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“The total movement of this disorder is its order”: Investment and utilization dynamics in long-run disequilibrium

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Abstract: Recently economists have developed Kaleckian-Harrodian models, in which non-capacity-creating autonomous demand acts as a stabilizing force that drives long-run growth. But critics have questioned the plausibility of the stability conditions for these models. Motivated by this controversy, in this paper I formulate an alternative framework, in which stable equilibria need not exist, and solution trajectories can perpetually fluctuate in violent and aperiodic ways, but the long-run dynamics can be understood in terms of time averages. On this basis I argue that key findings in the Kaleckian-Harrodian literature can be sustained even if the stability conditions are rejected.

To see simulations of the model in this paper, with animations showing how the results change when different parameter inputs are selected, check out this web app:

<https://sthompsonchicago.github.io/macro-disequilibrium/>

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1. Introduction

What is the driving force behind long-run economic growth? In several recent contributions, economists have explored the role of non-capacity creating (NCC) autonomous demand as a possible answer to this question. Here, the qualifier *non-capacity creating* refers to expenditures that do not directly add to the stock of productive capital, while the term *autonomous* describes components of aggregate demand that are (at least approximately, and over long periods of time) independent of other economic variables. One particularly interesting source of NCC autonomous demand may be residential investment, the importance of which for sustaining growth was revealed in dramatic fashion during the 2007-08 global crisis (Fiebiger and Lavoie 2019). More generally, to the extent that autonomous NCC demand plays a role in driving the long-run expansion of a capitalist economy, one can speak of *NCC-demand-driven growth*. The now-flourishing literature on these issues has established interesting results, but has also stirred controversy.

Models of NCC-demand-driven growth are rooted in the idea that businesses adjust output, capital accumulation and employment to meet changes in product-market demand. From this perspective, aggregate fixed-capital investment varies over time as a result of firms' attempts to keep capacity utilization at a particular targeted level, consistent with Harrod's (1939) analysis. The key theoretical result for the literature on NCC-demand-driven growth is that, under these conditions, and absent the existence of some other stabilizing force, perpetually expanding NCC autonomous demand is a prerequisite for sustained (and non-explosive) macroeconomic expansion. Key

contributions include the supermultiplier models developed by Serrano (1995) and Freitas and Serrano (2015), and the Harrodian-Kaleckian models of Allain (2015) and Lavoie (2016). For the models developed in this literature, the growth rate for NCC autonomous expenditures determines both the GDP growth rate and the business investment share of GDP in the long run.

These theoretical findings have been bolstered by the results of recent econometric studies. Indeed, several authors—by looking at the role of particular NCC expenditures such as exports, residential investment and government spending—have provided formal evidence to support models of NCC-demand-driven growth. For example, Pérez-Montiel and Pariboni (2021) present evidence that residential investment Granger causes both short-run and long-run output growth in the US, while Girardi and Pariboni (2020) look at a panel of 20 OECD countries and provide evidence that NCC autonomous expenditures drive the share of productive investment in GDP. In a similar vein, Girardi and Pariboni (2016) find that autonomous residential investment, government spending and exports collectively Granger cause both output growth and the business investment share in the US. Haluska, Braga and Summa (2019) give further evidence for the importance of NCC autonomous expenditures in the US, while Braga (2020) gives evidence for Brazil. In view of these empirical findings, and the theoretical results mentioned above, the literature on NCC-demand-driven growth may be seen as a promising basis for future research in the theory of long-run economic dynamics.

All this clearly differs from the standard neoclassical analysis, exemplified by Solow (1997), in which the demand side of the economy plays a completely passive role

in the long run. Nevertheless, models of NCC-demand-driven growth are part of a long tradition in the history of economic thought. Indeed, Luxemburg ([1913] 2015) argued over a century ago that exports to non-capitalist spheres are essential for sustaining capitalism, and ideas like this have long played an important role in Marxist theory (Thompson 2020a, p. 436). Similarly, some Keynesian economists have developed theories of export-led growth (Kaldor 1996, p. 67-69). And Godley and Lavoie (2006), in their seminal work on stock-flow consistent macroeconomics, make a forceful case regarding the importance of government spending for sustaining aggregate demand. The literature on NCC-demand-driven growth raises the prospect of uniting together and clarifying these ideas within a common framework.

However, this approach has also faced important challenges. Following standard practice in economic theory, advocates of the NCC-demand-driven growth perspective have made their arguments in terms of steady-state equilibrium models, and the central theoretical results have been expressed in the form of local stability theorems. But it is far from clear that NCC expenditures in the real world tend to actually play the type of stabilizing role posited by these models. Indeed, some components of NCC demand may be quite volatile, and residential investment, rather than being autonomous, seems to have *interacted* with the global economy in a way that has been *destabilizing*. Thus, as I discuss below, Skott (2019) and others have raised serious doubts about the usefulness of Harrodian-Kaleckian and supermultiplier models.

In the present paper, I seek to contribute to this debate by drawing a distinction between two separate claims: first, the assertion that NCC demand plays a stabilizing role, and second, the proposition that NCC demand drives long-run growth. The existing

literature tends to conflate these two claims, but I will argue that, in fact, while the first claim may well be questionable, this does not actually undermine the second. Indeed, NCC demand may be most relevant, not in a hypothetical steady-state equilibrium, but instead in the (arguably more realistic) context of an unstable capitalist economy that lurches from one crisis to the next.

In support of this idea, below I develop a model in which, although stable equilibria need not exist, under general conditions it is possible to derive formulas showing how the long-run average values of different variables depend on model parameters. I then show that the long-run average rate of economic growth in the model is determined by the trend growth rate of NCC demand. As a core component of this analysis, I track monetary flows between sectors, and show how financial constraints can impose bounds on the level of economic activity. On this basis I seek to demonstrate that under plausible assumptions, NCC demand can determine the long-run rate of expansion for a capitalist economy, even if the growth process is highly volatile and NCC demand plays a destabilizing role.

The rest of this paper is organized as follows. In Section 2, I provide background for the analysis in this paper by reviewing the controversy mentioned above, and linking this with broader questions about economic modeling methodology. Section 3 develops the model that is the focus of this paper. Section 4 then formulates the main results of the paper, while proofs are given in Section 5. Section 6 then illustrates these results with numerical simulations. Section 7 shows how the long-run average behavior of the model relates to steady-state trajectories in already existing frameworks. Section 8 concludes.

2. Background

2.1. *Models of NCC-demand-driven growth*

A key reference point for the NCC-demand-driven growth literature is the supermultiplier framework developed by Serrano (1995). Serrano's analysis is based on a macroeconomic model in which aggregate demand consists of three parts: (i) a portion of consumption that is induced, in the sense that it is proportional to income; (ii) productive investment, which is also induced in the sense that it adjusts to keep the rate of capacity utilization at a particular targeted value; and (iii) a component consisting of autonomous NCC expenditures that grow at a constant rate. Because of the induced nature of (i) and (ii), neither can ultimately *determine* the long-run rate of economic growth. But the autonomous NCC expenditures that make up (iii) *do* play such a role, and Serrano illustrates this by showing the existence of a growth path in which the rate of expansion is determined by the growth rate of autonomous NCC demand.

Although Serrano's initial paper focused on a steady-state path, more recent contributions have extended this analysis beyond the steady state. In particular, Allain (2015) and Lavoie (2016) have developed Kaleckian-Harrodian frameworks in which the short-run rate of capacity utilization is free to take any value, but the long-run rate of capital accumulation adjusts over time as a result of firms' attempts to keep capacity utilization at a targeted level. In these models, if the NCC autonomous expenditure component is sufficiently large in relation to national income, and the investment function is not too sensitive to changes in demand, then solution trajectories can

converge to a long-run growth path like the one described by Serrano, in which autonomous NCC expenditures determine the overall rate of expansion and capacity utilization stays at the level targeted by firms.

As I mentioned above, however, these models have faced sharp criticism.² Skott (2017) has pointed out that, if parameters are set at reasonable values, the Kaleckian-Harrodian stability conditions may be difficult to satisfy. A key issue is that, for stability to be guaranteed, the steady-state share of autonomous NCC demand must be sufficiently large. Since there are doubts regarding how much of NCC demand is truly autonomous in the real world (Nikiforos 2018), it is far from clear that the requisite condition can be expected to hold. Perhaps even more importantly, Skott (2017) further shows that, even if *local stability* does hold, *global stability* will not, and as a result, moderate perturbations of the steady state are likely to generate explosive growth away from it. This lack of global stability may be a major issue since significant components of NCC demand can be extremely volatile in practice (Skott 2019). Thus, fluctuations in the growth rate of NCC demand will constantly be knocking the economy out of its steady-state path; since these fluctuations may be large, and the Kaleckian-Harrodian steady-state path is not globally stable, this means that the steady-state may not have much relevance to the long-run dynamics. Finally, the conditions for local stability may

² The controversy over Kaleckian-Harrodian models is just one salvo in an ongoing debate over the role of aggregate demand in theories of long-run growth. Skott (1990) and Fazzari et al (2013) take an alternative approach, in which instability leads to cyclical growth, and labor-market constraints act as a ceiling for the level of economic activity. Fazzari, Ferri and Variato (2020) present a hybrid perspective, developing a framework that behaves like a NCC-demand-constrained model if the steady-state is stable, and behaves like a cyclical labor-market-constrained model if the steady-state is unstable. There are also many other perspectives on these issues. The present paper does not purport to give anything like a comprehensive overview, but for more discussion see Thompson (2020a: Section 6). For other perspectives see Rogé (2019), Setterfield (2018), Lavoie (2017), Kemp-Benedict (2020), Serrano, Freitas and Bhering (2019) and Hein, Lavoie and van Treeck (2011).

require that investment exhibits implausibly slow reactions to changes in aggregate demand (Skott, Santos and Oreiro 2021: p. 20).

Lavoie (2017) responded to Skott by pointing out that a simple model can provide useful insight into the nature of capitalism, and provide a basis for further research, even if it lacks realism in important respects. And since then, Hein and Woodgate (2021) have extended the earlier Kaleckian-Harrodian framework to incorporate debt dynamics, but their results suggest that this realistic extension of their model has the effect of further *restricting* the set of parameters in which stability obtains. Recent elaborations of the Sraffian supermultiplier approach constitute another class of models that generate similar results, but as Skott, Santos and Oreiro (2021) show, these models are subject to essentially the same criticisms as their Kaleckian-Harrodian counterparts. Therefore, we are left with important questions about the role of NCC demand in long-run economic dynamics. A key issue is that economists have sought to defend this approach by formulating it in terms of *stability*, and it is not clear that the stabilization mechanism they have proposed is plausible.

2.2. *Understanding the long-run behavior of an unstable economy*

Although the controversy described above concerns a particular theory of long-run growth, it also pertains to more general (and longstanding) questions about the relevance of steady-state models to real-world macroeconomic dynamics. Indeed, capitalism is a complex system exhibiting aperiodic fluctuations, spatially uneven development, occasional crashes, and constant structural change. When we seek to

develop a mathematical representation for the dynamics of such a system, with the aim untangling the relationships between its different parts, it is not obvious what approach is best.

Kaleckian-Harrodian and supermultiplier models follow a widely used strategy by describing macroeconomic dynamics in terms of a stable long-run equilibrium growth path. But, as Skott's critique emphasizes, it is difficult to not be skeptical of this approach.³ Market economies do not show any clear tendency to move toward equilibria (Miller and Page, 2007: p. 222-23). Steady-state models, by ignoring the inherent instability of capitalism, can create a misleading picture of the way the system works (Skott, 2010; Goodwin, 1953; Robinson, 1956).

Of course, economists have also developed other approaches. A large literature has sought to show that the boom-and-bust cycles of a capitalist economy can be understood in terms of periodic solutions of dynamical systems (for discussions of various examples, see Taylor (2004) and Skott (1990)). In some cases, economists have also developed modeling frameworks that go beyond simple periodic oscillations, employing ideas like chaos (e.g., Dávila-Fernández and Sordi, 2019; Jiang, 2015) and catastrophe theory (Varian, 1979).

But there are reasons to be dissatisfied with these alternatives to steady-state modeling as well. Theories of cyclical growth are typically based on highly schematized, two-dimensional "toy models" that, while yielding important insights, can only capture a

³ In economics, the term "equilibrium" can mean different things. But the term *stable long-run equilibrium* is more restrictive. It stipulates that, if all the variables treated as parameters in a theory are held constant, then the economy will move toward a steady-state growth path and stay there forever. In practical terms this means that, although an economy might be frequently perturbed from its long-run equilibrium, forces within the economy will tend to bring it back again, and one can corroborate this by verifying that empirically measured parameter values meet appropriate stability conditions.

small subset of the dynamic mechanisms driving a capitalist economy at any one time (Taylor, 2004: p. 306). These simple periodic systems seem to miss much of what is important about the way capitalist economies actually behave (Foley, 1986: p. 48). On the other hand, more sophisticated cyclical and chaotic models represent an intriguing alternative, but formal analysis of such models can run into formidable technical obstacles, so that in practice one often cannot do much more than study simulations.⁴ And while catastrophe theory may be useful for understanding periods of crisis, there is little indication that it can provide a basis for a general theory of long-run economic dynamics. Thus there is no agreement regarding the appropriate modeling approach for understanding the long-run behavior of capitalism.

Given the drawbacks of these alternatives, it is not surprising that—despite the substantial limitations of such frameworks—economists still frequently rely on models of steady-state growth. Unfortunately, this state of affairs seems to make controversies like the one over Kaleckian-Harrodian models all but inevitable. After all, if there is no practical method for formalizing ideas about the dynamics of a capitalist economy in a realistic way, then *all* macroeconomic theories must be subject to serious doubts.

This situation is untenable and calls for a different approach. More specifically, we need a modeling approach that allows us to characterize the long-run behavior of key variables in economic models, and understand how this behavior depends on different assumptions about underlying parameters, without being forced to restrict our

⁴ For example, although the Lorenz system has long been treated as a paradigmatic example of chaos, it took mathematicians nearly *four decades*, and a creative new proof technique, just to establish that the behavior of the Lorenz system observed in computer simulations really does correspond to a chaotic attractor (Stewart, 2000). And despite this progress, we are still “decades (if not centuries) away” from understanding the different phenomena that occur in the model when parameters are changed (Hirsch, Smale and Devaney, 2013: p. 312). Attempts to develop macroeconomic theories involving chaos have also run up against the limitations of available data and econometric methods (Foley 2005).

attention to simplified situations in which we can guarantee local stability or periodicity. Thus, in this paper, I propose an alternative to the more commonly used modeling methods in economics.

In the next section, I will formulate a macroeconomic model in which demand can fluctuate in violent, aperiodic ways. Moreover, I allow for the possibility that NCC demand can interact with other economic variables. We will see that solution trajectories need not ever settle into any steady state or a periodic cycle, but I will show how to calculate the trend rates of growth for the capital stock and aggregate demand. I will also derive formulas for the long-run average values of the accumulation rate, the profit rate and capacity utilization rate, and show that financing constraints impose bounds on the size of the model's macro-fluctuations. The end result will be a new perspective on the conditions under which long-run economic development is determined by the growth of NCC demand.

A key characteristic of the analysis below is the fact that I employ a probabilistic concept—time averages—to understand the dynamics of a deterministic model. In this respect the approach taken here is similar in spirit to the methods sometimes used for analyzing chaotic systems (Berkovitz, Frigg and Kronz, 2006; Matsumoto, 2001). Importantly, however, the analysis below does not require the assumption that the model is actually chaotic, and this makes it possible to sidestep some major technical barriers. Moreover, by focusing on time averages instead of equilibria or cycles, I can adopt more general assumptions than one would typically find in analyses that rely on, say, local stability analysis or the Poincare-Bendixson theorem. I will establish simple

formulas describing the long-run behavior of the system which are valid irrespective of whether solution trajectories exhibit periodicity, chaos or stability.

3. A model of fluctuating macroeconomic dynamics

3.1. A sketch of the model

As I discussed during the brief literature review in the Introduction, several different types of NCC demand may be important in practice; for some countries export demand could be the key driving force behind growth, whereas for other countries it could be debt-financed consumer spending, or fiscal policy, or some combination of these. But it is not the purpose of this paper to furnish a general theory that treats all of the different possible special cases. Instead, the aim here is more limited: to show how NCC demand can drive the growth process even in an economy that tends to perpetually fluctuate in violent and aperiodic ways. To illustrate this point, I will focus on the role of government spending as a source of NCC demand (as in Allain (2015)), although in principle the analysis could be extended and generalized in various ways.⁵ A table summarizing the model is provided at the end of this section.

Let us now sketch the model. The equations below describe a closed economy in which workers, capitalists and the public sector interact. Workers earn a money wage from capitalists, which they spend on consumption goods. Capitalists purchase both

⁵ A more complete model would incorporate exports and imports, consumer credit and more. Any thorough discussion of these issues would probably need to start with a perspective—such as Panitch and Gindin (2012) or Vasudevan (2020)—on how international trade, finance, and politics relate to each other within global capitalism today.

consumption goods and fixed capital equipment; they also hire workers to produce a flow of output which they then seek to sell at a profit. (Thus “capitalists” should be seen as being a sector that comprises both businesses and their owners.) The public sector makes purchases from capitalists and collects income taxes. There is a single type of financial asset, which pays an interest rate i and circulates as money, and a single type of produced commodity, which can be used both as a consumption good and as fixed capital.

Capitalists vary the level of capacity utilization so that output matches demand at all times, and they accumulate capital with the aim of keeping the rate of capacity utilization—measured as the ratio of output to fixed capital equipment—at a particular desired level u^d . This investment behavior could be seen as arising from an explicit maximization problem faced by firms, as in Skott (1990), or could be seen as a formalization of Marx’s ([1867] 1990: pg. 739) contention that capitalists are forced by “external and coercive laws” to accumulate capital, as in Thompson (2020a, 2020b). In either case, this means that, as long as aggregate demand is growing, capitalists will be under pressure to accumulate more and more capital so that u does not stray too far from u^d . It also means that, although a component of investment demand is fixed in the short run, in the long run investment is completely induced, and represents a potentially destabilizing force, as in Harrod’s (1939) analysis.

Financing constraints play a role in the model as well. With respect to investment, this means that, although the rate of capital accumulation fluctuates within a certain range, the level of investment cannot rise too far in relation to capitalists’ expectations regarding future cashflow. On the other hand, because there is a drive to

perpetually accumulate capital, and a need to finance a portion of capital accumulation with internal funds, there are also limits on how much capitalists can consume.⁶ In the context of the model, this will mean that capitalists tend to increase or decrease their consumption levels depending on whether or not their net financial wealth is above or below a particular targeted level. And since capitalists' net financial assets will rise when they are running a financial surplus with other sectors, this means that financial flows between different parts of the economy can have an important influence on aggregate demand. Consequently, the public sector can play a particularly important role, and deficit spending by the state will tend to stimulate the economy (Taylor, Proano, de Carvalho and Barbosa 2012).

This brings us, finally, to the question of how to best model public sector spending. In the real world, government spending will fluctuate over time in response to a variety of different factors, both economic and political. For this reason, one can raise doubts about any analysis that depends on the assumption that government outlays grow at a fixed, exogenously given rate. And other sources of non-capacity-creating demand could be even more volatile. Thus it may be useful to analyze what happens when NCC demand is free to fluctuate within wide limits. Motivated by these considerations, I will assume that government spending has a well-defined *trend* rate of growth, but can fluctuate around this trend in complex ways. The aim will then be to analyze the implications of this for the economy as a whole.

⁶ The basic economic mechanism at work here was explained well by Kaldor (1966). He argued that businesses must retain a significant fraction of their profits because, in the presence of economies of scale, firms must grow in order to survive, and in the presence of financing constraints, firms must retain some of their profits in order to keep expanding their productive capacity. For a more discussion of the theory of consumption and investment in this paper, see Thompson (2020a).

3.2. Notation, aggregate demand and output

We can now begin to build the model by setting up some notation. In what follows, if x is any variable depending on time, then \dot{x} denotes the time derivative, \hat{x} denotes the growth rate, and \bar{x} denotes the long-run average value (starting at time $t = 0$). Thus

$$\dot{x} = \frac{dx}{dt}, \quad \hat{x} = \frac{\dot{x}}{x}, \quad \text{and} \quad \bar{x} = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t x[s] ds. \quad (1)$$

The model describes a single-commodity world, and the commodity unit price is normalized to one, so that real and nominal magnitudes are identical. Total demand at any given time, which I will denote by Y , is the sum of workers' consumption spending C_W , capitalists' consumption spending C_C , capitalists' gross fixed-capital investment expenditures I , and government spending A . Thus

$$Y = C_W + C_C + I + A. \quad (2)$$

Capitalists adjust output to meet changes in demand, so that real output Q is determined by the equation

$$Q = Y. \quad (3)$$

3.3. Financial stocks and flows

Let us now set up the equations describing financial stocks and flows between different sectors, essentially following Godley and Lavoie (2006). Capitalists' net financial wealth is denoted by V_C . The dynamics of V_C are determined by

$$\dot{V}_C = R + (1 - \tau)iV_C - C_C - I \quad (4)$$

where R is the flow of capitalists' after-tax profits, and τ is the tax rate (a fixed number strictly between zero and one). Workers, on the other hand, do not own fixed capital, so their net financial wealth, V_W , satisfies the equation

$$\dot{V}_W = W + (1 - \tau)iV_W - C_W \quad (5)$$

where W is workers' after-tax wage income. Finally, the net financial assets of the public sector are V_G , and

$$\dot{V}_G = T + iV_G - A \quad (6)$$

where T gives the value of total taxes at any given time:

$$T = \tau Q + \tau iV_C + \tau iV_W. \quad (7)$$

3.4. *Profits and capital accumulation*

To set up the description of productive investment, let us begin with some definitions.

The fixed-capital stock, K , evolves according to the equation

$$\dot{K} = I - \delta K \quad (8)$$

where I is gross investment and δ is the depreciation rate. Capacity utilization is

$$u = \frac{Q}{K}. \quad (9)$$

(Note that, following standard practice in models of demand-driven growth, I use the output-capital ratio to measure the degree of capacity utilization at a point in time.) The flow of total after-tax profit is

$$R = (1 - \tau)(1 - \psi)Q, \quad (10)$$

where ψ represents unit labor costs. The rate of profit on fixed capital is

$$r = \frac{R}{K}. \quad (11)$$

Finally, g denotes the gross accumulation rate, and we have

$$I = gK. \quad (12)$$

For the investment function I will adopt the basic form

$$g = b + \theta r. \quad (13)$$

Here, θ is a positive constant strictly less than one, which quantifies the way current profitability influences the rate of accumulation. And as in Harrodian models such as Allain (2015), I will assume that the investment function tends to shift in response to changes in u . This means b exhibits a tendency to rise when capacity utilization is above u^d , and fall when capacity utilization is below u^d , reflecting firms' efforts to keep capacity utilization at the desired rate.

There is also an upper limit to how high b can rise in this model. To understand why, we need to keep in mind a few important points. First, investment decisions made at particular point in time will often entail a long-run commitment; if capitalists decide to increase the rate of accumulation, they will not necessarily be able to quickly reverse that decision later if there is a sudden fall in profitability. Second, capitalists are constantly adjusting their investment plans in an effort to keep u near u^d , and, to the extent that these efforts are successful, capitalists can reasonably expect that the profit rate will fluctuate around the value $(1 - \tau)(1 - \psi)u^d$ in the long run. Finally, capitalists will experience a financial deficit if g is too high in relation to the profit rate. For these reasons, if b rises too far in relation to $(1 - \tau)(1 - \psi)u^d$, capitalists put themselves at risk of financial difficulties.

Therefore, there is a ceiling, B , for the possible values of b . To ensure that gross investment can never become negative in the model, I also assume that b cannot go below zero. To formalize these ideas, I adopt the equation

$$\dot{b} = \phi[b](u - u^d), \quad (14)$$

where

$$\phi[b] = \eta b(B - b). \quad (15)$$

The parameter η is positive and describes the speed with which capitalists react to changes in u . Thus b fluctuates in a Harrodian fashion, but equation (15) implies that its rate of change will slow down if it approaches either its upper limit ($b = B$) or lower limit ($b = 0$).

This model of investment decisions, while novel in certain respects, also has important commonalities with other approaches. First, the time dimension of investment, and more specifically the idea that investment decisions entail a long-run commitment, is a characteristic of a Hicksian traverse (Hicks 1987), although Hicks' own analysis ignored the demand-side issues that play a central role here.⁷ Second, the concern with financing constraints on investment is somewhat reminiscent of Lavoie and Godley's (2001–02) approach, although the formal aspects of their investment function are quite different from the one developed above. Third, this investment function follows Peter Skott in rejecting the possibility of a steady-state growth path in which u differs from u^d ; as Skott (2010, p. 114) writes, "it is hard to conceive of a steady-growth scenario in which firms are content to accumulate at a constant rate despite having significantly more (or less) excess capacity than they desire." The main novelty of the investment

⁷ I am grateful to an anonymous referee for pointing this connection out to me.

function developed in this paper lies in the way that these three ideas are synthesized together.

3.5. *The labor market*

Following Harrod (1939), one can define the *natural rate of growth* to be the growth rate of the labor force plus the growth rate of labor productivity. Clearly, output growth cannot exceed the natural rate forever, because if it did the employment rate would eventually exceed one. Similarly, if output growth were perpetually below the natural rate, the employment rate would fall to zero. Thus in the long run, we expect that the natural rate of growth will be equal to the rate of output growth. But this leaves open the question of how this equality is enforced.

A long tradition in economics argues that—within certain limits—the natural rate of growth will tend to adjust to accommodate changes in the growth rate of GDP. For example, the famous Kaldor-Verdoorn effect, in which the labor productivity growth rate becomes an increasing function of the output growth rate due to the presence of static and dynamic economies of scale, provides an important channel through which the natural rate of growth can vary in response to changes in the rest of the economy (Kaldor 1966). On the other hand, Girardi, Meloni and Stirati (2021) provide evidence that demand expansions have lasting effects on the labor force participation rate, because when economic booms occur, this induces more workers to join the labor force. Michl (2009, p. 50-51) makes a similar point, arguing that even in advanced capitalist economies like the United States, the labor supply will tend to adjust in the

long run, both by means of immigration and changes in the labor force participation rate, to meet the demand for labor.

Fazzari, Ferri and Variato (2020) offer a particularly fruitful way to formalize these ideas. To capture the idea that a tight labor market will induce increases in the natural rate of growth, they assume that the growth rate of the labor force is a decreasing function of the unemployment rate. They also assume that labor productivity growth is determined by the Kalder-Verdoorn effect. After calibrating their model with data for the United States, they find that, as a result of these mechanisms, there is a continuum of different long-run growth rates compatible with empirically plausible rates of unemployment:

For example, consider an economy with stagnant growth of 1.5% and unemployment of 6%. A 1-percentage point increase in autonomous demand growth raises growth to 2.5%. To induce supply growth to adapt to the higher demand path, unemployment must decline to just 5.3% with the benchmark parameters. The accommodation of demand by supply, what might be called 'reverse Say's Law', is not just theoretically possible but empirically relevant (Fazzari, Ferri and Variato 2020, p. 598).

Thus, within certain limits, the economy can grow at different rates without bumping up against the limits of the available labor supply. In view of these ideas, I will simply assume in most of what follows that the labor force is adequate at all times to accommodate changes in the rate of output growth, but it is also straightforward to

explicitly incorporate Fazzari *et al.*'s model of the labor market into the system of equations set up here, and for the purposes of illustration I will do this in one of the simulations in Section 6.

Let us now turn to the dynamics of the real wage. I will denote the real wage by w and denote the output-labor ratio by q . Unit labor costs are

$$\psi = \frac{w}{q} \quad (16)$$

and the flow of total (after-tax) wage income is

$$W = (1 - \tau)\psi Q. \quad (17)$$

Labor productivity and the wage may fluctuate over time in a variety of different ways, but for simplicity I will assume that they stay constant in relation to each other, so that ψ takes a fixed value strictly between zero and one. The assumption here is that, since the labor supply adjusts to meet the demand for new workers, the state of the economy is relatively unimportant for determining the distribution of income, and thus the wage share can be treated as a parameter reflecting the role of social norms and institutions. Thus this paper adopts a “conventional wage share model” as in Foley and Michl (1999, p. 104).

This formulation is clearly a simplification of reality, since it ignores the possibility that changes in the level of economic activity will affect the distribution of income. But the supermultiplier and Harrodian-Kaleckian models reviewed above all adopt a conventional wage share model as above, and since the purpose here is to extend those analyses to the context in which NCC demand fails to play a stabilizing role, concerns about the distribution of income are largely peripheral to the aims of this paper. Moreover, in models of NCC-demand-driven growth, the long-run rate of growth

is independent of the distribution of income, and consequently, for the purposes of this analysis there is not much reason to worry here about the details of how income distribution is actually determined. Nevertheless, the reader should keep in mind that if a less simplistic description of wage dynamics were incorporated into the model, this could generate a variety of interesting possibilities and technical challenges not addressed here.

3.6. Consumer spending

Next we come to the determination of consumption decisions. Serrano (1995) argues that, because capitalists can draw upon their vast financial resources to make purchases, a portion of their consumption is independent of their current income, and can therefore be modeled as an autonomous expenditure component. But, as Nikiforos (2018) points out, in the long run, financial flows will fluctuate with the state of the economy, and presumably this will affect consumption decisions by changing the amount of financial wealth available to support new expenditures. Thus, rather than treating capitalists' consumption as being truly autonomous, it may be more fruitful to formulate a model in which consumption rises or falls in reaction to changes in net financial assets.

There is reason to believe that these considerations are relevant to the real world. Skott and Ryoo (2008) point out that the ratio of households' net worth to income in the US has been broadly stable over time, and on this basis formulate a model in which households adjust their spending to keep stocks of financial wealth at targeted

levels in relation to GDP. Similarly, Godley (1999), looking at data for the United States, argues that private sector spending tends to fluctuate around a level that balances financial inflows with outflows. All this can be understood in terms of a process that is both simple and intuitive: if spending grows in relation to income by too much for too long, then eventually debts will pile up, and financing constraints will limit additional spending, while if spending grows too slowly in relation to income for too long, then stocks of financial wealth will be rising in relation to income, and this will eventually encourage households to increase their spending. Thus, although private-sector spending may diverge from income flows for extended periods, the fluctuations in net financial assets ultimately exert a powerful limit on this process, and can be expected to bring about changes in the growth rate of consumption expenditures.

With these considerations in mind, let us now set up the equations to model capitalists' consumption expenditures. The variable

$$m = \frac{V_c}{K}, \quad (18)$$

which measures capitalists' net financial wealth in relation to the size of the economy, will be an important factor in capitalists' consumption decisions. I will assume that consumption expenditures tend to grow when $V_c > \sigma Q$, and decay when $V_c < \sigma Q$, where σ is a positive constant corresponding to the idea of a stock-flow norm (Godley and Lavoie, 2006). Scaling V_c and Q to the size of the economy, we obtain the equation

$$\widehat{C}_c = \xi(m - \sigma u) \quad (19)$$

for the growth rate of capitalists' consumption, where ξ is a positive constant. Thus, capitalists' consumption spending is fixed in the short run, but will show a tendency to adjust over time in reaction to changes in financial stock-flow variables. This formulation

is admittedly still rather simple, but it has the advantage of capturing Serrano's (1995) point that capitalists' consumption need not grow in lockstep with their current incomes, while also accepting Nikiforos' (2018) argument about the need to incorporate stock-flow norms into the analysis.

Let us now turn to workers' consumption. Heterodox macroeconomic theories, such as Kalecki (1933), often assume that workers spend exactly what they earn. Such an assumption may be supported by the observation that the bottom 60% of households in the US have very little net worth (Taylor 2020, chapter 3). In fact, there may be a tendency, at least in some years, for workers to spend *more* than they earn: Taylor (2014) looks at data for the US in 2008, and reports that the richest 1% of households had an average saving rate of 81%, while the remaining 99% of households had a negative saving rate. But for the sake of simplicity, in what follows I stick to the usual assumption that workers' consumption is equal to their wage income at all times:

$$C_W = W. \quad (20)$$

Consistent with the assumption that workers spend what they earn, I also assume when implementing the model that the initial value for workers' net financial wealth is zero. Combining these assumptions with equation (5), it is easy to verify that $V_W[t] = 0$ for all t , and thus workers do not earn any interest income in this model.⁸ Altogether, equations (19) and (20) describe, albeit in a simplistic way, a situation in which total financial holdings influence aggregate spending, but there is an unequal distribution of financial holdings across social groups.

⁸ Combining equations (5) and (20), we see that $dV_W/dt = (1 - \tau)iV_W$, which describes exponential growth at the rate $(1 - \tau)i$. Solving this differential equation and using the assumption that the initial value for workers' net financial wealth is zero, we obtain $V_W[t] = 0$ for all t .

3.7. Government spending as a source of semi-autonomous NCC demand

As mentioned above, I will assume that A , the level of government spending, can fluctuate in complex ways but has a well-defined long-run trend rate of growth. I denote this trend rate of growth by γ . Note that this leaves open the possibility that A is a function of other variables. In principle, even the long-run trend rate of growth for A could be determined endogenously through interactions with the rest of the economic system. We will see some simulations in Section 6 that demonstrate these possibilities.

To capture these different possibilities, I assume A satisfies

$$\hat{A} = \Gamma \quad (21)$$

where Γ is a differentiable function of time with a bounded derivative. To make concrete the idea that A fluctuates around the long-run trend rate of growth γ , I also assume there are positive constants κ_1 and κ_2 such that

$$\kappa_1 e^{\gamma t} < A[t] < \kappa_2 e^{\gamma t} \quad (22)$$

for all t . These assumptions leave open the possibility that A can fluctuate in complex ways, making it impossible for the model to ever converge to a long-run equilibrium or periodic trajectory, but of course it is also possible that A could play a stabilizing role. The condition (22) also includes as a special case the situation in which Γ is constant, i.e., $\Gamma[t] = \gamma$ for all t . It is easy to show that, as long as (22) is satisfied, the long-run average value of Γ is γ .⁹

⁹ We have $\bar{\Gamma} = \lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s \Gamma[t] dt = \lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s \frac{d}{dt} \ln[A[t]] dt = \lim_{s \rightarrow \infty} \frac{1}{s} [\ln[A[s]] - \ln[A[0]]] \leq \lim_{s \rightarrow \infty} \frac{1}{s} [\ln[\kappa_2 e^{\gamma s}] - \ln[A[0]]] = \gamma$, so $\bar{\Gamma} \leq \gamma$. A similar argument shows $\bar{\Gamma} \geq \gamma$ as well. Thus $\bar{\Gamma} = \gamma$.

The condition (22) imposes limits on how far A can rise or fall at any given time. This means public spending in this model plays a special role, as suggested by Skott (2019: p. 242), by preventing a complete collapse of aggregate demand during economic downturns. For this reason, although A can fluctuate in complex ways and may depend on other variables, it nevertheless plays a role that is similar to the autonomous expenditure components in the Harrodian-Kaleckian and supermultiplier models reviewed earlier in this paper. Motivated by these observations, and borrowing terminology from Fiebiger (2018), one can therefore refer to government spending in this model as a form of *semi-autonomous non-capacity-creating demand*.

3.8. The full system

The system of equations (2)–(21) constitutes the model that is the focus of this paper. For the convenience of the reader, Table 1 at the end of this section collects together and summarizes the different components of the model.

This system can also be expressed in reduced form. Using equations (2)–(21), we can derive the following set of equations:

$$\dot{a} = a(\Gamma - g + \delta) \quad (23)$$

$$\dot{b} = \phi[b](u - u^d) \quad (24)$$

$$\dot{c} = c(\xi(m - \sigma u) - g + \delta) \quad (25)$$

$$\dot{m} = r + (1 - \tau)im - c - g - m(g - \delta) \quad (26)$$

where

$$u = \Lambda(a + b + c) \quad (27)$$

$$\Lambda = \frac{1}{\tau + (1 - \theta)(1 - \tau)(1 - \psi)} \quad (28)$$

$$r = (1 - \tau)(1 - \psi)u \quad (29)$$

$$a = \frac{A}{K} \quad (30)$$

$$c = \frac{C_c}{K}. \quad (31)$$

Thus the model (2)–(21) can be expressed as a four-dimensional nonlinear dynamical system consisting of equations (23)–(26). We have four state variables: a is the ratio of government spending to fixed capital, b is the Harrodian term in the investment function, c is the ratio of capitalists' consumption to fixed capital, and m (originally defined in equation (18)) is the ratio of capitalists' net financial wealth to fixed capital. Equation (27) gives a convenient formula for capacity utilization u , which shows how a multiplier effect (expressed by the constant Λ) determines output, while equation (29) shows how the profit rate r can be expressed in terms of capacity utilization, thus illustrating how the demand side of the economy can affect profitability. As will be discussed in more detail below, these variables all interact with each other in complex ways.

The reduced-form system (23)–(26) is straightforward to derive. First, to derive equation (23), we take growth rates of both sides of equation (30), and then apply equations (21), (8) and (12) to get $\hat{a} = \hat{A} - \hat{K} = \Gamma - \frac{1}{K}(I - \delta K) = \Gamma - g + \delta$. Equation (24) comes straight from equation (14). We obtain equation (25) by applying growth rates to both sides of equation (31), and then applying equations (19), (8) and (12) to see that $\hat{c} = \hat{C}_c - \hat{K} = \xi(m - \sigma u) - g + \delta$. To derive equation (26), we take growth rates of both sides of equation (18), and then note (using equations (4) and (11)) that $\hat{m} =$

$\widehat{V}_C - \widehat{K} = \frac{1}{V_C}(R + (1 - \tau)iV_C - C_C - I) - g + \delta = \frac{1}{m}(r + (1 - \tau)im - c - g) - g + \delta$. It is

easy to derive equation (27), for the utilization rate, if we note that $u = \frac{Q}{K} =$

$\frac{1}{K}(C_W + C_c + I + A) = \frac{1}{K}((1 - \tau)\psi Q + cK + gK + aK) = (1 - \tau)\psi u + c + g + a$. Finally,

we can derive equation (29) for the profit rate if we note that $r = \frac{R}{K} = \frac{(1 - \tau)(1 - \psi)Q}{K} =$

$(1 - \tau)(1 - \psi)u$.

The system (23)–(26) is *nonautonomous*, in the sense that it has a parameter Γ (representing the growth rate of government expenditures) that may depend on time. However, as is well known, one can always convert a nonautonomous system to an autonomous one by adding an additional state variable (Meiss, 2007: p. 4). More specifically, if we let n satisfy the differential equation $\dot{n} = 1$ with $n[0] = 0$, and replace $\Gamma[t]$ with $\Gamma[n]$, then we obtain a five-dimensional autonomous system in the variables a, b, c, m and n that is equivalent to the nonautonomous system (23)–(26). It will become convenient in one of the proofs later to transform (23)–(26) in this way.

Finally it is important to emphasize that, at any given time, the profit rate, given by equation (29), is determined in a standard Kaleckian fashion; unit labor costs are fixed, and capacity utilization varies to meet changes in demand. If there is an increase in capitalists' consumption, fixed-capital investment or the government budget deficit, then this will increase profits, as expected (Kalecki 1933). But we will see below that the *long-run average* rate of profit is governed by other forces, so that the Kaleckian theory of profits gives an incomplete picture. To understand these “other forces”, let us now move to the next section, which gives the key results of the paper.

Table 1: Summary of the Model

Equation number	Verbal description	Equation
(2)	Aggregate demand Y is the sum of workers' consumption C_w , capitalists' consumption C_c , fixed-capital investment I and government expenditures A . The economy is closed to trade.	$Y = C_w + C_c + I + A$
(3)	Output Q equals aggregate demand Y .	$Q = Y$
(4)	The rate of change for capitalists' net financial assets V_c is profits plus interest income $R + (1 - \tau)iV_c$, minus consumption and investment spending.	$\frac{dV_c}{dt} = R + (1 - \tau)iV_c - C_c - I$
(5)	The rate of change for workers' net financial assets V_w is wage income plus interest income $W + (1 - \tau)iV_w$, minus consumption spending.	$\frac{dV_w}{dt} = W + (1 - \tau)iV_w - C_w$
(6)	The rate of change for the governments' net financial assets V_g is taxes T plus interest income iV_g minus spending A .	$\frac{dV_g}{dt} = T + iV_g - A$
(7)	Proportional tax rate τ on all income.	$T = \tau Q + \tau iV_c + \tau iV_w$
(8)	The rate of change in fixed capital K is gross investment I minus depreciation δK .	$\frac{dK}{dt} = I - \delta K$
(9), (27)	The rate of capacity utilization u is output Q divided by fixed capital K . Equation (27) gives alternative formula.	$u = \frac{Q}{K} = \Lambda(a + b + c)$
(10)	Definition of capitalists' gross profits R , where τ is the tax rate and ψ is the labor share.	$R = (1 - \tau)(1 - \psi)Q$
(11), (29)	The rate of profit r is gross profits R divided by fixed capital K . Equation (29) gives alternative formula.	$r = \frac{R}{K} = (1 - \tau)(1 - \psi)u$
(12)	Identity: gross investment I equals the gross accumulation rate g multiplied by the fixed capital stock K .	$I = gK$
(13)	The gross accumulation rate g consists of a term b that is fixed in the short run, and another term θr .	$g = b + \theta r$
(14)	The rate of change for the b -term in the gross accumulation function depends on the difference between actual and desired utilization (Harrodian investment mechanism).	$\frac{db}{dt} = \phi[b](u - u^d)$
(15)	The rate of change of b slows down if it approaches its upper limit ($b = B$) or lower limit ($b = 0$).	$\phi[b] = \eta b(B - b)$
(16)	The labor share ψ is the real wage w divided by labor productivity q .	$\psi = \frac{w}{q}$
(17)	Definition of workers' after-tax wage income, W .	$W = (1 - \tau)\psi Q$
(18)	Definition: m is the ratio of capitalists' net financial assets V_c to fixed capital K .	$m = \frac{V_c}{K}$
(19)	The growth of capitalists' consumption depends on whether their net financial wealth is above or below the target level.	$\frac{1}{C_c} \frac{dC_c}{dt} = \xi(m - \sigma u)$
(20)	Workers spend what they earn.	$C_w = W$
(21)	The growth rate of government spending, A , is Γ , which may change over time.	$\frac{1}{A} \frac{dA}{dt} = \Gamma$
(22)	The trend growth rate of government spending is γ .	$\kappa_1 e^{\gamma t} < A[t] < \kappa_2 e^{\gamma t}$
(23)–(26)	Model in reduced form with state variables a , b , c and m	See (23)–(26)
(28)	The multiplier effect.	$\Lambda = \frac{1}{\tau + (1 - \theta)(1 - \tau)(1 - \psi)}$
(30)	a is the ratio of government expenditures A to fixed capital.	$a = \frac{A}{K}$
(31)	c is the ratio of capitalists' consumption to fixed capital.	$c = \frac{C_c}{K}$

4. Capital accumulation, utilization and profitability in the long run

4.1. *Some assumptions regarding model parameters and initial conditions*

To begin, we need to make some assumptions. First, I will assume that the long-run average growth rate of government spending is higher than the after-tax interest rate:

$$(1 - \tau)i < \gamma. \quad (32)$$

We will see below that the long-run trend rate of output growth for the model is γ . In this sense, the above condition is similar to the usual assumption that the (after-tax) interest rate does not exceed the growth rate of the economy.

Second, we need to make an assumption regarding the parameter B . The reader should recall that this parameter represents the upper limit for the term b in the accumulation function (13), and reflects the existence of financing constraints on investment. I will assume that

$$B < (\tau + (1 - \theta)(1 - \tau)(1 - \psi))u^d. \quad (33)$$

Note that, when $u = u^d$, the above condition is equivalent to the assumption that $g < \tau u^d + (1 - \tau)(1 - \psi)u^d$. Since $(1 - \tau)(1 - \psi)u^d$ is the profit rate when capacity is utilized at its normal rate, this means that the financing constraint on investment may be relatively lax; the inequality (33) can be satisfied even for solution trajectories in which investment perpetually exceeds profits. The assumption (33) simply imposes an upper limit on the *degree to which* investment can exceed profits in the long run. Although the financing constraint on investment could be modeled in different ways, I will adopt (33) as a reasonable starting point.

Third, we need the initial conditions to be chosen so that they are consistent with the economic interpretation of the model. Since the model describes a closed system with three sectors, I will assume that initially the net financial claims held by those three sectors sum to zero:

$$V_c[0] + V_w[0] + V_g[0] = 0. \quad (34)$$

If (34) did not hold, it would imply the existence of a financial relationship with some entity outside the model, thus violating the assumption that the equations above describe a closed system. I also assume that the initial values $C_c[0]$ (capitalists' consumption) and $K[0]$ (the capital stock) are positive. This means $c[0]$ must be positive, and because of (22), $a[0]$ must be positive as well. I also assume $b[0]$ is strictly between its limiting values 0 and B . Recall that we also made the assumption earlier in the paper that $V_w[0] = 0$, consistent with the idea that workers spend what they earn.

Finally, we need to assume that the investment function (13) is able to sustain growth at the rate γ when productive capacity is utilized at the normal rate.¹⁰

This means:

$$b^* + \theta(1 - \tau)(1 - \psi)u^d = \gamma + \delta, \quad \text{for some } b^* \text{ strictly between 0 and } B. \quad (35)$$

Note that the inequality (33) imposes an upper limit on the possible values for b^* .

¹⁰ Note that it is possible for (35) to hold for an extended period of time, but then become violated if unit labor costs rise or the normal output-capital ratio falls. After this occurred, capacity utilization would be chronically above the desired level, potentially creating an inflationary situation. It is easy to imagine policy responses that would bring γ down in response to inflation in this scenario, so that (35) would tend to stay satisfied. This suggests an avenue through which long-run profitability could impose an upper limit on the growth rate of demand, and consequently the growth of the economy itself, but I will not attempt to formalize any of these ideas here.

4.2. Main theorems

Now we can state the main results of the paper.

Theorem 1: *For each set of parameters and initial conditions meeting the assumptions above, the system (23)(26) has a unique solution $(a[t], b[t], c[t], m[t])$. The solution is bounded and exists for all $t \geq 0$. Moreover, the variables $a[t], b[t]$ and $c[t]$ stay positive for all $t \geq 0$, and in addition $b[t] < B$ for all $t \geq 0$.*

Theorem 1 takes care of some important technical preliminaries. It asserts that the solution for the model exists, and is bounded, so ratios like C_c/K will never become infinite. It also confirms that b , the fluctuating term in the investment function, actually stays within its limiting values 0 and B .

Next we can consider Theorem 2, which is stated below. It says that the long-run average rate of growth for the capital stock is γ . The theorem also states, with the condition $\pi_1 e^{\gamma t} \leq K[t] \leq \pi_2 e^{\gamma t}$, that the capital stock fluctuates within a band $[\pi_1 e^{\gamma t}, \pi_2 e^{\gamma t}]$ that grows at the rate γ . The same can be said about long-run output growth (see (36)). Thus, although solution trajectories in the model need not ever gravitate toward a steady-state growth path or periodic cycle, Theorem 2 provides a way to uncover the trend rate of growth for the economy. Long-run economic development is determined by the growth of NCC semi-autonomous demand.

Theorem 2: *Under the assumptions stated in Theorem 1, the long-run average growth rate for the capital stock, $\bar{g} - \delta$, is equal to γ . Moreover, there are positive constants π_1 , π_2 , π_3 and π_4 such that, for all $t \geq 0$,*

$$\pi_1 e^{\gamma t} \leq K[t] \leq \pi_2 e^{\gamma t} \quad (36)$$

and

$$\pi_3 e^{\gamma t} \leq Q[t] \leq \pi_4 e^{\gamma t}. \quad (37)$$

Finally, Theorem 3 shows how to determine the long-run average values for capacity utilization and the rate of profit:

Theorem 3: *If we make the assumptions in Theorem 1, then the long-run average rate of capacity utilization is $\bar{u} = u^d$, and the long-run average rate of profit is $\bar{r} = (1 - \tau)(1 - \psi)u^d$.*

Thus, in the long run, the profit rate fluctuates around a center of gravity determined by labor costs, taxes and the output-capital ratio at normal utilization of capacity. If there is an increase in government spending or the level of investment, this will boost profits in the short run, but—in contrast to what standard Kaleckian theory predicts—will have no effect on the long-run average profit rate. Capitalists seeking to increase the profit rate in the long run would need to either decrease unit labor costs or adopt a production technology that increases the normal output-capital ratio u^d . This is consistent with a classical-Marxian theory of profits, as in Duménil and Lévy (1993).

Although the model developed here clearly simplifies reality in important ways, Theorems 2 and 3 are consistent with the results of some recent empirical studies. As I mentioned in the introduction, several papers have found evidence that autonomous NCC demand drives long-run GDP growth. This is precisely what we would expect from Theorem 2, which shows that in this model, the trend growth rate of GDP is equal to γ . On the other hand, Theorem 3 shows that the utilization rate tends to fluctuate around the desired level u^d in this model, and thus long-run utilization is *not* determined on the demand side of the economy. This fits with Skott (2012), who reviews data for the United States and finds it inconsistent with theories in which the demand side determines long-run utilization. Also relevant is Gahn (2021), who finds that demand shocks have only temporary effects on the utilization rate, but for an alternative perspective see Nikiforos (2016).

4.3. *An intuitive explanation of the theorems*

Theorems 1-3 will be proved below in Section 4. But before getting into all the technical details, it is useful to informally think through the mechanisms behind the main results stated above. In fact, although the proofs below involve some rather intricate mathematics, the basic reasoning behind them can be understood on an intuitive level.

To begin, it is useful to consider the counterfactual scenario in which a (the ratio of government spending to productive capacity) and c (the ratio of capitalists' consumption to productive capacity) are held fixed at zero. In that situation, if capacity utilization is at the normal rate (so $u = u^d$), the ratio of total saving to productive

capacity would be $\tau u^d + (1 - \tau)(1 - \psi)u^d$. If investment demand were to absorb total saving, then the gross rate of accumulation would need to equal $\tau u^d + (1 - \tau)(1 - \psi)u^d$, implying that the gross accumulation rate would exceed the profit rate by τu^d . If parameters are chosen to have realistic values, then this implies that capitalists would be running a huge financial deficit, and to keep capacity utilization at its normal level, capitalists would have to sustain this huge deficit in perpetuity.¹¹ But because of the existence of financing constraints, formalized in the model by the inequality (33), this cannot happen; instead, the accumulation rate would be below the level $\tau u^d + (1 - \tau)(1 - \psi)u^d$, with the consequence that u would be below the level u^d . Thus, the accumulation rate would be perpetually falling (due to equation (14)), ultimately leading to a total economic collapse.

These observations suggest that, in order for the model to sustain a positive rate of growth, a or c will need to be positive. But, owing to equation (19), c will tend to fall toward zero if the economy is growing and m (the ratio of capitalists' net financial assets to productive capacity) is not positive. And m , in turn, cannot stay positive unless a is positive, simply as a matter of accounting. The upshot is that if a does not stay above a certain level, the system will crash. This means that, in view of equation (23), the growth rate of the capital stock cannot stay perpetually above the trend growth rate of semi-autonomous NCC demand, γ . The growth rate of capital also cannot stay *below* γ forever, because if it did, demand would eventually become larger and larger in relation

¹¹ In a more realistic model of an economy open to international trade, in which export demand forms a component of A , and μ denotes the import propensity, the ratio of available saving to productive capacity in this scenario would be $(\tau + \mu)u^d + (1 - \tau)(1 - \psi)u^d$. Thus, if investment demand absorbed total saving at normal capacity utilization, the accumulation rate would exceed the profit rate by $(\tau + \mu)u^d$. For realistic values of μ and τ , this would imply that, in the steady state, capitalists would run a financial deficit equal to half of GDP or more.

to productive capacity, and this would drive up the rate of accumulation in relation to γ . Thus, in the long run, the net accumulation rate will have to fluctuate around γ ; the long-run average value for $g - \delta$ must be γ . Similar observations imply that key ratios such as $A[t]/K[t]$ must stay bounded between positive constants, and this makes it possible to establish the bounds (36) and (37).

Theorem 3, on the other hand, is a consequence of the above mechanisms and the investment dynamics in equations (13) through (15). If u rises above the targeted utilization rate u^d , then this will cause the accumulation rate to rise. Because of the Harrodian nature of the investment dynamics, the accumulation rate can keep rising, stimulating demand and pushing u further and further upward in the process. But, by pushing up the accumulation rate, this will eventually cause productive capacity to rise in relation to government spending, which will push a down and, in accord with the processes described in the previous paragraphs, will eventually push u back down again. Similarly, if capacity utilization is below the normal rate, then this will set off destabilizing forces going in the opposite direction, but will ultimately cause a to rise, which in turn will lead to progressive increases in u . Together, these dynamics will cause u to fluctuate around u^d , so that the long-run average rate of capacity utilization is $\bar{u} = u^d$, and the long-run average rate of profit is $\bar{r} = (1 - \tau)(1 - \psi)u^d$.¹²

¹² The simplifying assumption that ψ is a constant becomes important here. In a more complex model in which ψ is allowed to vary, the long-run average profit rate would be $\bar{r} = (1 - \tau)(1 - \bar{\psi})\bar{u} + (1 - \tau)\text{Cov}[1 - \psi, u]$, where $\bar{\psi}$ is the long-run average wage share and Cov denotes the covariance operator.

5. Proofs of Theorems 1-3

The proofs of Theorems 1-3 are somewhat more complicated than what one would generally find in a paper dealing solely with stable equilibria. Nevertheless, the results only rely on basic facts from calculus, differential equations, and a very small amount of real analysis. In fact the basic idea for the proofs comes from a brief comment in a paper by Skott and Ryoo (2008: p. 840).

To start, we need to establish the accounting consistency of the model. If we add together equations (4)–(6), and then note that

$$R + W + T + (1 - \tau)iV_C + (1 - \tau)iV_W = Q + i(V_C + V_W), \quad (38)$$

we can see that

$$\frac{d}{dt}(V_C + V_W + V_G) = Q + i(V_C + V_W + V_G) - (C_C + C_W + I + A). \quad (39)$$

Now, using the fact that $Q = Y = C_C + C_W + I + A$, we see that

$$\frac{d}{dt}(V_C + V_W + V_G) = i(V_C + V_W + V_G). \quad (40)$$

Using the initial value (34) and solving the above differential equation, we obtain

$$V_C[t] + V_W[t] + V_G[t] = 0 \quad (41)$$

for all t . Thus aggregate financial claims in the model sum to zero, as in the stock-flow consistent approach developed by Godley and Lavoie (2006).

Notice that (41) implies that $\frac{d}{dt}(V_C + V_W + V_G) = 0$, i.e.,

$$\dot{V}_C = -(\dot{V}_W + \dot{V}_G). \quad (42)$$

Plugging the expressions in equations (4), (5) and (6) into (42), applying (41), and doing a little algebra, we can see that

$$R + (1 - \tau)iV_C - I - C_C = A + (1 - \tau)iV_C - \tau Q. \quad (43)$$

Equation (43) will be useful later.

In what follows, the key difficulty lies in establishing appropriate bounds for the different variables in the model. These bounds will be established in the lemmata below using standard techniques (roughly as in Meiss, 2007). Once we do this, the proofs of the main results (i.e., the assertions in Theorems 2 and 3) are relatively easy and only require basic calculus.

In what follows, it will be useful to work in terms of the following variables:

$$w[t] = \frac{V_C[t]}{e^{\gamma t}} \quad (44)$$

$$x[t] = \frac{C_C[t]}{e^{\gamma t}} \quad (45)$$

$$y[t] = \frac{A[t]}{e^{\gamma t}} \quad (46)$$

$$z[t] = \frac{e^{\gamma t}}{K[t]}. \quad (47)$$

Of course we have $\dot{x} = x(\xi(m - \sigma u) - \gamma)$, $\dot{z} = z(\gamma - g + \delta)$, etc.

Lemma 1: Under the assumptions made in Theorem 1, suppose there is an interval $[0, T]$ on which the solution to (23)–(26) exists and is continuous. Suppose $0 < b[t] < B$ for all t in $[0, T)$, and also suppose $a[t]$ and $c[t]$ are positive for all t in $[0, T)$. Define the constants

$$w^{\text{up}} = \max \left\{ V_C[0] + \frac{\kappa_2}{\gamma - (1 - \tau)i}, \frac{\kappa_2}{\gamma - (1 - \tau)i} \right\} \quad (48)$$

$$x^{\text{up}} = \max \left\{ \frac{w^{\text{up}}}{\sigma \Lambda}, C_C[0] \right\} \quad (49)$$

$$y^{\text{up}} = \kappa_2 \quad (50)$$

$$z^{\text{up}} = \max \left\{ \frac{\gamma + \delta}{\theta(1-\tau)(1-\psi)\Lambda\kappa_1}, \frac{1}{K[0]} \right\}. \quad (51)$$

Then, for all t in $[0, T]$, $w[t] \leq w^{\text{up}}$, $x[t] \leq x^{\text{up}}$, $y[t] \leq y^{\text{up}}$ and $z[t] \leq z^{\text{up}}$. Moreover, defining

$$a^{\text{up}} = y^{\text{up}} z^{\text{up}} \quad (52)$$

$$c^{\text{up}} = x^{\text{up}} z^{\text{up}}, \quad (53)$$

we have $a[t] \leq a^{\text{up}}$ and $c[t] \leq c^{\text{up}}$ for all t in $[0, T]$.

Proof: To begin, we consider the dynamics of $K[t]$ on $[0, T]$. It is easy to verify that

$$K[t] = K[0] e^{\int_0^t (g[s] - \delta) ds} \quad (54)$$

is the solution to equation (8), where $g[s] = b[s] + \theta(1-\tau)(1-\psi)\Lambda(a[s] + b[s] + c[s])$.

Note that $g[s]$ is a continuous function on the compact set $[0, T]$ because a, b and c are continuous on $[0, T]$; consequently, on this interval $g[s]$ stays bounded and thus the exponential factor in (54) cannot approach zero or infinity. Since $K[0] > 0$, equation (54) implies that $K[t] > 0$ for all $t \in [0, T]$. Since a, b and c are continuous functions of t such that $a[t] > 0$, $0 < b[t] < B$, and $c[t] > 0$ for all t in $[0, T]$, we also know that $a[t] \geq 0$, $0 \leq b[t] \leq B$, and $c[t] \geq 0$ for all t in $[0, T]$. Since $Q = uK = \Lambda(a + b + c)K$, this shows

$$Q[t] \geq 0 \quad (55)$$

for all t in $[0, T]$.

Next we consider the dynamics of $V_c = mK$ on $[0, T]$. Notice that

$$\begin{aligned} \frac{d}{dt} \left[e^{-(1-\tau)it} V_c[t] \right] &= -(1-\tau)ie^{-(1-\tau)it} V_c + e^{-(1-\tau)it} \frac{d}{dt} [V_c] \\ &= -(1-\tau)ie^{-(1-\tau)it} V_c + e^{-(1-\tau)it} (R + (1-\tau)iV_c - C_c - I) \end{aligned}$$

$$\begin{aligned}
&= -(1-\tau)ie^{-(1-\tau)it}V_C + e^{-(1-\tau)it}(A + (1-\tau)iV_C - \tau Q) \\
&= e^{-(1-\tau)it}(A - \tau Q) \\
&\leq e^{-(1-\tau)it}A \\
&< e^{-(1-\tau)it}\kappa_2 e^{\gamma t} \\
&= \kappa_2 e^{(\gamma-(1-\tau)i)t}
\end{aligned}$$

where we have used (43), (55) and (22). Integrating both sides of the inequality

$\frac{d}{dt} \left[e^{-(1-\tau)it} V_C[t] \right] < \kappa_2 e^{(\gamma-(1-\tau)i)t}$ and applying the fundamental theorem of calculus then gives $e^{-(1-\tau)it} V_C[t] - V_C[0] < \frac{\kappa_2}{\gamma-(1-\tau)i} e^{(\gamma-(1-\tau)i)t}$. Thus

$$V_C[t] < V_C[0]e^{(1-\tau)it} + \frac{\kappa_2}{\gamma-(1-\tau)i} e^{\gamma t}. \quad (56)$$

If $V_C[0] \geq 0$, then this implies that $V_C[t] < \left(V_C[0] + \frac{\kappa_2}{\gamma-(1-\tau)i} \right) e^{\gamma t}$, while if $V_C[0] < 0$, the

above inequality implies $V_C[t] < \frac{\kappa_2}{\gamma-(1-\tau)i} e^{\gamma t}$. In either case we have

$$V_C[t] < e^{\gamma t} \max \left\{ V_C[0] + \frac{\kappa_2}{\gamma-(1-\tau)i}, \frac{\kappa_2}{\gamma-(1-\tau)i} \right\}. \quad (57)$$

This proves that $w[t] < w^{\text{up}}$ for all t in $[0, T]$, where w^{up} is defined by (48).

Next, we know that $y[t] \leq \kappa_2$ for all t because of the condition (22). This establishes that $y[t] \leq y^{\text{up}}$ for all t in $[0, T]$. Notice also that, again because of (22), we have $y[t] > \kappa_1 > 0$ for all t .

Now, let us consider the dynamics of x . Since we know already that $K[t] > 0$ for all t in $[0, T]$ by (54), and $e^{\gamma t} > 0$ for all t , we know $z[t] > 0$ for all t in $[0, T]$. Additionally, since $x[t] = \frac{c_C[t]}{e^{\gamma t}} = \frac{c[t]K[t]}{e^{\gamma t}}$, we know that $x[t] > 0$ for all t in $[0, T)$. We also have $y[t] > \kappa_1 > 0$ for all t by (22). With all this in mind, we can see that, on $[0, T)$,

$$\dot{x} = x(\widehat{C_C} - \gamma)$$

$$\begin{aligned}
&= x(\xi(m - \sigma u) - \gamma) \\
&= x(\xi(m - \sigma\Lambda(a + b + c)) - \gamma) \\
&= x(\xi(wz - \sigma\Lambda(yz + b + xz)) - \gamma) \\
&= xz(\xi(w - \sigma\Lambda(y + x)) - (b + \gamma)/z) \\
&\leq xz\xi(w - \sigma\Lambda(y + x)) \\
&< xz\xi(w - \sigma\Lambda x)
\end{aligned}$$

It follows that $\dot{x}[t] < 0$, and consequently x will be decreasing, whenever

$$x \geq \frac{w^{\text{up}}}{\sigma\Lambda}. \quad (58)$$

Thus, on $[0, T)$, x cannot exceed the maximum of the two values $\frac{w^{\text{up}}}{\sigma\Lambda}$ and $x[0]$. Since

$x[0] = \frac{c_c[0]}{e^{\gamma_0}} = c_c[0]$, this shows that for all t in $[0, T)$, $x[t] \leq x^{\text{up}}$, where x^{up} is defined by

(49). Since x is continuous, it follows from this that $x[t] \leq x^{\text{up}}$ on all of $[0, T]$.

Next, let us establish an upper bound for z , keeping in mind that $\dot{z} = z(\gamma - g + \delta)$. We have, on $[0, T)$,

$$\begin{aligned}
g &= b + \theta(1 - \tau)(1 - \psi)\Lambda(a + b + c) \\
&> \theta(1 - \tau)(1 - \psi)\Lambda a \\
&= \theta(1 - \tau)(1 - \psi)\Lambda yz \\
&> \theta(1 - \tau)(1 - \psi)\Lambda\kappa_1 z.
\end{aligned}$$

As a result, $\gamma + \delta < g$ whenever $\gamma + \delta < \theta(1 - \tau)(1 - \psi)\Lambda\kappa_1 z$. Thus $\dot{z} < 0$, and consequently z is decreasing, whenever

$$z > \frac{\gamma + \delta}{\theta(1 - \tau)(1 - \psi)\Lambda\kappa_1}. \quad (59)$$

Thus, for all t in $[0, T)$, $z[t]$ less than or equal to the value z^{up} given by (51). Since z is continuous on $[0, T]$, it follows that $z[t] \leq z^{\text{up}}$ on $[0, T]$.

Finally, note that $a[t] \leq a^{\text{up}}$ and $c[t] \leq c^{\text{up}}$ for all t in $[0, T]$, where a^{up} and c^{up} are defined by (52) and (53), because $a = yz$ and $c = xz$. ■

Lemma 2: Under the assumptions made in Theorem 1, suppose there is an interval $[0, T]$ on which the solution to (23)–(26) exists and is continuous. Suppose $0 < b[t] < B$ for all t in $[0, T)$, and also suppose $a[t]$ and $c[t]$ are positive for all t in $[0, T)$. Then there is an upper bound b^{up} for $b[t]$ on $[0, T]$ that is strictly less than B . Moreover, the upper bound can be chosen to depend only on model parameters and initial conditions; the upper bound does not depend on T .

Proof: Let b_1 be any number that is strictly greater than b^* and $b[0]$, but strictly less than B . Recall that $b^* + \theta(1 - \tau)(1 - \psi)u^d = \gamma + \delta$ by assumption (35). Since $b_1 > b^*$, this means

$$b_1 + \theta(1 - \tau)(1 - \psi)u^d - (\gamma + \delta) > 0. \quad (60)$$

Because of (60), there is a positive number T_1 for which

$$e^{T_1(b_1 + \theta(1 - \tau)(1 - \psi)u^d - (\gamma + \delta))} > \frac{z^{\text{up}}}{z^{\text{mb}}}, \quad (61)$$

where

$$z^{\text{mb}} = \frac{(u^d/\Lambda - B)}{(x^{\text{up}} + y^{\text{up}})}, \quad (62)$$

and x^{up} , y^{up} and z^{up} are the bounds established in Lemma 1. Note that z^{mb} is positive because of the assumption (33). Now let u^+ be any number that is greater than both u^d and $\Lambda(a^{\text{up}} + B + c^{\text{up}})$ and define

$$b^{\text{up}} = B - (B - b_1)e^{-\eta B(u^+ - u^d)T_1}. \quad (63)$$

This defines b^{up} entirely in terms of model parameters, the bounds established in the previous lemma, and the initial conditions. Since the bounds themselves were defined in terms of model parameters and initial conditions, this means b^{up} can be defined entirely in terms of model parameters and initial conditions, and in particular does not depend on the T in the statement of the lemma. Our goal is to prove that b^{up} is an upper bound for b .

To establish this upper bound, I will use a proof by contradiction. Thus, let us suppose that $b[t]$ does *not* stay less than or equal to b^{up} . Then there is a number b_{11} in the interval (b^{up}, B) and a number $t_{11} > 0$ such that $b[t_{11}] = b_{11}$. Notice that, by (63), since $b_{11} > b^{\text{up}}$, we must have

$$B - b_{11} < (B - b_1)e^{-\eta B(u^+ - u^d)T_1}. \quad (64)$$

Also note that, since $e^{-\eta B(u^+ - u^d)T_1}$ must be strictly between zero and one, (64) implies that that $b_1 < b_{11}$.

I claim that, since $b[t]$ is a continuous function and $b[0] < b_1 < b_{11} < B$, there is a positive real number t_1 such that $b[t_1] = b_1$, $b[t_{11}] = b_{11}$, and $b[t] > b_1$ for all t in $(t_1, t_{11}]$. To prove this claim, note first that the set $\{t \in [0, t_{11}]: b[t] = b_1\}$ is closed and bounded, and thus is compact by the Heine-Borel theorem (Bartle and Sherbert, 2011: Chapter 11). The set is nonempty by the intermediate value theorem since $b[0] < b_1 <$

$b[t_{11}]$. Since this is a compact and nonempty set of real numbers, it has a maximum element. Call this maximum element t_1 . Then the interval $[t_1, t_{11}]$ is such that

$$b[t_1] = b_1, b[t_{11}] = b_{11} \text{ and } b[t] > b_1 \text{ for all } t \text{ in } (t_1, t_{11}]. \quad (65)$$

Thus we have established the existence of a number t_1 with the desired properties.

Now we need to prove that $T_1 < (t_{11} - t_1)$ (recall that T_1 is a positive number satisfying (61)). To prove this, we first note that

$$\frac{d}{dt} \ln[B - b] = \frac{-\dot{b}}{B - b} = -\eta b(u - u^d) \geq -\eta B(u^+ - u^d). \quad (66)$$

If we integrate the inequality $\frac{d}{dt} \ln[B - b] \geq -\eta B(u^d - u^+)$ over the interval $[t_1, t_{11}]$, and apply the Fundamental Theorem of Calculus, we see that

$$\ln[B - b[t_{11}]] \geq \ln[B - b[t_1]] - \eta B(u^+ - u^d)(t_{11} - t_1). \quad (67)$$

Applying the natural exponential function to both sides of (67) then gives

$$B - b[t_{11}] \geq (B - b[t_1])e^{-\eta B(u^+ - u^d)(t_{11} - t_1)}. \quad (68)$$

Now, note that, by (64), $B - b[t_{11}] < (B - b[t_1])e^{-\eta B(u^+ - u^d)T_1}$. Combining this with (68), we obtain

$$(B - b[t_1])e^{-\eta B(u^+ - u^d)T_1} > (B - b[t_1])e^{-\eta B(u^+ - u^d)(t_{11} - t_1)}. \quad (69)$$

Dividing through by $B - b[t_1]$ and then taking the natural logarithm of both sides, we obtain

$$-\eta B(u^+ - u^d)T_1 > -\eta B(u^+ - u^d)(t_{11} - t_1). \quad (70)$$

It follows that

$$T_1 < t_{11} - t_1. \quad (71)$$

Next, we need to show that $\int_{t_1}^{t_{11}} u dt > (t_{11} - t_1)u^d$, i.e., the average value of u on $[t_1, t_{11}]$ is greater than u^d . This will be done in a series of steps. I will show first that

$$\int_{t_1}^{t_{11}} \eta(B - b)u dt < \int_{t_1}^{t_{11}} \eta(B - b)u^d dt. \text{ We have}$$

$$\frac{d}{dt} [\ln[b]] = \eta(B - b)(u - u^d) \quad (72)$$

and therefore

$$\begin{aligned} \ln[b[t_{11}]] - \ln[b[t_1]] &= \int_{t_1}^{t_{11}} \frac{d}{dt} [\ln[b[t]]] dt \\ &= \int_{t_1}^{t_{11}} \eta(B - b)(u - u^d) dt \\ &= \int_{t_1}^{t_{11}} \eta(B - b)u dt - \int_{t_1}^{t_{11}} \eta(B - b)u^d dt. \end{aligned}$$

Since $b_{11} > b_1$, $\ln[b[t_{11}]] > \ln[b[t_1]]$, and consequently, using the result above, we

$$\text{have } \int_{t_1}^{t_{11}} \eta(B - b)u dt = \ln[b[t_{11}]] - \ln[b[t_1]] + \int_{t_1}^{t_{11}} \eta(B - b)u^d dt > \int_{t_1}^{t_{11}} \eta(B - b)u^d dt.$$

Thus

$$\int_{t_1}^{t_{11}} \eta(B - b)u dt > \int_{t_1}^{t_{11}} \eta(B - b)u^d dt. \quad (73)$$

Now we need to show that $\int_{t_1}^{t_{11}} \eta b u dt > \int_{t_1}^{t_{11}} \eta b u^d dt$. To do this, we note that

$$\frac{d}{dt} \ln[B - b] = \frac{-\dot{b}}{B - b} = -\eta b(u - u^d), \text{ and therefore,}$$

$$\begin{aligned} \ln[B - b[t_1]] - \ln[B - b[t_{11}]] &= -(\ln[B - b[t_{11}]] - \ln[B - b[t_1]]) \\ &= - \int_{t_1}^{t_{11}} \frac{d}{dt} [\ln[B - b[t]]] dt \\ &= - \int_{t_1}^{t_{11}} -\eta b(u - u^d) dt \end{aligned}$$

$$\begin{aligned}
&= \int_{t_1}^{t_{11}} \eta b(u - u^d) dt \\
&= \int_{t_1}^{t_{11}} \eta b u dt - \int_{t_1}^{t_{11}} \eta b u^d dt.
\end{aligned}$$

Since $b_{11} > b_1$, $\ln[B - b[t_1]] > \ln[B - b[t_{11}]]$. Putting this all together, we have

$\int_{t_1}^{t_{11}} \eta b u dt = \int_{t_1}^{t_{11}} \eta b u^d dt + \ln[B - b[t_1]] - \ln[B - b[t_{11}]] > \int_{t_1}^{t_{11}} \eta b u^d dt$. Thus

$$\int_{t_1}^{t_{11}} \eta b u dt > \int_{t_1}^{t_{11}} \eta b u^d dt. \quad (74)$$

Now note that

$$\int_{t_1}^{t_{11}} \eta(B - b)u dt + \int_{t_1}^{t_{11}} \eta b u dt = \int_{t_1}^{t_{11}} (\eta(B - b)u + \eta b u) dt = \int_{t_1}^{t_{11}} \eta B u dt \quad (75)$$

and

$$\int_{t_1}^{t_{11}} \eta(B - b)u^d dt + \int_{t_1}^{t_{11}} \eta b u^d dt = \int_{t_1}^{t_{11}} (\eta(B - b)u^d + \eta b u^d) dt = \int_{t_1}^{t_{11}} \eta B u^d dt. \quad (76)$$

Putting (73), (74), (75) and (76) together, we see that

$$\begin{aligned}
\int_{t_1}^{t_{11}} \eta B u dt &= \int_{t_1}^{t_{11}} \eta(B - b)u dt + \int_{t_1}^{t_{11}} \eta b u dt \\
&> \int_{t_1}^{t_{11}} \eta(B - b)u^d dt + \int_{t_1}^{t_{11}} \eta b u^d dt \\
&= \int_{t_1}^{t_{11}} \eta B u^d dt.
\end{aligned}$$

Thus we have shown that $\int_{t_1}^{t_{11}} \eta B u dt > \int_{t_1}^{t_{11}} \eta B u^d dt$, and if we divide through by the

constant ηB , and then note that $\int_{t_1}^{t_{11}} u^d dt = (t_{11} - t_1)u^d$, we obtain the inequality

$$\int_{t_1}^{t_{11}} u dt > (t_{11} - t_1)u^d. \quad (77)$$

Now we have

$$K[t_{11}] = K[t_1]e^{\int_{t_1}^{t_{11}}(b[s]+\theta(1-\tau)(1-\psi)u[s]-\delta)ds} > K[t_1]e^{(t_{11}-t_1)(b_1+\theta(1-\tau)(1-\psi)u^d-\delta)} \quad (78)$$

where we used (77) and the fact from (65) that $b[t] \geq b_1$ for all t in $[t_1, t_{11}]$.

Consequently, using (78), (71), and (61), we see that

$$\begin{aligned} z[t_{11}] &= \frac{e^{\gamma t_{11}}}{K[t_{11}]} \\ &< \frac{e^{\gamma t_{11}}}{K[t_1]e^{(t_{11}-t_1)(b_1+\theta(1-\tau)(1-\psi)u^d-\delta)}} \\ &= \frac{e^{\gamma(t_{11}-t_1)}e^{\gamma t_1}}{K[t_1]e^{(t_{11}-t_1)(b_1+\theta(1-\tau)(1-\psi)u^d-\delta)}} \\ &= \frac{e^{\gamma t_1}}{K[t_1]e^{(t_{11}-t_1)(b_1+\theta(1-\tau)(1-\psi)u^d-(\gamma+\delta))}} \\ &< \frac{e^{\gamma t_1}}{K[t_1]e^{T_1(b_1+\theta(1-\tau)(1-\psi)u^d-(\gamma+\delta))}} \\ &= \frac{z[t_1]}{e^{T_1(b_1+\theta(1-\tau)(1-\psi)u^d-(\gamma+\delta))}} \\ &\leq \frac{z^{\text{up}}}{e^{T_1(b_1+\theta(1-\tau)(1-\psi)u^d-(\gamma+\delta))}} \\ &< z^{\text{mb}}. \end{aligned}$$

Thus $z[t_{11}] < z^{\text{mb}}$. It follows, using what has just been shown, together with (62), that

$$\begin{aligned} \dot{b}[t_{11}] &= \eta b[t_{11}](B - b[t_{11}])(\Lambda(a[t_{11}] + b[t_{11}] + c[t_{11}]) - u^d) \\ &= \eta b[t_{11}](B - b[t_{11}])(\Lambda(y[t_{11}]z[t_{11}] + b[t_{11}] + x[t_{11}]z[t_{11}]) - u^d) \\ &\leq \eta b[t_{11}](B - b[t_{11}])(\Lambda(y^{\text{up}}z[t_{11}] + B + x^{\text{up}}z[t_{11}]) - u^d) \\ &< \eta b[t_{11}](B - b[t_{11}])(\Lambda((y^{\text{up}} + x^{\text{up}})z^{\text{mb}} + B) - u^d) \end{aligned}$$

$$\begin{aligned}
&= \eta b[t_{11}](B - b[t_{11}]) \left(\Lambda \left((y^{\text{up}} + x^{\text{up}}) \frac{\left(\frac{u^d}{\Lambda} - B\right)}{(x^{\text{up}} + y^{\text{up}})} + B \right) - u^d \right) \\
&= \eta b[t_{11}](B - b[t_{11}]) \left(\Lambda \left(\frac{u^d}{\Lambda} - B + B \right) - u^d \right) \\
&= 0
\end{aligned}$$

Now recall that b_{11} was an arbitrary number between b^{up} and B , and t_{11} was any number such that $b[t_{11}] = b_{11}$. We have just shown that $\dot{b}[t_{11}] < 0$; we conclude that $\dot{b} < 0$ whenever b is between b^{up} and B .

On the other hand, since $b[0] < b^{\text{up}} < b[t_{11}]$ and b is a continuous function, there must be a number t_0 such that $b[t_0] = b^{\text{up}}$ and $b^{\text{up}} < b[t] < B$ for all t in $(t_0, t_{11}]$.

Hence, by the mean value theorem, there is a t in (t_0, t_{11}) such that $\dot{b}[t] = \frac{b_{11} - b^{\text{up}}}{t_{11} - t_0} > 0$.

But, by what we have just shown in the previous paragraph, $\dot{b}[t]$ must be negative. This is a contradiction. Thus b cannot exceed b^{up} . ■

Lemma 3: Under the assumptions made in Theorem 1, suppose there is an interval $[0, T]$ on which the solution to (23)–(26) exists and is continuous. Suppose $0 < b[t] < B$ for all t in $[0, T)$, and also suppose $a[t]$ and $c[t]$ are positive for all t in $[0, T)$. Then there exist real numbers w^{low} and z^{low} , where $z^{\text{low}} > 0$ (but w^{low} can be positive or negative), which depend only on model parameters and initial conditions and not on T , such that $z[t] \geq z^{\text{low}}$ and $w[t] \geq w^{\text{low}}$ for all t in $[0, T]$. In addition, there are real numbers m^{low} and m^{up} , which depend on model parameters and initial conditions but

not on T , such that $m^{\text{low}} \leq m[t] \leq m^{\text{up}}$ for all t in $[0, T]$. Finally, $c[t] > 0$, $a[t] > 0$ and $0 < b[t] < B$ for all t in $[0, T]$ (not just $[0, T)$).

Proof: This proof uses a strategy somewhat similar to the one used in the proof of Lemma 2. Let T_1 be any positive number such that

$$b^* > b^{\text{up}} e^{-\eta \Lambda(B-b^{\text{up}})^2 T_1}. \quad (79)$$

Define

$$z_1 = \min\{z^{\text{mb}}, z[0]\} \quad (80)$$

and

$$z^{\text{low}} = z_1 e^{T_1(\gamma + \delta - b^{\text{up}} - \theta(1-\tau)(1-\psi)u^d)} \quad (81)$$

where b^{up} is the bound we established in Lemma 2 at z^{mb} is as defined in (62). Recall from the proof of Lemma 2 that $b^{\text{up}} > b^*$. Thus by (35), $\gamma + \delta - b^{\text{up}} - \theta(1-\tau)(1-\psi)u^d$ is negative, and consequently $e^{T_1(\gamma + \delta - b^{\text{up}} - \theta(1-\tau)(1-\psi)u^d)}$, so (81) implies $z^{\text{low}} < z_1$.

Now suppose that z^{low} as defined above is not a lower bound for $z[t]$. Then there is a t_{11} such that $z[t_{11}] = z_{11}$, where z_{11} is a positive number such that $z_{11} < z^{\text{low}}$. Moreover, making the same argument we made in the proof of Lemma 2, we can find a positive number t_1 such that $z[t_1] = z_1$, and $z[t] \leq z_1$ for all t in $[t_1, t_{11}]$. Then, for all t in $[t_1, t_{11}]$,

$$\begin{aligned} u - u^d &= \Lambda(a[t] + b[t] + c[t]) - u^d \\ &= \Lambda(y[t]z[t] + b[t] + x[t]z[t]) - u^d \\ &\leq \Lambda(y^{\text{up}}z[t] + b^{\text{up}} + x^{\text{up}}z[t]) - u^d \\ &\leq \Lambda((y^{\text{up}} + x^{\text{up}})z^{\text{mb}} + b^{\text{up}}) - u^d \end{aligned}$$

$$\begin{aligned}
&= \Lambda \left((y^{\text{up}} + x^{\text{up}}) \frac{\left(\frac{u^d}{\Lambda} - B\right)}{(x^{\text{up}} + y^{\text{up}})} + b^{\text{up}} \right) - u^d \\
&= \Lambda \left(\frac{u^d}{\Lambda} - B + b^{\text{up}} \right) - u^d \\
&= \Lambda(b^{\text{up}} - B).
\end{aligned}$$

Note that we used the fact that, by (80), $z_1 \leq z^{\text{mb}}$, and thus $z[t] \leq z^{\text{mb}}$ for all t in $[t_1, t_{11}]$. Rewriting what has just been proved, we have

$$u \leq u^d - \Lambda(B - b^{\text{up}}) \quad (82)$$

on $[t_1, t_{11}]$. In particular, this means $u < u^d$ during the time interval $[t_1, t_{11}]$. It follows that

$$z[t_{11}] = z[t_1] e^{\int_{t_1}^{t_{11}} (\gamma - g + \delta) dt} \geq z[t_1] e^{\int_{t_1}^{t_{11}} (\gamma + \delta - b^{\text{up}} - \theta(1-\tau)(1-\psi)u^d) dt} \quad (83)$$

and consequently, since $\int_{t_1}^{t_{11}} (\gamma + \delta - b^{\text{up}} - \theta(1-\tau)(1-\psi)u^d) dt = (t_{11} - t_1) (\gamma + \delta - b^{\text{up}} - \theta(1-\tau)(1-\psi)u^d)$, the inequality (83) implies that

$$z[t_{11}] \geq z[t_1] e^{(t_{11}-t_1)(\gamma+\delta-b^{\text{up}}-\theta(1-\tau)(1-\psi)u^d)}. \quad (84)$$

Because $z[t_{11}] = z_{11} < z^{\text{low}}$, combining together (81) and (84) yields

$$z_1 e^{T_1(\gamma+\delta-b^{\text{up}}-\theta(1-\tau)(1-\psi)u^d)} > z_1 e^{(t_{11}-t_1)(\gamma+\delta-b^{\text{up}}-\theta(1-\tau)(1-\psi)u^d)}, \quad (85)$$

which implies

$$T_1 < t_{11} - t_1. \quad (86)$$

On the other hand, (82) implies

$$\frac{d}{dt} \ln[b[t]] = \frac{\dot{b}}{b} = \eta(B - b)(u - u^d) \leq -\eta\Lambda(B - b^{\text{up}})^2 \quad (87)$$

on $[t_1, t_{11}]$, so

$$b[t_{11}] \leq b[t_1] e^{-\eta\Lambda(B-b^{\text{up}})^2(t_{11}-t_1)}. \quad (88)$$

Combining together (86) and (88), we obtain

$$b[t_{11}] \leq b[t_1]e^{-\eta\Lambda(B-b^{\text{up}})^2T_1}, \quad (89)$$

and thus by (89) and Lemma 2,

$$b[t_{11}] \leq b^{\text{up}}e^{-\eta\Lambda(B-b^{\text{up}})^2T_1}. \quad (90)$$

Inequalities (79) and (90) together then imply

$$b[t_{11}] < b^*. \quad (91)$$

By inequalities (82) and (91), at time $t = t_{11}$, we know that $g < \gamma + \delta$. Since

$$\dot{z} = z(\gamma - g + \delta) \quad (92)$$

this implies

$$\dot{z}[t_{11}] > 0. \quad (93)$$

Now, let us take stock of what has been accomplished here. Recall that z_{11} was any positive number less than z^{low} for which $z[t] = z_{11}$ for some t , and t_{11} was any number such that $z[t_{11}] = z_{11}$. We have just shown that $\dot{z}[t_{11}]$ must be positive. Thus whenever $z[t]$ is strictly between z^{low} and 0, z must be increasing as a function of time. Thus, it is not possible for z to decrease from z^{low} to any positive value z_{11} less than z^{low} . But $z[0] \geq z^{\text{low}}$ by (80) and (81). Thus $z[t]$ can never reach any point z_{11} below z^{low} . This establishes bound $z[t] \geq z^{\text{low}}$ on $[0, T]$.

Now let us consider the dynamics of w . We have

$$\begin{aligned} \frac{d}{dt} \left[e^{-(1-\tau)it} V_c[t] \right] &= -(1-\tau)ie^{-(1-\tau)it} V_c[t] + e^{-(1-\tau)it} (R + (1-\tau)iV_c - C_c - I) \\ &= -(1-\tau)ie^{-(1-\tau)it} V_c[t] + e^{-(1-\tau)it} (A + (1-\tau)iV_c - \tau Q) \\ &= e^{-(1-\tau)it} (A - \tau Q) \\ &= e^{-(1-\tau)it} (A - \tau\Lambda(a + b + c)K) \end{aligned}$$

$$\begin{aligned}
&= e^{-(1-\tau)it}(A - \tau\Lambda(a + b + c)K) \\
&= e^{-(1-\tau)it}\left(A - \frac{\tau\Lambda(a + b + c)}{a}A\right) \\
&> e^{-(1-\tau)it}\left(A - \frac{\tau\Lambda(a^{\text{up}} + B + c^{\text{up}})}{a}\kappa_2 e^{\gamma t}\right) \\
&\geq e^{-(1-\tau)it}\left(A - \frac{\tau\Lambda(a^{\text{up}} + B + c^{\text{up}})}{\kappa_1 z^{\text{low}}}\kappa_2 e^{\gamma t}\right) \\
&> e^{-(1-\tau)it}\left(\kappa_1 e^{\gamma t} - \frac{\tau\Lambda(a^{\text{up}} + B + c^{\text{up}})}{\kappa_1 z^{\text{low}}}\kappa_2 e^{\gamma t}\right)
\end{aligned}$$

so

$$\frac{d}{dt}\left[e^{-(1-\tau)it}V_c[t]\right] > e^{(\gamma-(1-\tau)i)t}\left(\kappa_1 - \frac{\tau\Lambda(a^{\text{up}} + B + c^{\text{up}})}{\kappa_1 z^{\text{low}}}\kappa_2\right). \quad (94)$$

Integrating both sides, we then obtain

$$e^{-(1-\tau)it}V_c[t] > \frac{e^{(\gamma-(1-\tau)i)t}}{\gamma - (1-\tau)i}\left(\kappa_1 - \frac{\tau\Lambda(a^{\text{up}} + B + c^{\text{up}})}{\kappa_1 z^{\text{low}}}\kappa_2\right), \quad (95)$$

so after dividing through on both sides by $e^{(\gamma-(1-\tau)i)t}$, we obtain

$$w[t] > \frac{1}{\gamma - (1-\tau)i}\left(\kappa_1 - \frac{\tau\Lambda(a^{\text{up}} + B + c^{\text{up}})}{\kappa_1 z^{\text{low}}}\kappa_2\right). \quad (96)$$

Thus $w[t] \geq w^{\text{low}}$ for all t in $[0, T]$, where

$$w^{\text{low}} = \frac{1}{\gamma - (1-\tau)i}\left(\kappa_1 - \frac{\tau\Lambda(a^{\text{up}} + B + c^{\text{up}})}{\kappa_1 z^{\text{low}}}\kappa_2\right). \quad (97)$$

Now let us consider a . We know that $a = yz$, $y[t] > \kappa_1$ and $z[t] \geq z^{\text{low}}$ on $[0, T]$.

Thus $a[t] = y[t]z[t] > 0$ on $[0, T]$.

Next we look at the dynamics of m . We know that $m = wz$, where $w^{\text{low}} \leq w[t] \leq w^{\text{up}}$ and $z^{\text{low}} \leq z[t] \leq z^{\text{up}}$ on $[0, T]$, where w^{low} can be less than or equal to zero but

w^{up} , z^{low} and z^{up} are positive. Thus we have $m^{\text{low}} \leq m[t] \leq m^{\text{up}}$ for all t in $[0, T]$,

where

$$m^{\text{low}} = \min\{w^{\text{low}}z^{\text{low}}, w^{\text{low}}z^{\text{up}}\} \quad (98)$$

$$m^{\text{up}} = w^{\text{up}}z^{\text{up}}. \quad (99)$$

Finally, we look at the variables a, b and c . One can easily verify that (by equation (25)) c satisfies

$$c[t] = c[0]e^{\int_0^t (\xi(m[s] - \sigma u[s]) - g[s] + \delta) ds}. \quad (100)$$

Since a, b, c and m are continuous on $[0, T]$, and $g = b + \theta(1 - \tau)(1 - \psi)\Lambda(a + b + c)$, the exponential factor $e^{\int_0^t (\xi(m[s] - \sigma u[s]) - g[s] + \delta) ds}$ cannot go to zero on $[0, T]$. Thus, since $c[0] > 0$, it follows from (100) that $c[t] > 0$ for all t in $[0, T]$. Since we also have (using equations (23) and (24))

$$a[t] = a[0]e^{\int_0^t (\gamma - g[s] - \delta) ds} \quad (101)$$

and

$$b[t] = b[0]e^{\int_0^t \eta(B - b[s])(u[s] - u^d) ds}, \quad (102)$$

similar arguments also show $a[t] > 0$ and $b[t] > 0$ for all t in $[0, T]$. And of course we know $b[t] < B$ for all t in $[0, T]$ by Lemma 2. ■

Proof of Theorem 1: Using the bounds established in Lemma 1, Lemma 2 and Lemma 3, let us define the following subset of \mathbb{R}^4 :

$$\Delta = \{(a, b, c, m): 0 < a \leq a^{\text{up}}, 0 < b < B, 0 < c \leq c^{\text{up}}, m^{\text{low}} \leq m \leq m^{\text{up}}\}. \quad (103)$$

It is easy to check that $(a[0], b[0], c[0], m[0])$ is in Δ . For example, (52) asserts that

$a^{\text{up}} = y^{\text{up}}z^{\text{up}}$, while (50) says $y^{\text{up}} = \kappa_2$ and (51) implies $z^{\text{up}} \geq \frac{1}{\kappa[0]}$. Thus $a[0] = \frac{A[0]}{\kappa[0]} <$

$\frac{\kappa_2 e^{\gamma_0}}{K[0]} = y^{\text{up}} z^{\text{up}}$. On the other hand $a[0] > 0$, because (22) implies $A[0] > 0$ and our

assumptions about initial conditions in Section 3 included that $K[0] > 0$.

Moreover, by the nonautonomous existence-uniqueness theorem for ordinary differential equations (Meiss 2007: pg. 92 and pg. 99), the system (23)–(26) has a unique solution that is continuous and exists on some time interval $[0, \epsilon)$, where ϵ could be a finite positive number or could be ∞ (with the latter case indicating that the solution exists for all time). Now, suppose that there is some t in $[0, \epsilon)$ such that $a[t] = 0$. In that case, we know that, since the functions $(a[t], b[t], c[t], m[t])$ are continuous with $a[0] > 0$, $0 < b[0] < B$, and $c[0] > 0$, there is some T such that $a[t] > 0$, $0 < b[t] < B$, and $c[t] > 0$ for all t in $[0, T)$, but $a[T] = 0$. But then by Lemma 3, under these assumptions, $a[T] > 0$, which is a contradiction. Thus $a[t]$ must be positive on all of $[0, \epsilon)$. Similar arguments show that $c[t] > 0$ and $0 < b[t] < B$ for all t in $[0, \epsilon)$.

Now, T be any number in the interval $(0, \epsilon)$. We know, from what we have just proved, that $[0, T]$ is an interval on which the solution to (23)–(26) exists, is bounded, and is continuous, with $0 < b[t] < B$ for all t in $[0, T)$, and $a[t]$ and $c[t]$ positive for all t in $[0, T)$. Therefore, by the lemmas proved above, we know that for every t in $[0, T]$, $(a[t], b[t], c[t], m[t])$ is in Δ . But since this is true for every T in $(0, \epsilon)$, this means that $(a[t], b[t], c[t], m[t])$ is in Δ for all t in $[0, \epsilon)$. In other words, for any interval on which a solution to (23)–(26) exists and is continuous, and meets the assumptions of Theorem 1, we know that $(a[t], b[t], c[t], m[t])$ must stay in Δ .

To complete the proof, we need to transform (23)–(26) into a five-dimensional autonomous system using the procedure described at the end of Section 2. This means we replace $\Gamma[t]$ with $\Gamma[n]$, where $\dot{n} = 1$ and $n[0] = 0$. Now clearly, on any time interval

$[0, \epsilon)$, it will be true that $0 \leq n[t] < \epsilon$. And we just saw, in the previous paragraph, that the state variables a, b, c and m must stay within the fixed bounds defining the set Δ . Thus on every bounded time interval $[0, \epsilon)$, the state $(a[t], b[t], c[t], m[t], n[t])$ stays bounded. Therefore, by Theorem 3.18 in Meiss (2007: pg. 100), the system has a unique solution $(a[t], b[t], c[t], m[t], n[t])$ that exists for all $t \geq 0$. Moreover, by what we have proved already, we know $(a[t], b[t], c[t], m[t])$ is in Δ for all t , and this establishes the bounds in the statement of Theorem 1. ■

Proof of Theorem 2: We know from the Lemma 1 and Lemma 3, together with Theorem 1, that there is a pair of *positive* numbers z^{low} and z^{up} such that

$$z^{\text{low}} \leq z[t] \leq z^{\text{up}} \quad (104)$$

for all $t \geq 0$. It follows that

$$\ln z^{\text{low}} \leq \ln z[t] \leq \ln z^{\text{up}} \quad (105)$$

for all $t \geq 0$, and that $\ln z^{\text{low}}$ and $\ln z^{\text{up}}$ are real numbers since z^{low} and z^{up} are positive.

As a result, $\lim_{s \rightarrow \infty} \frac{1}{s} [\ln z^{\text{low}}] = 0$ and $\lim_{s \rightarrow \infty} \frac{1}{s} [\ln z^{\text{up}}] = 0$. So by the Squeeze Theorem,

$$\lim_{s \rightarrow \infty} \frac{1}{s} [\ln z[s]] = 0 \quad (106)$$

Finally, by the fundamental theorem of calculus, we have $\int_0^s \frac{d}{dt} \ln z \, dt = \ln z[s] - \ln z[0]$.

Putting all this together, we see that

$$\begin{aligned} 0 &= \lim_{s \rightarrow \infty} \frac{1}{s} [\ln z[s] - \ln z[0]] \\ &= \lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s \frac{d}{dt} \ln z \, dt \end{aligned}$$

$$\begin{aligned}
&= \lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s (\gamma - g + \delta) dt \\
&= \gamma - (\bar{g} - \delta)
\end{aligned}$$

In other words, $\gamma - (\bar{g} - \delta) = 0$, i.e., $\gamma = \bar{g} - \delta$.

Finally, let us prove (36) and (37). Since $z^{\text{low}} < \frac{e^{\gamma t}}{K[t]} < z^{\text{up}}$ for all t , the condition (36) is true if $\pi_1 = 1/z^{\text{up}}$ and $\pi_2 = 1/z^{\text{low}}$. Using the bounds established in the proof of the previous theorem, as well as the fact that $Q = \Lambda(a + b + c)K$, we can see that (37) holds if we set $\pi_3 = \Lambda\kappa_1 z^{\text{low}}/z^{\text{up}}$ and $\pi_4 = \Lambda(a^{\text{up}} + b^{\text{up}} + c^{\text{up}})/z^{\text{low}}$. ■

Lemma 4: Under the assumptions of Theorem 1, there is a positive number b^{low} such that $b[t] \geq b^{\text{low}}$ for all $t \geq 0$.

Proof: The proof is very similar to the proof of Lemma 3, but with some key details changed and some inequalities reversed (since now we are establishing a lower bound for b , rather than an upper bound).

First, we need some definitions. Let b_1 be any positive number less than both b^* and $b[0]$. (Recall from assumption (35) that b^* is a number strictly between 0 and B such that $b^* + \theta(1 - \tau)(1 - \psi)u^d = \gamma + \delta$.) Because $b_1 < b^*$, we know that

$$b_1 + \theta(1 - \tau)(1 - \psi)u^d - (\gamma + \delta) < 0. \quad (107)$$

Therefore, we can choose a positive number T_1 for which

$$e^{T_1(b_1 + \theta(1 - \tau)(1 - \psi)u^d - (\gamma + \delta))} < \frac{e^{\gamma T_1}}{K[t_1]z^{\text{up}}}. \quad (108)$$

Define the number b_{11} to be

$$b_{11} = b_1 e^{-\eta B u^d T_1}. \quad (109)$$

Now, let us suppose that b has no positive lower bound. I claim that, in that case, since $b[t]$ is a continuous function, $b[t]$ has no positive lower bound, and $b_{11} < b_1 < b[0]$, there is some time interval $[t_1, t_{11}]$ such that $b[t_1] = b_1$, $b[t_{11}] = b_{11}$, and $b[t] \leq b_1$ for all t in $[t_1, t_{11}]$. To prove this claim, note first that, by the intermediate value theorem and the supposition that b has no positive lower bound, there is some positive t for which $b[t] = b_{11}$. Choose any such t to be t_{11} . Now notice that the set

$$\{t \in [0, t_{11}]: b[t] = b_1\} \quad (110)$$

is closed and bounded, and thus is compact by the Heine-Borel theorem. The set is nonempty by the intermediate value theorem. Since this is a compact and nonempty set of real numbers, it has a maximum element. Call this maximum element t_1 . Then the interval $[t_1, t_{11}]$ is such that $b[t_1] = b_1$, $b[t_{11}] = b_{11}$, and $b[t] \leq b_1$ for all t in $[t_1, t_{11}]$.

Thus the claim has been proved.

Next, I will show that $T_1 \leq t_{11} - t_1$. To prove this, we first note that

$$\frac{d}{dt} \ln b = \frac{\dot{b}}{b} = \eta(B - b)(u - u^d) \geq -\eta B u^d. \quad (111)$$

If we integrate the inequality $\frac{d}{dt} \ln b \geq -\eta B u^d$ over the interval $[t_1, t_{11}]$, and apply the Fundamental Theorem of Calculus, we see that $\ln[b[t_{11}]] - \ln[b[t_1]] \geq -\eta B u^d(t_{11} - t_1)$. Thus

$$\ln[b[t_{11}]] \geq \ln[b[t_1]] - \eta B u^d(t_{11} - t_1). \quad (112)$$

Applying the natural exponential function to both sides then gives

$$b[t_{11}] \geq b[t_1] e^{-\eta B u^d(t_{11} - t_1)}. \quad (113)$$

Now, noting that $b[t_{11}] = b_{11} = b_1 e^{-\eta B u^d T_1} = b[t_1] e^{-\eta B u^d T_1}$, we can rewrite (113) as

$$b[t_1]e^{-\eta Bu^d T_1} \geq b[t_1]e^{-\eta Bu^d(t_{11}-t_1)}. \quad (114)$$

Dividing through by $b[t_1]$ and then taking the natural logarithm of both sides, we obtain

$$-\eta Bu^d T_1 \geq -\eta Bu^d(t_{11} - t_1). \quad (115)$$

Thus

$$T_1 \leq t_{11} - t_1. \quad (116)$$

Next, we need to show that $\int_{t_1}^{t_{11}} u dt < (t_{11} - t_1)u^d$, i.e., the average value of u on

$[t_1, t_{11}]$ is less than u^d . This will be done in a series of steps. I will show first that

$\int_{t_1}^{t_{11}} \eta(B - b)u dt < \int_{t_1}^{t_{11}} \eta(B - b)u^d dt$. We have

$$\frac{d}{dt} [\ln[b]] = \eta(B - b)(u - u^d) \quad (117)$$

and therefore

$$\begin{aligned} \ln[b[t_{11}]] - \ln[b[t_1]] &= \int_{t_1}^{t_{11}} \frac{d}{dt} [\ln[b[t]]] dt \\ &= \int_{t_1}^{t_{11}} \eta(B - b)(u - u^d) dt \\ &= \int_{t_1}^{t_{11}} \eta(B - b)u dt - \int_{t_1}^{t_{11}} \eta(B - b)u^d dt. \end{aligned}$$

Consequently,

$$\int_{t_1}^{t_{11}} \eta(B - b)u dt = \ln[b[t_{11}]] - \ln[b[t_1]] + \int_{t_1}^{t_{11}} \eta(B - b)u^d dt < \int_{t_1}^{t_{11}} \eta(B - b)u^d dt. \quad (118)$$

Thus

$$\int_{t_1}^{t_{11}} \eta(B - b)u dt < \int_{t_1}^{t_{11}} \eta(B - b)u^d dt. \quad (119)$$

Now we need to show that $\int_{t_1}^{t_{11}} \eta b u dt < \int_{t_1}^{t_{11}} \eta b u^d dt$. To do this, we note that

$$\frac{d}{dt} \ln[B - b] = \frac{-\dot{b}}{B - b} = -\eta b(u - u^d). \quad (120)$$

Therefore,

$$\begin{aligned} \ln[B - b[t_1]] - \ln[B - b[t_{11}]] &= -(\ln[B - b[t_1]] - \ln[B - b[t_{11}]])) \\ &= -\int_{t_1}^{t_{11}} \frac{d}{dt} [\ln[B - b[t]]] dt \\ &= -\int_{t_1}^{t_{11}} -\eta b(u - u^d) dt \\ &= \int_{t_1}^{t_{11}} \eta b(u - u^d) dt \\ &= \int_{t_1}^{t_{11}} \eta b u dt - \int_{t_1}^{t_{11}} \eta b u^d dt. \end{aligned}$$

Since $b_{11} < b_1$, $\ln[B - b[t_1]] < \ln[B - b[t_{11}]]$. Putting this all together, we have

$$\int_{t_1}^{t_{11}} \eta b u dt = \int_{t_1}^{t_{11}} \eta b u^d dt + \ln[B - b[t_1]] - \ln[B - b[t_{11}]] < \int_{t_1}^{t_{11}} \eta b u^d dt. \quad (121)$$

Thus

$$\int_{t_1}^{t_{11}} \eta b u dt < \int_{t_1}^{t_{11}} \eta b u^d dt. \quad (122)$$

Now note that

$$\int_{t_1}^{t_{11}} \eta(B - b)u dt + \int_{t_1}^{t_{11}} \eta b u dt = \int_{t_1}^{t_{11}} (\eta(B - b)u + \eta b u) dt = \int_{t_1}^{t_{11}} \eta B u dt \quad (123)$$

and

$$\int_{t_1}^{t_{11}} \eta(B - b)u^d dt + \int_{t_1}^{t_{11}} \eta b u^d dt = \int_{t_1}^{t_{11}} (\eta(B - b)u^d + \eta b u^d) dt = \int_{t_1}^{t_{11}} \eta B u^d dt. \quad (124)$$

Putting (119), (122), (123) and (124) together, we see that

$$\begin{aligned}
\int_{t_1}^{t_{11}} \eta B u dt &= \int_{t_1}^{t_{11}} \eta (B - b) u dt + \int_{t_1}^{t_{11}} \eta b u dt \\
&< \int_{t_1}^{t_{11}} \eta (B - b) u^d dt + \int_{t_1}^{t_{11}} \eta b u^d dt \\
&= \int_{t_1}^{t_{11}} \eta B u^d dt.
\end{aligned}$$

Thus we have shown that $\int_{t_1}^{t_{11}} \eta B u dt < \int_{t_1}^{t_{11}} \eta B u^d dt$, and if we divide through by the constant ηB , and then note that $\int_{t_1}^{t_{11}} u^d dt = (t_{11} - t_1)u^d$, we obtain the inequality

$$\int_{t_1}^{t_{11}} u dt < (t_{11} - t_1)u^d. \quad (125)$$

Now, let us look at the implications of all this for the variable $z[t]$, which the reader should recall is equal to $e^{rt}/K[t]$. Notice that

$$\ln[K[t_{11}]] - \ln[K[t_1]] = \int_{t_1}^{t_{11}} \frac{d}{dt} \ln[K[t]] dt = \int_{t_1}^{t_{11}} \widehat{K}[t] dt = \int_{t_1}^{t_{11}} (g[t] - \delta) dt. \quad (126)$$

Thus $\ln[K[t_{11}]] = \ln[K[t_1]] + \int_{t_1}^{t_{11}} (g[t] - \delta) dt$, and applying the exponential function to

both sides gives $K[t_{11}] = K[t_1] e^{\int_{t_1}^{t_{11}} (g[s] - \delta) ds}$. Therefore,

$$\begin{aligned}
K[t_{11}] &= K[t_1] e^{\int_{t_1}^{t_{11}} (g[s] - \delta) ds} \\
&= K[t_1] e^{\int_{t_1}^{t_{11}} (b[s] + \theta(1-\tau)(1-\psi)u[s] - \delta) ds} \\
&< K[t_1] e^{\int_{t_1}^{t_{11}} (b_1 + \theta(1-\tau)(1-\psi)u^d - \delta) ds} \\
&= K[t_1] e^{(t_{11}-t_1)(b_1 + \theta(1-\tau)(1-\psi)u^d - \delta)}.
\end{aligned}$$

Notice that in the third line we used the inequality (125). Consequently, using what we have just shown, together with (116) and (108), we can see that

$$\begin{aligned}
z[t_{11}] &= \frac{e^{\gamma t_{11}}}{K[t_{11}]} \\
&> \frac{e^{\gamma t_{11}}}{K[t_1]e^{(t_{11}-t_1)(b_1+\theta(1-\tau)(1-\psi)u^d-\delta)}} \\
&= \frac{e^{\gamma(t_{11}-t_1)}e^{\gamma t_1}}{K[t_1]e^{(t_{11}-t_1)(b_1+\theta(1-\tau)(1-\psi)u^d-\delta)}} \\
&= \frac{e^{\gamma t_1}}{K[t_1]e^{(t_{11}-t_1)(b_1+\theta(1-\tau)(1-\psi)u^d-(\gamma+\delta))}} \\
&> \frac{e^{\gamma t_1}}{K[t_1]e^{T_1(b_1+\theta(1-\tau)(1-\psi)u^d-(\gamma+\delta))}} \\
&\geq \frac{e^{\gamma t_1}}{K[t_1](e^{\gamma t_1}/(K[t_1]z^{\text{up}}))} \\
&= z^{\text{up}}.
\end{aligned}$$

Therefore, based on the supposition that b has no positive lower bound, we have proved that there is a positive number t_{11} such that $z[t_{11}] > z^{\text{up}}$. This contradicts the upper bound for z established in Lemma 1. Thus, there must be some *positive* number b^{low} such that $b[t] \geq b^{\text{low}}$ for all t . ■

Proof of Theorem 3: We know, because of the bounds established in Lemma 2 and Lemma 4, that $b^{\text{low}} \leq b[t] \leq b^{\text{up}}$ and $B - b^{\text{up}} \leq B - b[t] \leq B - b^{\text{low}}$ for all t , where b^{low} , b^{up} , $B - b^{\text{up}}$ and $B - b^{\text{low}}$ are all positive numbers. It follows that the quantities $\ln[b[t]]$ and $\ln[B - b[t]]$ must stay bounded between fixed real numbers (not necessarily positive). Consequently, $\frac{1}{t} \ln[b[t]]$ and $\frac{1}{t} \ln[B - b[t]]$ both approach zero as $t \rightarrow \infty$. Thus

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln b[t] = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{1}{t} \ln(B - b[t]) = 0. \tag{127}$$

Therefore,

$$\begin{aligned}
0 &= \lim_{s \rightarrow \infty} \frac{1}{s} (\ln b[s] - \ln b[0]) + \lim_{s \rightarrow \infty} \frac{1}{s} (\ln(B - b[s]) - \ln(B - b[0])) \\
&= \lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s \frac{d}{dt} \ln b[t] dt + \lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s \frac{d}{dt} \ln(B - b[t]) dt \\
&= \lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s \eta(B - b[t])(u[t] - u^d) dt + \lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s \eta b[t](u[t] - u^d) dt \\
&= \lim_{s \rightarrow \infty} \frac{1}{s} \left(\int_0^s \eta(B - b[t])(u[t] - u^d) dt + \int_0^s \eta b[t](u[t] - u^d) dt \right) \\
&= \lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s (\eta(B - b[t])(u[t] - u^d) + \eta b[t](u[t] - u^d)) dt \\
&= \lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s (\eta B(u[t] - u^d)) dt \\
&= \eta B \left(\lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s u[t] dt - \lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s u^d dt \right) \\
&= \eta B(\bar{u} - u^d).
\end{aligned}$$

Since this long chain of equalities started with zero, we know $\bar{u} = u^d$. Moreover, since

$r = (1 - \tau)(1 - \psi)u$, we have $\bar{r} = (1 - \tau)(1 - \psi)\bar{u} = (1 - \tau)(1 - \psi)u^d$. ■

6. Numerical simulations

To get a sense of the possibilities contained in the theorems proved above, I will now briefly describe the results from a couple numerical simulations of this model. The simulations were constructed using the program Minsky, and details are provided in the Appendix.¹³ Although an effort has been made to choose parameter values that can

¹³ I thank Steve Keen for his help with setting up the model in Minsky.

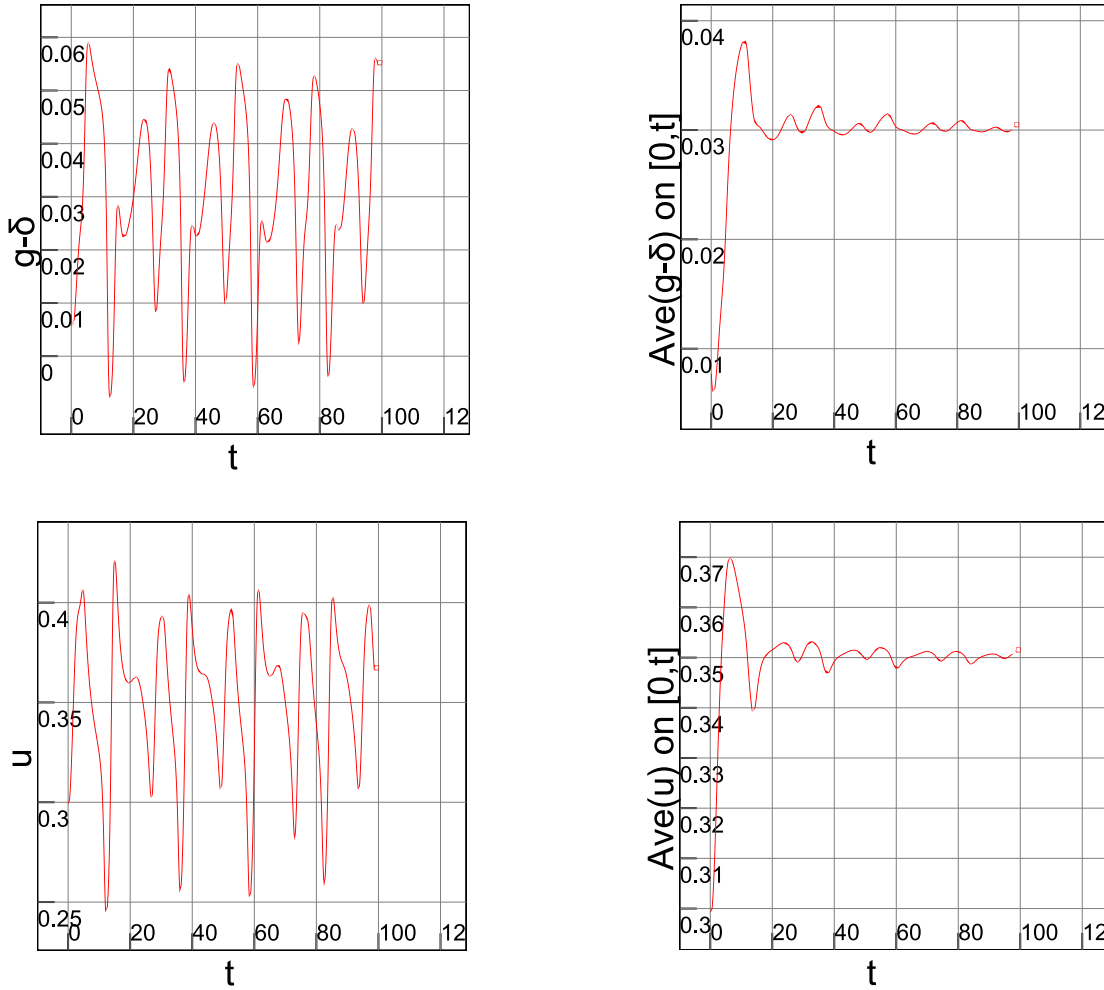
plausibly describe an advanced capitalist country, the aim here is not to exactly match real-world macroeconomic trajectories. Instead, the purpose of these simulations is simply to illustrate how key claims in the theory of NCC-demand-driven growth can survive even after issues raised in Section 2 are taken into account.

To this end, let us first consider a simulation of the model in which government spending depends in part on the level of government debt. More specifically, I define

$$A[t] = f \left[-\frac{V_G[t]}{K[t]} \right] e^{\gamma t}, \quad (128)$$

where f is a decreasing function that stays bounded between two positive constants, and $-\frac{V_G[t]}{K[t]}$ is government debt scaled by the size of the economy. One can imagine that γ , the trend growth rate for public expenditures, reflects long-term policy considerations, while the factor $f \left[-\frac{V_G[t]}{K[t]} \right]$ represents the fact that government officials feel pressure to curtail outlays to some degree when debts are relatively high. This is obviously a simplified model of fiscal policy, and one could consider a variety of other variables that might affect government spending. Nonetheless, this specific effect may have real-world relevance; Mason and Jayadev (2018) suggest that a fiscal policy rule in which spending adjusts to target a particular ratio of debt to potential GDP, together with an interest rate rule, has given rise to “policy generated cycles” in the US economy.

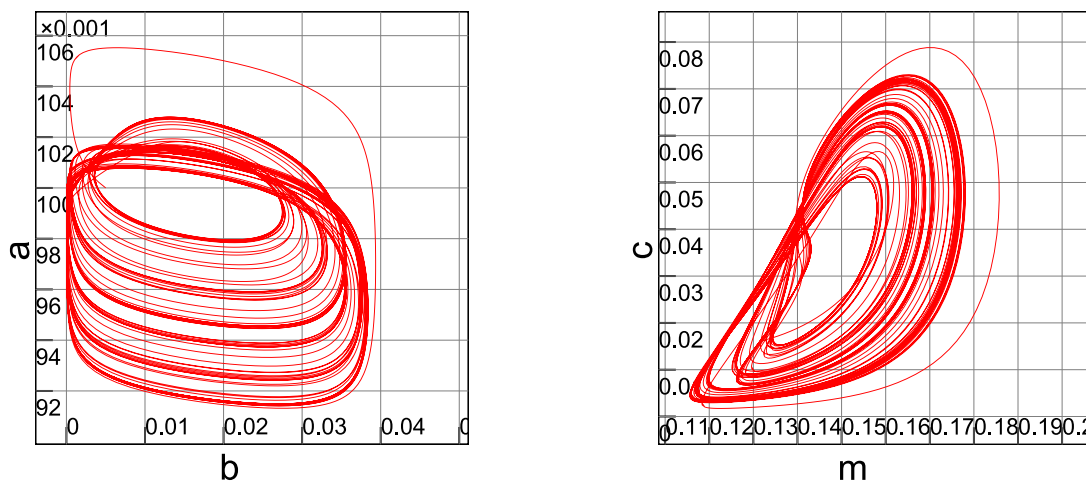
Figure 1: Investment and Utilization Dynamics for Simulation 1



Let us now consider the effects of equation (128) in the model developed here. Since f stays between two positive constants, the condition (22) is satisfied, and consequently, we can apply Theorems 1-3. Although the theorems in Section 4 provide some clarity regarding the long-run average behavior, the dynamics of the system can be quite wild, and this is illustrated by Figure 1. It is evident that the system alternates between booms and crashes, generating business cycles that last five to ten years and exhibit varying degrees of severity. There is no indication that the system ever approaches a steady state or periodic cycle. Indeed, Figure 2 illustrates the dynamics of

the four state variables a , b , c and m for a period of 800 years; it is evident that the variables perpetually wander around a complicated attractor. Still, as can be seen in Figure 1, the long-run average value for $g - \delta$ (the capital stock growth rate) moves toward γ (which takes the value 0.03 in this simulation), as we would expect from Theorem 2. (Note that the value $\text{Ave}(g - \delta)$ in Figure 1 represents $\frac{1}{t} \int_0^t (g[s] - \delta) ds$, the average value of $g - \delta$ over the first t years of the simulation, which by definition approaches the long-run average value of $g - \delta$ as $t \rightarrow \infty$; an analogous comment applies to $\text{Ave}(u)$.) Similarly, the long-run average value for the output-capital ratio u converges to the target value u^d (which in this simulation is 0.35). The business cycles shown in this simulation are somewhat extreme, but this serves to underline the fundamental point of the paper: that semi-autonomous NCC demand can drive the growth process even in the presence of severe and ongoing macroeconomic turbulence.

Figure 2: Dynamics of a, b, c and m for 800 Years in Simulation 1



For the second simulation, I now assume that government officials pursue a deliberate counter-cyclical spending policy. More specifically, I assume that Γ , the growth rate of government spending, is determined by the rule

$$\Gamma = \gamma_0 + \gamma_1 v, \quad (129)$$

where γ_1 and γ_2 are parameters, v denotes the unemployment rate, and γ_1 is positive.

Thus when the employment rate falls, public officials try to boost the economy in response. In line with the model of the labor market discussed in Section 3, I assume that the natural rate of growth is

$$n = \alpha_0 - \alpha_1 v + \alpha_2 \hat{Y}, \quad (130)$$

where α_1 , α_2 and α_3 are parameters, the last two of which must be positive. As in Fazzari, Ferri and Variato (2020), the term $\alpha_1 v$ represents the tendency for the growth rate of the labor force to increase when the labor market tightens, while $\alpha_2 \hat{Y}$ represents the Kaldor-Verdoorn effect. Now, let N denote the size of the labor force, let q be the output/labor ratio, and let $k = K/(qN)$. Then we have $v = 1 - ku$ and

$$\dot{k} = k(g - \delta - \alpha_0 + \alpha_1(1 - ku) - \alpha_2 \hat{Y}). \quad (131)$$

To incorporate this into the original system, we also note that $\hat{Y} = \hat{u} + \hat{K}$, and therefore

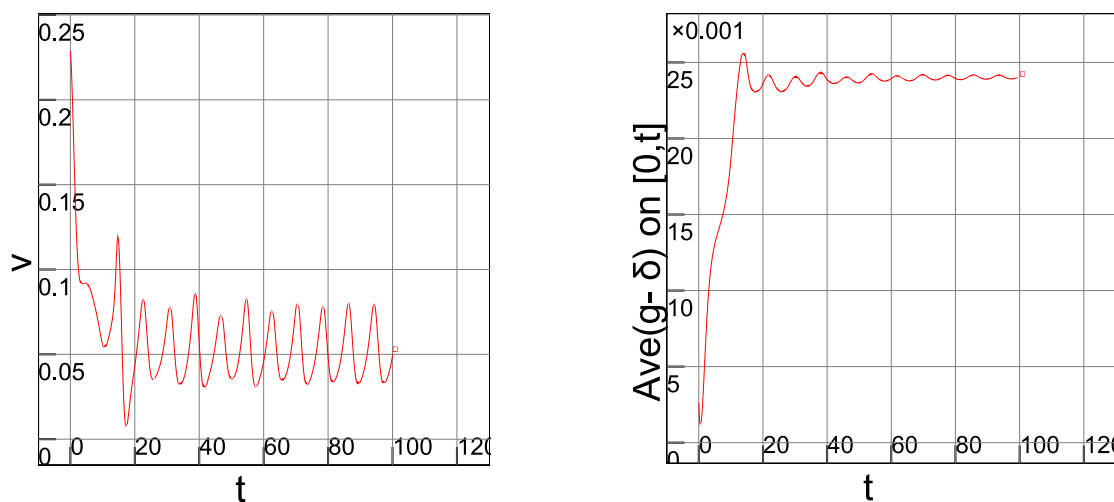
$$\hat{Y} = \frac{a(\Gamma - g + \delta) + \phi[b](u - u^d) + c(\xi(m - \sigma u) - g + \delta)}{a + b + c} + g - \delta. \quad (132)$$

By including these equations in the model, we can explore the possibility of interactions between the labor market and fiscal policy.

The results of this second simulation are illustrated in Figure 3, which shows the unemployment rate and the time average for the growth rate of the capital stock. Initially the economy is in a deep recession, with an unemployment rate of approximately 23%. The countercyclical spending program quickly lifts the economy out of the slump, but

overall this policy has limited success: the employment rate continues to fluctuate forever, and the system as a whole approaches a limit cycle trajectory. As before, the long-run average values of the capacity utilization rate (not shown) and accumulation rate converge, consistent with what we expect from Theorems 2 and 3. The difference is that, in this second simulation, the long-run trend rate of growth for government spending (and thus for the rest of the economy as well) is determined endogenously through the interaction between fiscal policy and the labor market—public expenditures are not autonomous. But, despite important differences, the basic properties of the model are similar to what one finds in autonomous-expenditure driven models such as Allain (2015); government spending serves to prop up the economy and sustain long-run growth.

Figure 3: Unemployment Rate and Investment in Simulation 2



7. An equilibrium version of the model

We have now seen how various ideas—macroeconomic instability, financial stock-flow norms, a labor market, and different types of fiscal policy—can be incorporated into a theory of NCC-demand-driven growth. However, all this has come at a cost: the model developed here was complicated and the analysis required a significant effort. For some purposes, this degree of complexity may not be helpful. Thus, it is natural to ask how this model might relate to simpler frameworks.

In fact, there is a sense in which the long-run average behavior of this model can be represented by a model with a stable long-run equilibrium. To see this, let us assume now that government spending grows at a constant rate, so $\Gamma[t] = \gamma$ for all t . Let us also assume that the financial stock-flow adjustment process happens very rapidly, so we can take the limit as $\xi \rightarrow \infty$, and the consumption function (19) reduces to the condition

$$V_C = \sigma Q. \quad (134)$$

Consequently, the rate of change for aggregate demand (Y) is

$$\dot{Y} = \dot{Q} = \frac{1}{\sigma} \dot{V}_C = \frac{1}{\sigma} (R + (1 - \tau)iV_C - I - C_C) = \frac{1}{\sigma} (A + (1 - \tau)iV_C - \tau Y) \quad (135)$$

and it follows that $\hat{Y} = \frac{1}{\sigma} (A/Y + (1 - \tau)i\sigma - \tau)$. The model reduces to the system

$$\dot{d} = d \left(\gamma - \frac{1}{\sigma} (d + (1 - \tau)i\sigma - \tau) \right) \quad (136)$$

$$\dot{u} = u \left(\frac{1}{\sigma} (d + (1 - \tau)i\sigma - \tau) - b - \theta(1 - \tau)(1 - \psi)u + \delta \right) \quad (137)$$

$$\dot{b} = \eta b (B - b)(u - u^d) \quad (138)$$

where d denotes the ratio of government spending to GDP, A/Y . Recall that u is the utilization rate and b is the Harrodian term in the investment function.

The assumption (134), although clearly a simplification of reality, has a stabilizing effect on the model. Indeed one can show easily that the positive steady state for the model is locally stable when (134) holds. In this steady state, the properties of this simplified model are identical with the long-run average behavior of the more general framework developed earlier in the paper: output and capital stock grow at the rate γ , while the rate of capacity utilization is u^d . For this reason, the special case in which $\Gamma = \gamma$ and $\xi \rightarrow \infty$ can be viewed as an equilibrium approximation of the main model.

Models similar to the system (136)–(138) have already appeared in the economics literature. Thompson (2020a) studies a nearly identical model, using it to shed light on Rosa Luxemburg’s theory of accumulation, the Marxist theory of overproduction, and the phenomenon of “profits without investment”. Thompson (2020b) also develops a related model but with induced technical change and non-steady-state growth paths, to revisit Marx’s argument that there is a tendency for the rate of profit to fall. If evidence can be found to support the model developed in Section 3, then these simpler equilibrium versions may provide a useful approximation to reality, not because the concept of long-run equilibrium itself is empirically plausible, but because the equilibrium model has the same *long-run average* behavior as the more realistic disequilibrium version.

8. Conclusion: what can we learn from a disequilibrium analysis of capitalism?

In the academic economics literature, questions of the form “does an economy have the property P in the long run?” are often treated as being reducible to questions of the form “does the economy have a stable long-run equilibrium with the property P ?”. But if we doubt the plausibility of stable long-run economic equilibria to begin with, then these two types of questions must be carefully distinguished, with the latter type potentially having very limited relevance to the former.

These issues have become particularly relevant in the debate over NCC-demand-driven growth. As described above, economists have developed supermultiplier and Kaleckian-Harrodian models which seek to show that, by stabilizing a long-run equilibrium path, NCC autonomous demand can drive the rate of expansion. The problem is that the stability conditions are questionable; as critics have pointed out, the amount of *autonomous* NCC demand may be too small in relation to GDP to play a stabilizing role, and volatility in some NCC expenditure components may undermine the stabilization argument as well. Motivated by this controversy, in this paper I sought to show how NCC demand can drive growth even in the absence of macroeconomic stability.

To this end, I showed that, starting with a model in which solution trajectories can exhibit persistently violent and aperiodic fluctuations, it is possible to characterize the long-run behavior by calculating time averages of key variables. In essence, by looking at the cumulative effect of the disorderly fluctuations in the model, in Theorems 2 and 3 I obtained simple formulas showing how a type of order emerges in the long run. This

calls to mind an old Marxist dictum (Marx and Engels 1902, quoted in Mandel, 2015: pg. 81) about how to best think about the dynamics of capitalism: “The total movement of this disorder is its order.”

Using this approach, I was able to describe the trend around which the model’s solution trajectories fluctuate. We saw that in the long run, under some general assumptions, long-run capital accumulation and output growth are determined by the growth of semi-autonomous non-capacity-creating demand, and the long-run average utilization rate is equal to the one targeted by firms. Thus, a key contribution of this paper is to show that one can recover Kaleckian-Harrodian results even if NCC demand fails to play a stabilizing role.

In connection with this more general point, this paper also put forward a position regarding the bounds for the level of economic activity in capitalist economies. Past models of NCC-demand-driven growth, particularly Fazzari, Ferri and Variato (2020), suggest that if an economy fails to stabilize along a steady-state path, then labor-market limits will have to act as the binding constraint on economic fluctuations; thus the long-run rate of growth would be determined by the supply side, and the economy would periodically approach full utilization of available resources. This paper offered an alternative perspective, showing how demand-side problems (particularly involving financial constraints) can impose a ceiling on the level of economic activity. Thus, the arguments in this paper suggest that, even in the presence of instability, there is no mechanism that guarantees a capitalist economy will ever approach its productive potential, although full utilization of resources *might* be achieved either by happenstance or by means of policy.

Certainly, in formulating these arguments, I made simplifying assumptions that could be reconsidered in future work. Possible avenues for further research—such as incorporating multiple commodities with fluctuating relative prices into the model, letting income distribution change over time, bringing in other types of financial assets, allowing for more complex consumer spending dynamics, looking at the implications of capital mobility between different sectors (or countries), or empirically calibrating the parameters—remain unexplored. But, even with these caveats in mind, the theorems proved in this paper can be seen as a step toward developing a new way of analyzing long-run economic development, which gives precise statements regarding how different parameter values affect the way the system behaves, without relying on the idea that capitalist economies ever gravitate toward long-run equilibria.

Appendix: notes on the simulations

For both simulations, I set $\tau = 0.3$, $\theta = 0.6$, $\psi = 0.5$, $i = 0.01$, $\delta = 0.06$, $u^d = 0.35$, $\eta = 1,100$, $B = 0.04$, $\xi = 25$, $\sigma = 0.4$, $a[0] = 0.1$, $b[0] = 0.005$, $c[0] = 0.0162$, and $m[0] = 0.132$. The reader may notice that one parameter, η , which describes how the investment function shifts in response to changes in $u - u^d$, is especially large. But this parameter is multiplied by $b(B - b)$, which tends to be a small number, so the overall effect is moderate. For example, since $B = 0.04$ in the simulations, when $b = 0.02$ we have $\dot{b} = \eta b(B - b)(u - u^d) = 0.44(u - u^d)$.

For the model of fiscal policy in the first simulation, I set $\gamma = 0.03$, and in equation (128) I used the function $f(x) = 1 + \frac{0.5}{1+e^{10x}}$. In implementing the simulation it is also

useful to note that, since $V_W[t] = 0$ for all t , we have $-\frac{V_G[t]}{K[t]} = m[t]$. For the second simulation, I set $\Gamma = 0.019 + 0.1v$, $n = 0.02 - 0.2v + 0.6\hat{Y}$ and $k[0] = 2.7$. All simulation code is available from the author upon request.

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