geq 0, y \geq 0, z \geq 0.
\end{aligned} \tag{4.2.3}$$

Define an unknown return vector $R \in \mathbb{R}^n$ as $R = e + r = (e + \mu) + V^T f + \epsilon$. Then the multi-period factor model is defined as the following:

$$y_i = Ax_i + \epsilon_i, \quad \text{for all} \quad i = 1, \ldots, n,$$

where $y_i = (1 + r_i^1, \ldots, 1 + r_i^p)^T$, $A = (e\ B^T)$, $x_i = (1 + \mu_i, V_{1i}, \ldots, V_{mi})^T$, and $\epsilon_i = (\epsilon_i^1, \ldots, \epsilon_i^p)^T$ for $i = 1, \ldots, n$. The least squares estimate $\bar{x}_i$ of the true parameter

x_i is given by $\bar{x}_i = (A^T A)^{-1} A^T y_i$. The joint ellipsoidal uncertainty set is defined as

$$S_{m,v}(w) = \left\{ (\mu, V) \in \mathbb{R}^n \times \mathbb{R}^{m \times n} : \sum_{i=1}^{n} \frac{(x_i - \bar{x}_i)^T (A^T A)(x_i - \bar{x}_i)}{s_i^2 (m+1)} \leq c(w) \right\} \quad (4.2.4)$$

for some $c(w)$, where $x_i = (1+\mu_i, V_{1i}, V_{2i}, \ldots, V_{mi})^T$ for $i = 1, \ldots, n$. Since the return $R \sim \mathcal{N}(e + \mu, V^T F V + D)$, the probability constraint is changed to

$$\begin{aligned}
\mathbf{P}(c^T R \leq 1 - \delta) \leq \epsilon &\Leftrightarrow \mathbf{P}((e+\mu)^T c + \mathscr{Z} \sqrt{c^T (V^T F V + D) c} \leq 1 - \delta) \leq \epsilon \\
&\Leftrightarrow \mathbf{P}\left(\mathscr{Z} \leq \frac{-(e+\mu)^T c + 1 - \delta}{\sqrt{c^T (V^T F V + D) c}} \right) \leq \epsilon \\
&\Leftrightarrow \frac{-(e+\mu)^T c + 1 - \delta}{\sqrt{c^T (V^T F V + D) c}} \leq \mathcal{F}^{-1}(\epsilon) \\
&\Leftrightarrow -\mathcal{F}^{-1}(\epsilon) \sqrt{c^T (V^T F V + D) c} \leq (e+\mu)^T c - (1 - \delta) \\
&\Leftrightarrow \ \epsilon_0^2 c^T (V^T F V + D) c \leq ((e+\mu)^T c - \delta_0)^2
\end{aligned}$$

where $\epsilon_0 = -\mathcal{F}^{-1}(\epsilon)$ and $\delta_0 = 1 - \delta$. By the above equivalent constraint and introducing new variable ν on the objective function, the problem (4.2.3) can be derived as follow

$$
\begin{aligned}
\max_{c,y,z,\nu} \quad & \nu \\
\text{s.t} \quad & \epsilon_0^2 c^T(V^T F V + D)c - ((e+\mu)^T c - \delta_0)^2 \leq 0, && \forall (\mu, V) \in S_{m,v}, \\
& \nu - (e+\mu)^T c \leq 0, && \forall (\mu, V) \in S_{m,v}, \\
& x_0^0 = -(1 - c_{sell}) e^T y + (1 + c_{buy}) e^T z, \\
& \bar{x} c = x^0 - y + z, \\
& c \geq 0, y \geq 0, z \geq 0.
\end{aligned}
\tag{4.2.5}
$$

Let δ^* is the minimum of the WVaR over c, i.e.,

$$
\delta^* := \max_{c} \max_{(\mu,V) \in S_{m,v}} \epsilon_0 \sqrt{c^T(V^T F V + D)c} - \mu^T c.
$$

By the definition of δ^*, for any $\delta > \delta^*$, there exists a solution c such that

$$
\max_{(\mu,V) \in S_{m,v}} \epsilon_0^2 c^T(V^T F V + D)c - (\mu^T c + \delta)^2 \leq 0.
$$

First two constraints in problem (4.2.5) can be changed to semidefinite constraints, which are shown in the next lemma.

Lemma 4.2.1. *Let $S_{m,v}$ be an w-confidence uncertainty set given in (4.2.4) for*

$w \in (0,1)$. *Then, the first two constraints in problem (4.2.5) are equivalent to*

$$\begin{pmatrix} \tau_1 R - (cc^T) \otimes \begin{pmatrix} -1 & 0 \\ 0 & \epsilon_0 F \end{pmatrix} & \tau_1 h - \delta_0 q \\ \tau_1 h^T - \delta_0 q^T & \tau_1 \eta - \epsilon_0^2 t + \delta_0^2 \end{pmatrix} \succeq 0, \tag{4.2.6}$$

$$\begin{pmatrix} \tau_2 R & \tau_2 h + q \\ \tau_2 h^T + q^T & \tau_2 \eta - 2\nu \end{pmatrix} \succeq 0, \ \ \tau_1, \tau_2 \geq 0,$$

where

$$R = \begin{pmatrix} \frac{A^T A}{s_1^2(m+1)} & & \\ & \ddots & \\ & & \frac{A^T A}{s_n^2(m+1)} \end{pmatrix} \in \mathbb{R}^{[(m+1)n] \times (m+1)n}, \qquad \eta = \sum_{i=1}^{n} \bar{x}_i^T \left(\frac{A^T A}{s_i^2(m+1)} \right) \bar{x}_i - c(w),$$

$$h = \begin{pmatrix} -\frac{A^T A \bar{x}_1}{s_1^2(m+1)} \\ \vdots \\ -\frac{A^T A \bar{x}_n}{s_1^2(m+1)} \end{pmatrix} \in \mathbb{R}^{(m+1)n}, \qquad q = (c_1, 0 \ldots, c_n, 0)^T \in \mathbb{R}^{(m+1)n}.$$

Proof. Given any $(t, \nu, c) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$, we define

$H(\mu, V) = \epsilon_0^2 c^T V^T F V c - ((1+\mu)^T c)^2 + 2\delta_0 (1+\mu)^T c - \delta_0^2 + \epsilon_0^2 t \leq 0$ and

$L(\mu, V) = \nu - (1+\mu)^T c \leq 0.$

Since $x_i = (1+\mu_i, V_{1i}, V_{2i}, \ldots, V_{mi})^T$ for $i = 1, \ldots, n$,

$$\frac{\partial H}{\partial x_i} = \begin{pmatrix} 2\delta c_i - 2(\mu^T c)c_i \\ \\ 2\epsilon_0^2 c_i F V c \end{pmatrix}, \quad \frac{\partial^2 H}{\partial x_i \partial x_j} = \begin{pmatrix} -2c_i c_j & 0 \\ \\ 0 & 2\epsilon_0^2 c_i c_j F \end{pmatrix}$$

$$\frac{\partial H}{\partial x_i}(0,0) = \begin{pmatrix} 2\delta c_i \\ \\ 0 \end{pmatrix}, \quad \frac{\partial^2 H}{\partial x_i \partial x_j}(0,0) = \begin{pmatrix} -2c_i c_j & 0 \\ \\ 0 & 2\epsilon_0^2 c_i c_j F \end{pmatrix}$$

By the Taylor series expansion for $H(e+\mu, V)$ at $x = 0$, we get

$$\begin{aligned} H(\mu, V) &= \frac{1}{2} \sum_{i,j=1}^{n} x_i^T \begin{pmatrix} -2c_i c_j & 0 \\ \\ 0 & 2\epsilon_0^2 c_i c_j F \end{pmatrix} x_j + \sum_{i=1}^{n} \begin{pmatrix} 2\delta_0 c_i \\ \\ 0 \end{pmatrix} x_i + \epsilon_0^2 t - \delta_0^2 \\ &= \sum_{i,j=1}^{n} x_i^T \begin{pmatrix} -c_i c_j & 0 \\ \\ 0 & \epsilon_0^2 c_i c_j F \end{pmatrix} x_j + 2 \sum_{i=1}^{n} \begin{pmatrix} \delta_0 c_i \\ \\ 0 \end{pmatrix} x_i + \epsilon_0^2 t - \delta_0^2 + \nu. \end{aligned}$$

Similarly, the Taylor series expansion for $L(\mu, V)$ at $x = 0$, we obtain

$$\frac{\partial L}{\partial x_i}(0,0) = \begin{pmatrix} -c_i \\ \\ 0 \end{pmatrix}, L(\mu, V) = \sum_{i=1}^{n} \begin{pmatrix} -c_i \\ \\ 0 \end{pmatrix} x_i + \nu.$$

The joint uncertainty set in (4.2.4), $S_{m,v}$ can be written as

$$\begin{aligned} S_{m,v} = \Bigg\{ &(\mu, V) \in \mathbb{R}^n \times \mathbb{R}^{m \times n} : \sum_{i=1}^{n} x_i^T \left(\frac{A^T A}{(m+1)s_i^2} \right) x_i \\ &+ 2 \sum_{i=1}^{n} \left(\frac{-A^T A \bar{x}_i}{(m+1)s_i^2} \right)^T x_i + \sum_{i=1}^{n} \bar{x}_i^T \left(\frac{A^T A}{(m+1)s_i^2} \right) \bar{x}_i - c(w) \leq 0 \Bigg\}. \quad (4.2.7) \end{aligned}$$

We see that $x = \bar{x}$ satisfies the strict inequality given in (4.2.7). By using Lemma

3.3.1, we can conclude that $H(\mu, V) \leq 0$ and $L(\mu, V) \leq 0$ for all $(\mu, V) \in S_{m,v}$ if and only if there exist $\tau_1, \tau_2 \in \mathbb{R}$ such that

$$\tau_1 \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} - \begin{pmatrix} (cc^T) \otimes \begin{pmatrix} -1 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} & \delta_0 q \\ \delta_0 q^T & \epsilon_0^2 t - \delta_0^2 + \nu \end{pmatrix} \succeq 0, \quad \tau_1 \geq 0,$$

$$\tau_2 \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} - \begin{pmatrix} 0 & -q \\ -q^T & 2\nu \end{pmatrix} \succeq 0, \quad \tau_2 \geq 0.$$

$\square$

By the lemma, the VaR constraint in problem (4.2.5) has the equivalent semidefinite constraint as the one in (4.2.6). For simplicity, we replace the quadratic variable cc^T by semidefinite matrix $S \in \mathbb{R}^n_+$ so that $S = cc^T$. We relax the constraint by $S \succeq cc^T$ and add more conditions to reduce the gap between S and cc^T. Since cc^T is a doubly non-negative matrix constraint, so is S. By the fact that $c^T e \leq 1$,

$$Se = c(c^T e) \leq c, \; e^T Se \leq e^T c \leq 1.$$

The estimated constraint of (4.2.6) is

$$\begin{pmatrix} \tau_1 R - S \otimes \begin{pmatrix} -1 & 0 \\ 0 & \epsilon_0 F \end{pmatrix} & \tau_1 h - \delta_0 q \\ \tau_1 h^T - \delta_0 q^T & \tau_1 \eta - \epsilon_0^2 t + \delta_0^2 \end{pmatrix} \succeq 0, \; S \succeq cc^T, \; Se \leq c, \; S_{ij} \geq 0.$$

Theorem 4.2.2. *Let $S_{m,v}$ be an w-confidence uncertainty set given in (4.2.4) for $w \in (0,1)$. Then, problem (4.2.5) with estimation on VaR constraint is equivalent to*

$$
\begin{aligned}
\max_{\nu,c,y,z,\tau_1,\tau_2,S,t} \quad & \nu \\
s.t \quad & \begin{pmatrix} \tau_1 R - S \otimes \begin{pmatrix} -1 & 0 \\ 0 & \epsilon_0 F \end{pmatrix} & \tau_1 h - \delta_0 q \\ \tau_1 h^T - \delta_0 q^T & \tau_1 \eta - \epsilon_0^2 t + \delta_0^2 \end{pmatrix} \succeq 0, \\
& \begin{pmatrix} \tau_2 R & \tau_2 h + q \\ \tau_2 h^T + q^T & \tau_2 \eta - 2\nu \end{pmatrix} \succeq 0, \\
& \begin{pmatrix} 1 & c^T \\ c & S \end{pmatrix} \succeq 0, \qquad (4.2.8) \\
& \begin{pmatrix} 1+t \\ 1-t \\ 2D^{1/2}c \end{pmatrix} \in \mathcal{L}^{n+2}, \\
& c - Se \geq 0, \\
& S_{ij} \geq 0, \ \forall i,j, \\
& x^0 - y + z - \bar{x}c = 0, \\
& x_0^0 + (1 - c_{sell})e^T y - (1 + c_{buy})e^T z \geq 0, \\
& c, y, z \geq \ 0, \tau_1, \tau_2 \geq 0,
\end{aligned}
$$

where R, h, η, and q are the same as those defined in Lemma 4.2.1.

4.3 Real Market Data Simulation

We compute the problem (4.2.8) on real market data and compare to the VaR model without considering transactions costs in Section 3. The same assets and factors are used from Section 3.4. In total, there are $n = 36$ assets in this set (see Table A.1.3). The set of factors are $m = 6$ major market indices (see Table A.1.2). The time period is from January 1, 2013 through December 31, 2016, containing 12 periods of length $p = 84$ trading days over 4 years, so there are 11 investment periods. For each investment period t, the factor covariance matrix F is computed on the factor returns of the previous trading period, and the upper bound of the variance $\bar{d}_i$ of the residual return is computed to be $\bar{d}_i = s_i^2$, where s_i^2 is given in Proposition 1.4.1. Let the confidence level w be 97.5%. The symbol 'HP2' represents the robust VaR constraint problem solved using (3.2.10). The symbol 'HPtc' represents problem (4.2.8) that maximizes transactions costs-adjusted return while imposing threshold on the transactions costs-adjusted return.

We define transactions costs as the 1-norm of difference between the weight vectors of current period and previous period. The transactions costs percentage of the pervious wealth for the 'HP2' model are the from the results of Section 3.3. For the models considering transactions costs, the cost can be directly obtained from variables y, the amount to sell, and z, the amount to buy. The *Rebalancing ratio*, transactions costs percentage of the wealth on period p, $p = 1, \ldots, 11$, is defined by

$$\frac{e^T(y+z)}{\sum_{i=1}^{n} x_i^p + x_0^p} \times 100\ (\%).$$

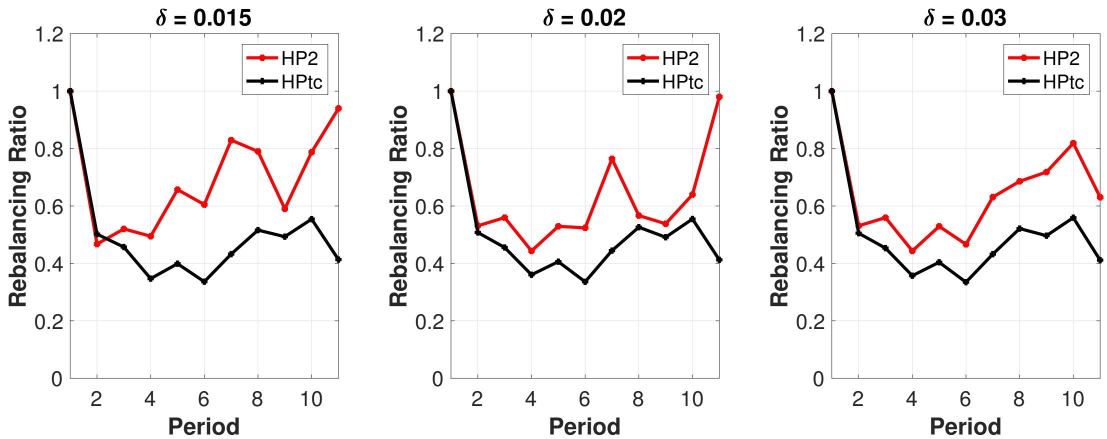


Figure 4.3.1: Rebalancing ratio with no transactions costs

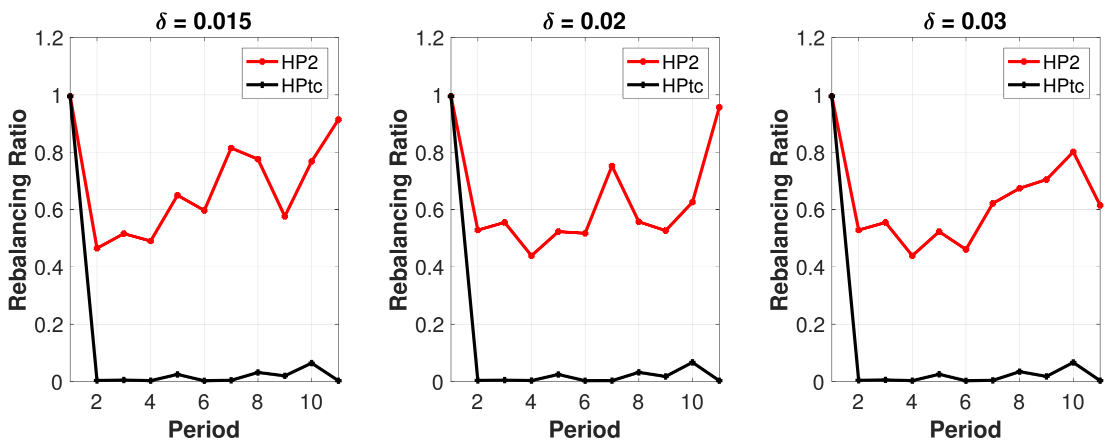


Figure 4.3.2: Rebalancing ratio with 0.5% transactions costs

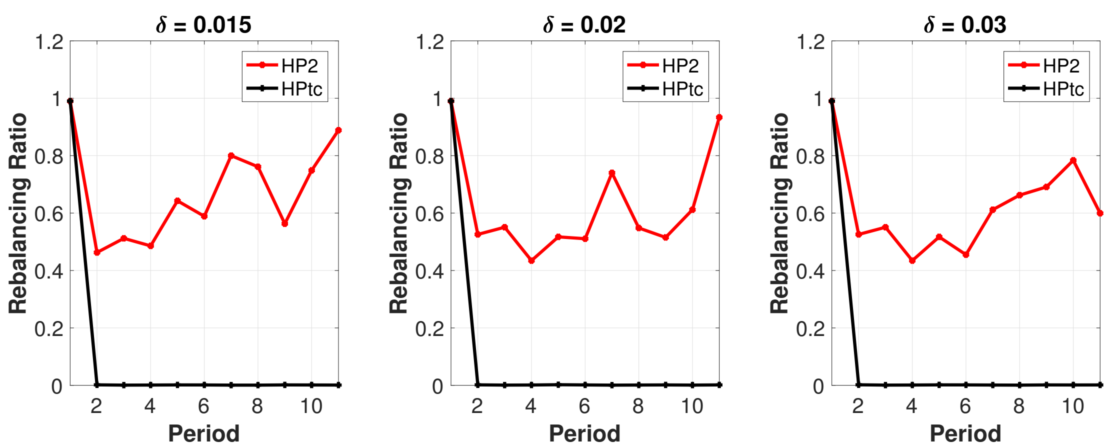


Figure 4.3.3: Rebalancing ratio with 1% transactions costs

Figure 4.3.1 represents the rebalancing ratio when transactions costs are 0. Figure 4.3.2 represents the ratio when transactions costs have rate 0.5%. The ratio of the 'HPtc' model decreases significantly since the portfolio does not make profit more than the cost when rebalanced. When transactions costs rate is 1%, the rebalancing ratio is negligible compared to the 'HP2' model since the portfolio does not change for this same reason. As transactions costs rate increases, rebalancing ratio of the 'HPtc' model decreases but it does not affect the 'HP2' model. Table 4.3.1 is the average transactions costs over 10 periods excluding the costs for the first period for the different threshold on return δ and transactions costs rate. Both the 'HPtc' model decreases as transactions costs rate increases.

Table 4.3.1: Mean rebalancing ratio

	0.015	0.02	0.03
0	0.6681	0.6075	0.6014
0.1	0.6658	0.6056	0.5995
0.5	0.6566	0.5981	0.5922
1	0.6454	0.5887	0.5831

(a) HP2

	0.015	0.02	0.03
0	0.4450	0.4494	0.4475
0.1	0.1689	0.1697	0.1712
0.5	0.0165	0.0165	0.0170
1	0.0011	0.0013	0.0013

(b) HPtc

The overall wealth growth return on the period $k = 1, \ldots, 11$ for the 'HPtc' models are defined as

$$\text{owgr}_k = \prod_{1 \le t \le k} \left[(r^t + e)^T c^t \right] - 1 \;=\; \prod_{1 \le t \le k} \left[1 + ((r^t)^T c^t - tcr^t) \right] - 1$$

where $tcr^t = (c_{sell} e^T y_i^t + c_{buy} e^T z_i^t)/\bar{x}^{t-1}$ is the portion of transactions costs incurred from stock i on wealth on the previous period. The overall wealth growth return for the 'HP2' model is in Figure 3.3.1(a), 3.3.2(a), and 3.3.3(a). Figure 4.3.4 plots the overall wealth growth return when there is no transactions costs, Figure 4.3.5 shows the same plots with a transactions costs rate of 0.5% amount of buying or selling stocks, and Figure 4.3.6 shows the same plots with a transactions costs rate of 1%. When there is no transaction fee, the owgr graphs of both models coincide. When transactions costs exist, the owgr graph of the 'HP2' model is under the 'HPtc' model for the most period of time. As the costs rate increases, the difference of the two models gets bigger.

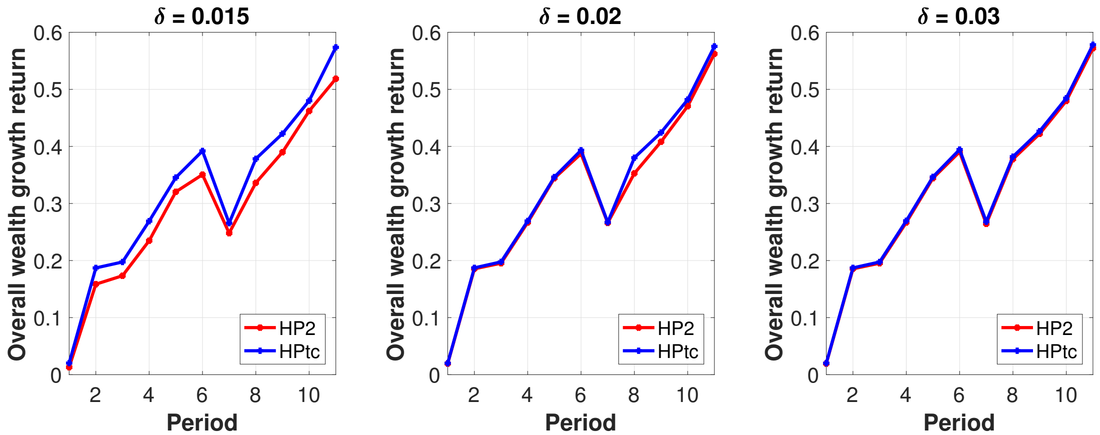


Figure 4.3.4: Overall wealth growth return with no transactions costs

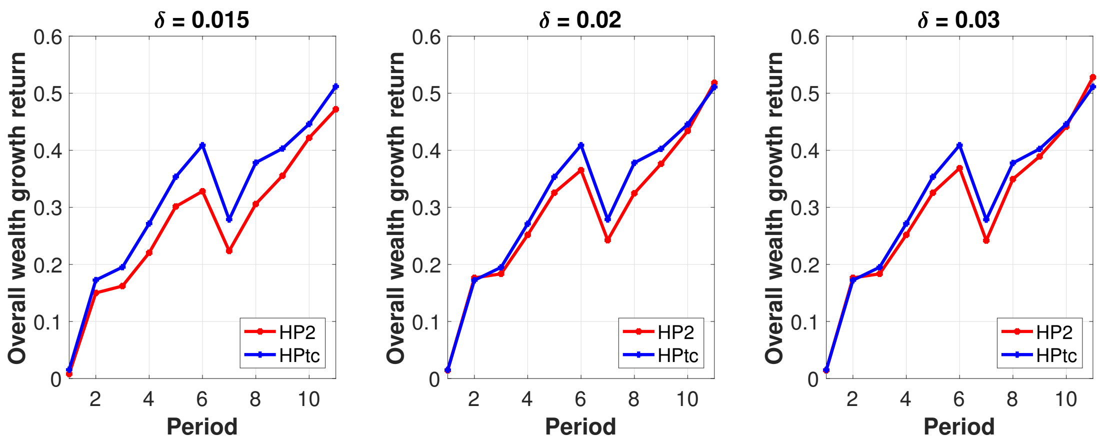


Figure 4.3.5: Overall wealth growth return with 0.5% transactions costs

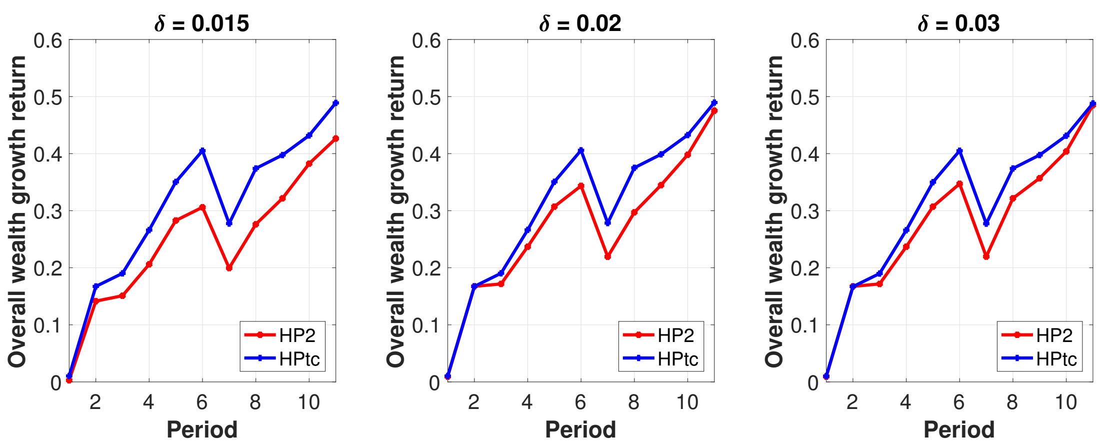


Figure 4.3.6: Overall wealth growth return with 1% transactions costs

4.4 The RMRAR Problem with Transactions Costs

In this section, we extend the RMRAR problem (1.4.27) under the joint ellipsoidal uncertainty set to the problem in the presence of transactions costs. The problem is

$$\max_{\phi\in\Phi} \min_{(\mu,V)\in S_{m,v}} \mathbf{E}[\tilde{r}^T\phi] - \theta\mathbf{Var}[\tilde{r}^T\phi] \tag{4.4.1}$$

where $\theta \geq 0$ is the risk-aversion parameter. The definition of transactions costs-adjusted return is defined same as in (4.2.2). Using the definition and notations from Section 4.2, problem 4.4.1 can be rewritten as

$$\begin{aligned}
&\max_{c,y,z} \min_{(\mu,V)\in S_{m,v}} \mathbf{E}[c^T(e+\tilde{r})] - \theta\mathbf{Var}[c^T(e+\tilde{r})], \\
&\text{s.t} \;\; x_0^0 = -(1-c_{sell})e^T y + (1+c_{buy})e^T z, \\
&\qquad c = (x^0 - y + z)/\bar{x}, \\
&\qquad c \geq 0, y \geq 0, z \geq 0,
\end{aligned} \tag{4.4.2}$$

where $\bar{x} = \sum_{i=0}^n x_i^0$ is the initial wealth. Using the definition of stock returns from a factor model (1.4.13), the transactions costs-adjusted portfolio return is normally distributed with mean $c^T(e+\mu)$ and variance $c^T(V^TFV+D)c$. Then, the objective function in problem (4.4.2) is changed to

$$\min_{(\mu,V)\in S_{m,v}} \left\{ c^T(e+\mu) - \theta c^T V^T F V c \right\} - \theta c^T D c.$$

Using the slack variables ν and t in the objective function, problem (4.4.2) is equivalent to

$$\begin{aligned}
&\max_{c,y,z,\nu,t} \ \nu - \theta t \\
&\text{s.t} \quad \min_{(\mu,V)\in S_{m,v}} c^T(e+\mu) - \theta c^T V^T F V c \geq \nu, \\
&\qquad c^T D c \leq t, \\
&\qquad x_0^0 = -(1-c_{sell})e^T y + (1+c_{buy})e^T z, \\
&\qquad c = (x^0 - y + z)/\bar{x}, \\
&\qquad c \geq 0, y \geq 0, z \geq 0.
\end{aligned} \tag{4.4.3}$$

The first constraint in (4.4.3) has an equivalent semidefinite constraint, which can be easily seen from Lemma 4.3 of [29].

Lemma 4.4.1. *The first constraint in (4.4.3) is equivalent to*

$$\begin{pmatrix} \tau R - 2\theta S \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} & \tau h + q \\ \tau h^T + q^T & \tau\eta - 2\nu + 2(c^T e) \end{pmatrix} \succeq 0, \ \ S \succeq \phi\phi^T, \ \ \tau \geq 0.$$

Proof. This is easy to get by moving $c^T e$ to the right hand side:

$$\min_{(\mu,V)\in S_{m,v}} c^T\mu - \theta c^T V^T F V c \geq (\nu - c^T e).$$

$\nu - c^T e$ is according to ν in Lemma [29]. □

Theorem 4.4.2. *Let $S_{m,v}$ be an w-confidence uncertainty set given in (3.2.1) for $w \in (0,1)$. Then, problem (4.4.3) is equivalent to*

$$
\begin{aligned}
\max_{c,y,z,S,\tau,\nu,t} \quad & \nu - \theta t \\
s.t \quad & \begin{pmatrix} \tau R - 2\theta S \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} & \tau h + q \\ \tau h^T + q^T & \tau\eta + 2c^T e - 2\nu \end{pmatrix} \succeq 0, \\
& \begin{pmatrix} 1 & c^T \\ c & S \end{pmatrix} \succeq 0, \\
& \begin{pmatrix} 1+t \\ 1-t \\ 2D^{1/2}c \end{pmatrix} \in \mathcal{L}^{n+2}, \\
& c = (x^0 - y + z)/\bar{x} \\
& x_0^0 = -(1-c_{sell})e^T y + (1+c_{buy})e^T z \\
& c, y, z \geq 0, \tau \geq 0,
\end{aligned}
\tag{4.4.4}
$$

where R, h, η, and q are the same as those defined in Lemma 3.2.4

4.5 Real Market Data Simulation

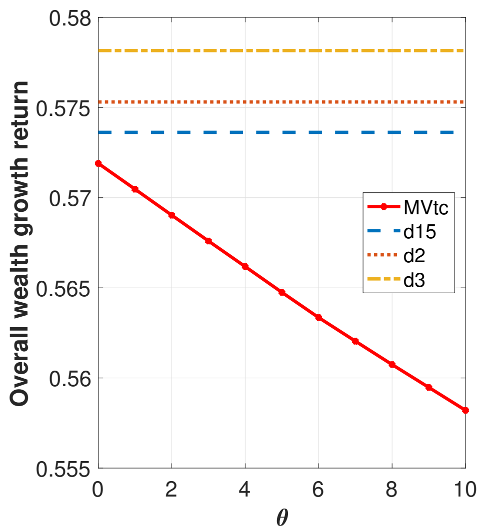
We compute the problem (4.4.4) on the same real market data in Section 4.3. The symbol 'MVtc' represents the RMRAR model under the joint uncertainty set in the presence of transactions costs, and the symbol 'HPtc' represents the robust

VaR model under the same set in the presence of the costs, that were derived in Section 4.2. The overall wealth growth return is defined as in Section 4.3. The *average diversification number* is defined as $\sum_{p=1}^{11} I(\phi^p)/11$, where $I(\phi^p)$ denotes the diversification number of the portfolio ϕ^p. The *average transactions costs* is defined as

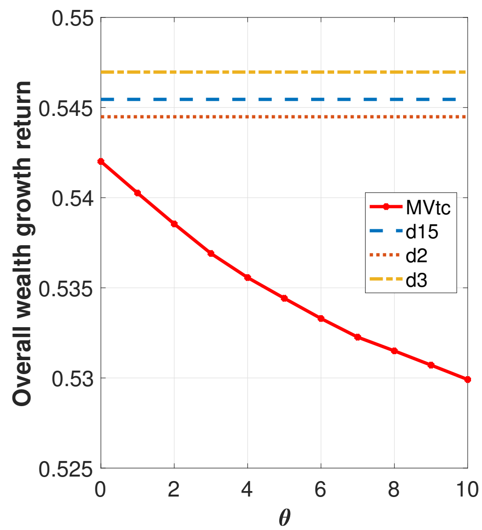
$$\frac{1}{10}\sum_{p=2}^{11}\frac{c_{sell}y^t + c_{buy}z^t}{\sum_{i=1}^{n} x_i^p + x_0^p} \times 100(\%).$$

Figure 4.5.1 is total return over the 11 periods with respect to the risk aversion parameter θ. Low levels of risk associated with low potential returns and high levels of risk are associated with high potential returns. It is expected that the overall return might be higher when the risk aversion parameter is low, and all considered transactions costs rates of 0%, 0.1%, 0.5%, and 1% fulfill this expectation. Also, for a given θ, the owgr of the portfolio decreases as transactions costs rate increases.

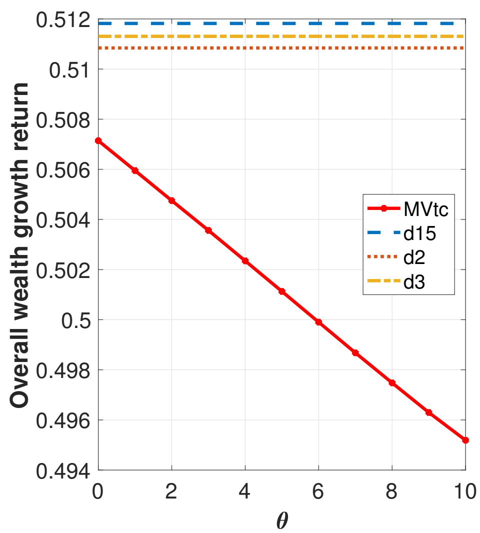
Table 4.5.1(a) is the average diversification number of the 'MVtc' model with respect to θ from 0 to 10 and the 'HPtc' model with respect to $\delta = 0.015, 0.02, 0.03$ when transactions costs rate is $0\%, 0.1\%, 0.5\%$ and 1%. Both models construct well-diversified portfolio with 26 to 30 number of assets. Table 4.5.1(b) is the average transactions costs in the percentage units. The risk aversion parameter θ does not affect the atc of 'MVtc' model and also the threshold δ does not affect the atc of the 'HPtc' model. However, both models tend to change portfolio less as transactions costs rate increases in order to reduce the cost.



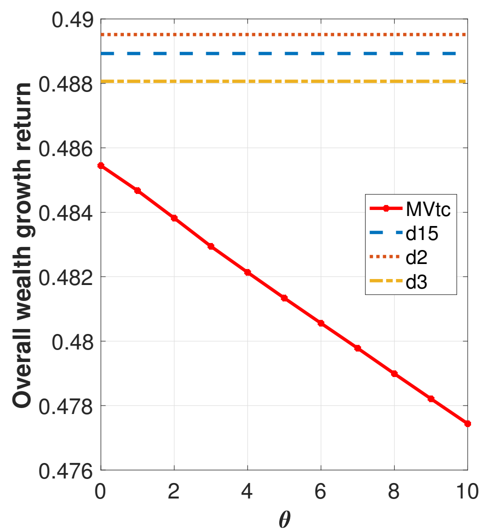
(a) No transactions costs



(b) 0.1% tc



(c) 0.5% tc



(d) 1% tc

Figure 4.5.1: Overall wealth growth return of the 'MVtc' model compare to the 'HPtc' model with $\delta = 0.015, \delta = 0.02$, and $\delta = 0.03$ as the risk aversion parameter θ ranges from 0 to 10: (a) tcb = tcs = 0, (b) tcb = tcs = 0.001, (c) tcb = tcs = 0.005, (d) tcb = tcs = 0.01.

Table 4.5.1: The comparison of the 'MVtc' and 'HPtc' models with $w = 0.975$, $\epsilon = 0.025$. (a) Average diversification number (b) Average transactions costs except for the first period

(a) Adn

MVtc

θ	0%	0.1%	0.5%	1%
0	30.3	32.0	27.5	26.5
1	30.4	32.0	27.5	26.3
2	30.3	31.8	26.7	26.1
3	30.3	31.8	26.6	26.0
4	30.0	32.0	26.5	26.0
5	29.8	31.9	26.4	26.0
6	29.7	31.6	26.4	25.9
7	29.7	31.6	26.4	25.9
8	29.5	31.6	26.4	25.9
9	29.3	31.5	26.4	25.9
10	29.2	31.5	26.4	25.9

HPtc

δ	0%	0.1%	0.5%	1%
1.5	30.5	32.0	29.0	27.4
2	30.5	32.1	28.8	27.6
3	30.4	32.1	29.0	27.4

(b) Atc : (%)

MVtc

θ	0%	0.1%	0.5%	1%
0	0.4648	0.1664	0.0151	0.0001
1	0.4647	0.1673	0.0153	0.0001
2	0.4645	0.1682	0.0157	0.0001
3	0.4644	0.1691	0.0160	0.0002
4	0.4643	0.1702	0.0163	0.0002
5	0.4644	0.1713	0.0166	0.0002
6	0.4646	0.1724	0.0170	0.0002
7	0.4647	0.1735	0.0173	0.0002
8	0.4650	0.1747	0.0177	0.0002
9	0.4656	0.1760	0.0181	0.0002
10	0.4664	0.1774	0.0185	0.0002

HPtc

δ	0%	0.1%	0.5%	1%
1.5	0.4450	0.1689	0.0165	0.0011
2	0.4494	0.1697	0.0165	0.0013
3	0.4475	0.1712	0.0170	0.0013

4.6 Discussion

In this chapter, we derive the robust VaR models in the presence of transactions costs using the notations from the multi-period portfolio selection problem. Investors consider not only the expected return of each stock but also the costs of the transactions, which impact the actual return. There are two models we derive, both of which have the same objective function, which maximizes the worst expected transactions costs-adjusted return, but have a slight difference on the VaR constraint. The 'HPtc' model restricts the transactions costs-adjusted return to be less than a certain threshold. The two models construct similar portfolios regardless of transactions costs rate and threshold δ. When the transactions costs rate is large, the 'HPtc' model does not change the portfolio that much in order to reduce the cost for any δ.

We also extend the RMRAR problem to the model in the presence of transactions costs. For an investor who defines the risk as the standard deviation of the portfolio, the 'MVtc' model is applicable. For an investor who defines the risk as VaR, the 'HPtc' model is applicable.

Assume a portfolio consists of 30 stocks and a flat fee of \$5 is on each transaction. If an investor has a small capital less than \$15,000 so that the transactions costs rate is more than 1%, then the 'HPtc' and 'MVtc' models will provide better portfolios. However, if an investor has a huge capital so that the transactions costs is negligible, then all of the models we derive in this dissertation will produce similar portfolios.

Appendix A

Appendix

A.1 Market Data for Simulation

This section has the assets and factors used for the computations in Chapters 2,3, and 4.

Table A.1.1: 4-week U.S. Treasury bill rate (%) per year

Month	2013	2014	2015	2016
Jan	0.07	0.01	0.02	0.17
Feb	0.02	0.04	0.01	0.17
Mar	0.07	0.04	0.02	0.29
Apr	0.06	0.02	0.02	0.20
May	0.03	0.02	0.00	0.10
Jun	0.03	0.04	0.02	0.27
Jul	0.01	0.03	0.01	0.23
Aug	0.02	0.01	0.02	0.19
Sep	0.03	0.02	0.01	0.26
Oct	0.10	0.01	-0.01	0.24
Nov	0.03	0.03	0.01	0.24
Dec	0.02	0.01	0.19	0.31

Table A.1.2: 6 factors

GSPC	S&P 500 Index
DJI	Dow Jones 30 Industrial Average
IXIC	NASDAQ Composite
NYA	NYSE Composite index
XAX	NYSE AMEX Composite
RUT	Russell 2000 Index

Table A.1.3: 36 assets

Financials		Health care	
BRK-A	Berkshire Hathaway	MCK	McKesson
FNMA	Fannie Mae	UNH	UnitedHealth Group
JPM	J.P Morgan Chase	ABC	AmerisourceBergen
BAC	Bank of America Corp.	CAH	Cardinal Health
Energy		Food and Drug Stores	
XOM	Exxon Mobil	CVS	CVS
CVX	Chevron	KR	Kroger
PSX	Phillips 66	WBA	Walgreens Boots Alliance
Retailing		Technology	
WMT	Walmart	AAPL	Apple
COST	Costco	AMZN	Amazon.com
HD	Home Depot	HPQ	HP
Aerospace & Depence		Food, Beverages & Tobacco	
BA	Boeing	ADM	Archer Danils Midland
UTX	United Technologies	PEP	PepsiCo
Industrials		Motor Vehicles and Parts	
GE	General Electric	GM	General Motors
CAT	Caterpillar	F	Ford Motor
Telecommunications		Transportation	
T	AT&T	UPS	UPS
VZ	Verizon	FDX	FedEx
Chemicals		Household Products	
DOW	Dow Chemical	PG	Proctor&Gamble
Media		Wholesalers	
DIS	Disney	SYY	Sysco

A.2 Dual Problems and Dual Variables Of Models in Chapter 3

All the models in this dissertation are solved using MOSEK, a tool for solving mathematical optimization problems, with a personal academic license in Matlab [20]. The duals of each models are provided in this chapter and are used for computation. All the dual problems contain linear, quadratic cone, and semidefinite constraints for which a primal solution can be easily obtained with MOSEK. The dual of the robust VaR problem with estimation (3.2.9) is provided, and the dual variables corresponding to the constraints in the problem are listed in Table A.2.1. Note that $\epsilon_0 = -\mathcal{F}^{-1}(\epsilon)$.

$$
\begin{aligned}
\min_{Y^1,Y^2,Y^3,x^4,x^5} \quad & \delta^2 Y^1_{22} + Y^3_{11} + x^1_1 + x^1_2 + x^2, \\
\text{s.t} \quad & 2\delta\Psi(Y^1_{21}) + 2\Psi(Y^2_{12}) + 2Y^3_{21} + 2D^{1/2}x^1_3 - x^2 e \leq 0, \\
& \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} \odot Y^1_{11} - Y^3_{22} \geq 0, \\
& \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \cdot Y^k \leq 0, \;\; k = 1, 2 \\
& - \epsilon_0^2 Y^1_{22} + x^1_1 - x^1_2 = 0, \\
& 2Y^2_{22} = 1, \\
& Y^1 \succeq 0, Y^2 \succeq 0, Y^3 \succeq 0, x^1 \in \mathcal{L}^{n+2},
\end{aligned}
\tag{A.2.1}
$$

where $Y^1_{11}, Y^2_{11} \in \mathbb{R}^{[(m+1)n]\times[(m+1)n]}$, $Y^1_{12}, Y^2_{12} \in \mathbb{R}^{(m+1)n}$, $Y^3_{22} \in \mathbb{R}^{n\times n}$, $x^4 \in \mathbb{R}^{(n^2+n)/2}$, $Y^3_{21}, x^1_3 \in \mathbb{R}^n$, $Y^1_{22}, Y^2_{22}, Y^3_{11}, x^1_1, x^1_2, x^2 \in \mathbb{R}$, $\Psi : \mathbb{R}^{(m+1)n} \to \mathbb{R}^n$ is defined as $\Psi(x) =$

$(x_1, x_{m+2}, \ldots, x_{(n-2)(m+1)+1}, x_{(n-1)(m+1)+1})^T$ for any $x \in \mathbb{R}^{(m+1)n}$, and

$$\begin{pmatrix} 0 & 0 \\ & \\ 0 & \epsilon_0^2 F \end{pmatrix} \odot X \equiv \left[\begin{pmatrix} 0 & 0 \\ & \\ 0 & \epsilon_0^2 F \end{pmatrix} \cdot X_{ij} \right] \in \mathbb{R}^{n \times n}$$

for any $X = (X_{ij}) \in \mathbb{R}^{[(m+1)n] \times [(m+1)n]}$ with $X_{ij} \in \mathbb{R}^{(m+1) \times (m+1)}$ for $i, j = 1, \cdots, n$.

The dual of the robust VaR problem with estimation (3.2.10) is provided, and the

Table A.2.1: Constraints of (3.2.11) and its dual variables

The constraints of (3.2.11)	Dual variables
$\begin{pmatrix} \tau_1 R - S \otimes \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} & \tau_1 h + \delta q \\ \tau_1 h^T + \delta q^T & \tau_1 \eta - \epsilon_0^2 t + \delta^2 \end{pmatrix} \succeq 0$	$Y^1 = \begin{pmatrix} Y_{11}^1 & Y_{12}^1 \\ Y_{21}^1 & Y_{22}^1 \end{pmatrix} \succeq 0$
$\begin{pmatrix} \tau_2 R & \tau_2 h + q \\ \tau_2 h^T + q^T & \tau_2 \eta - 2\nu \end{pmatrix} \succeq 0$	$Y^2 = \begin{pmatrix} Y_{11}^2 & Y_{12}^2 \\ Y_{21}^2 & Y_{22}^2 \end{pmatrix} \succeq 0$
$\begin{pmatrix} 1 & \phi^T \\ \phi & S \end{pmatrix} \succeq 0$	$Y^3 = \begin{pmatrix} Y_{11}^3 & Y_{12}^3 \\ Y_{21}^3 & Y_{22}^3 \end{pmatrix} \succeq 0$
$\begin{pmatrix} 1 + t \\ 1 - t \\ 2D^{1/2} \end{pmatrix} \in \mathcal{L}^{n+2}$	$\begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \end{pmatrix} \in \mathcal{L}^{n+2}$
$e^T \phi = 1$	$x^2 (\in \mathbb{R})$

dual variables corresponding to the constraints in the problem are listed in Table

A.2.2.

$$
\begin{aligned}
\min_{Y^1,Y^2,Y^3,x^1,x^2,x^3,x^4} \quad & \delta^2 Y^1_{22} + Y^3_{11} + x^1_1 + x^1_2 + x^2, \\
\text{s.t} \quad & 2\delta\Psi(Y^1_{21}) + 2\Psi(Y^2_{12}) + 2Y^3_{21} + 2D^{1/2}x^1_3 + x^2 - x^3 e \leq 0, \\
& \begin{pmatrix} -1 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} \odot Y^1_{11} - Y^3_{22} + x^2 e^T + x^4 = 0, \\
& \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \cdot Y^k \leq 0, \;\; k = 1, 2 \\
& -\epsilon_0^2 Y^1_{22} + x^1_1 - x^1_2 = 0, \\
& 2Y^2_{22} = 1, \\
& Y^1 \succeq 0, Y^2 \succeq 0, Y^3 \succeq 0, x^1 \in \mathcal{L}^{n+2},
\end{aligned}
\tag{A.2.2}
$$

where $Y^1_{11}, Y^2_{11} \in \mathbb{R}^{[(m+1)n]\times[(m+1)n]}$, $Y^1_{12}, Y^2_{12} \in \mathbb{R}^{(m+1)n}$, $Y^3_{22} \in \mathbb{R}^{n\times n}$, $x^4 \in \mathbb{R}^{(n^2+n)/2}$, $Y^3_{21}, x^1_3, x^2 \in \mathbb{R}^n$, $Y^1_{22}, Y^2_{22}, Y^3_{11}, x^1_1, x^1_2, x^3 \in \mathbb{R}$.

Table A.2.2: Constraints of (3.2.12) and its dual variables

The constraints of (3.2.12)	Dual variables
$\begin{pmatrix} \tau_1 R - S \otimes \begin{pmatrix} -1 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} & \tau_1 h + \delta q \\ \tau_1 h^T + \delta q^T & \tau_1 \eta - \epsilon_0^2 t + \delta^2 \end{pmatrix} \succeq 0$	$Y^1 = \begin{pmatrix} Y^1_{11} & Y^1_{12} \\ Y^1_{21} & Y^1_{22} \end{pmatrix} \succeq 0$
$\begin{pmatrix} \tau_2 R & \tau_2 h + q \\ \tau_2 h^T + q^T & \tau_2 \eta - 2\nu \end{pmatrix} \succeq 0$	$Y^2 = \begin{pmatrix} Y^2_{11} & Y^2_{12} \\ Y^2_{21} & Y^2_{22} \end{pmatrix} \succeq 0$
$\begin{pmatrix} 1 & \phi^T \\ \phi & S \end{pmatrix} \succeq 0$	$Y^3 = \begin{pmatrix} Y^3_{11} & Y^3_{12} \\ Y^3_{21} & Y^3_{22} \end{pmatrix} \succeq 0$
$\begin{pmatrix} 1+t \\ 1-t \\ 2D^{1/2} \end{pmatrix} \in \mathcal{L}^{n+2}$	$\begin{pmatrix} x^1_1 \\ x^1_2 \\ x^1_3 \end{pmatrix} \in \mathcal{L}^{n+2}$
$\phi - Se \geq 0$	$x^2 (\in \mathbb{R}^n) \geq 0$
$e^T \phi = 1$	$x^3 (\in \mathbb{R})$

A.3 Dual Problems and Dual Variables Of Models in Chapter 4

The dual problem of the robust VaR problem (4.2.8) is provided and dual variables corresponding constraints in the problem are also listed in Table A.3.1. Note that $\delta_0 = 1 - \delta$, $\alpha \in \mathbb{R}^n$ represents the vector of initial amounts on stocks, $\alpha_0 \in \mathbb{R}$ represents the initial cash holding and $\bar{\alpha}$ is the sum of all initial amount on stocks and cash holding.

$$
\begin{aligned}
\min_{Y^1,Y^2,Y^3,x^1,x^2,x^3,x^4,x^5} \quad & \delta_0^2 Y_{22}^1 + Y_{11}^3 + x_1^1 - x_2^1 + \alpha^T x^3 + \alpha_0 x^4, \\
\text{s.t} \quad & -2\delta_0 \Psi(Y_{12}^1) + 2\Psi(Y_{12}^2) + 2Y_{21}^3 - 2D^{1/2} x_3^1 + x^2 - \bar{\alpha} x^3 \leq 0, \\
& \begin{pmatrix} -1 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} \odot Y_{11}^1 - Y_{22}^3 + x^2 e^T = 0, \\
& \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \cdot Y^k \leq 0, \ k = 1, 2, \\
& -x^3 + (1 + c_{sell}) x^4 \leq 0, \\
& x^3 - (1 + c_{buy}) x^4 \leq 0, \\
& -\epsilon_0^2 Y_{22}^1 + x_1^1 - x_2^1 = 0, \\
& 2Y_{22}^2 = 1, \\
& Y^1 \succeq 0, Y^2 \succeq 0, Y^3 \succeq 0, x^1 \in \mathcal{L}^{n+2}, x^2 \geq 0
\end{aligned}
\tag{A.3.1}
$$

The dual problem of the robust VaR problem (4.4.4) is provided and dual variables

Table A.3.1: Constraints of (4.2.8) and its dual variables

The constraints of (4.2.8)	Dual variables
$\begin{pmatrix} \tau_1 R - S \otimes \begin{pmatrix} -1 & 0 \\ 0 & \epsilon_0^2 F \end{pmatrix} & \tau_1 h - \delta_0 q \\ \tau_1 h^T - \delta_0 q^T & \tau_1 \eta - \epsilon_0^2 t + \delta_0^2 \end{pmatrix} \succeq 0$	$Y^1 = \begin{pmatrix} Y_{11}^1 & Y_{12}^1 \\ Y_{21}^1 & Y_{22}^1 \end{pmatrix} \succeq 0$
$\begin{pmatrix} \tau_2 R & \tau_2 h + q \\ \tau_2 h^T + q^T & \tau_2 \eta - 2\nu \end{pmatrix} \succeq 0$	$Y^2 = \begin{pmatrix} Y_{11}^2 & Y_{12}^2 \\ Y_{21}^2 & Y_{22}^2 \end{pmatrix} \succeq 0$
$\begin{pmatrix} 1 & c^T \\ c & S \end{pmatrix} \succeq 0$	$Y^3 = \begin{pmatrix} Y_{11}^3 & Y_{12}^3 \\ Y_{21}^3 & Y_{22}^3 \end{pmatrix} \succeq 0$
$\begin{pmatrix} 1+t \\ 1-t \\ 2D^{1/2} \end{pmatrix} \in \mathcal{L}^{n+2}$	$\begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \end{pmatrix} \in \mathcal{L}^{n+2}$
$c - Se \geq 0$	$x^2 (\in \mathbb{R}^n) \geq 0$
$\alpha - y + z - \bar{\alpha} c = 0$	$x^3 (\in \mathbb{R}^n)$
$\alpha_0 + (1 - c_{sell}) e^T y - (1 + c_{buy}) e^T z = 0$	$x^4 (\in \mathbb{R})$

corresponding constraints in the problem are also listed in Table A.3.2.

$$
\begin{aligned}
\min_{Y^1, Y^2, x^1, x^2, x^3} \quad & Y_{11}^2 + x_1^1 - x_2^1 + \alpha^T x^2 + \alpha_0 x^3, \\
\text{s.t} \quad & 2\Psi(Y_{12}^1) + 2Y_{21}^2 - 2D^{1/2} x_3^1 + \bar{\alpha} x^2 \leq 0, \\
& 2\theta \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \odot Y_{11}^1 - Y_{22}^3 = 0, \\
& \begin{pmatrix} R & h \\ h^T & \eta \end{pmatrix} \cdot Y^1 \leq 0, \\
& x_1^1 - x_2^1 = \theta \\
& -x^2 + (1 + c_{sell}) x^3 \leq 0, \\
& x^2 - (1 + c_{buy}) x^3 \leq 0, \\
& 2Y_{22}^1 = 1, \\
& Y^1 \succeq 0, Y^2 \succeq 0, x^1 \in \mathcal{L}^{n+2}.
\end{aligned}
\tag{A.3.2}
$$

Table A.3.2: Constraints of (4.4.4) and its dual variables

The constraints of (4.4.4)	Dual variables
$\begin{pmatrix} \tau R - 2\theta S \otimes \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} & \tau h + q \\ \tau h^T + q^T & \tau\eta + 2c^T e - 2\nu \end{pmatrix} \succeq 0$	$Y^1 = \begin{pmatrix} Y^1_{11} & Y^1_{12} \\ Y^1_{21} & Y^1_{22} \end{pmatrix} \succeq 0$
$\begin{pmatrix} 1 & c^T \\ c & S \end{pmatrix} \succeq 0$	$Y^3 = \begin{pmatrix} Y^3_{11} & Y^3_{12} \\ Y^3_{21} & Y^2_{22} \end{pmatrix} \succeq 0$
$\begin{pmatrix} 1+t \\ 1-t \\ 2D^{1/2} \end{pmatrix} \in \mathcal{L}^{n+2}$	$\begin{pmatrix} x^1_1 \\ x^1_2 \\ x^1_3 \end{pmatrix} \in \mathcal{L}^{n+2}$
$\alpha - y + z - \bar{\alpha}c = 0$	$x^2 (\in \mathbb{R}^n)$
$\alpha_0 + (1 - c_{sell})e^T y - (1 + c_{buy})e^T z = 0$	$x^3 (\in \mathbb{R})$

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