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Ultraslow light pulses in a nonlinear metamaterial

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We find the analytical expression for the threshold intensity necessary to launch ultraslow light pulses in a metamaterial with simultaneous cubic electric and magnetic nonlinearity. The roles played by the permittivity, the permeability, the electric cubic nonlinearity, the magnetic cubic nonlinearity and the pulse duration are clearly identified and discussed. © 2008 Optical Society of America

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1. INTRODUCTION

Temporal solitons, i.e., guided light pulses that propagate without dispersion due to the balancing between the group-velocity dispersion (GVD) and the self-phase modulation induced by a Kerr nonlinearity, play a fundamental role in optical communications systems [1,2]. In the past few years metamaterials, i.e., artificial composites assembled in such a way that they show both an effective electric and magnetic response although the constituent materials are nonmagnetic, have been the subject of intense theoretical and experimental investigations due to their vast range of potential applications from superresolution [3] to cloaking [4]. The study of optical solitons in metamaterials is a new and exciting field of research that has already produced some important theoretical results: we cite, for example, the possibility to excite bright and dark gap solitons in a metamaterial cavity [5]; a new nonlinear Schrödinger equation (NLSE) for metamaterials with only cubic electric nonlinearity [6]; and a generalized system of two coupled spatiotemporal NLSEs for metamaterials with simultaneous cubic electric and magnetic nonlinearity [7]. In [8,9] the spatiotemporal dynamics of cavities containing a nonlinear metamaterial has also been studied.

The aim of this paper is to arrive at an analytical expression for the threshold intensity needed to launch an ultraslow optical soliton in a metamaterial and to clearly put into evidence the roles played by the permittivity, the permeability, the electric cubic nonlinearity, the magnetic cubic nonlinearity, and the pulse duration. "Slow light" has recently received a great deal of attention in telecommunications for its numerous applications—ranging from all-optical storage to all-optical switching [10]. Before going into the mathematical analysis of the problem we would like to discuss briefly the linear properties of a metamaterial, which we describe by a Drude model [3] for both permittivity and permeability: $\varepsilon(\tilde{\omega}) \approx 1 - 1/\tilde{\omega}^2$, $\mu(\tilde{\omega}) \approx 1 - (\omega_{mp}/\omega_{ep})^2/\tilde{\omega}^2$, where $\tilde{\omega} = \omega/\omega_{ep}$ is the normalized frequency; ω_{ep} and ω_{mp} are, respectively, the electric plasma frequency and the magnetic plasma frequency of the metamaterial; and $\omega_{ep} = 2\pi c/\lambda_{ep}$, where λ_{ep} is the electric plasma wavelength. In Fig. 1(a) we show the refractive index $n = \pm\sqrt{\varepsilon\mu}$ and group velocity (GV) V_g for $\omega_{mp}/\omega_{ep} = 0.8$. The negative determination of the square root must be taken when ε and μ are simultaneously negative [11]. The material is characterized by an opaque region (no propagative modes allowed) between $\omega_{mp} < \omega < \omega_{ep}$ and two transparent regions (propagative modes allowed), respectively, for $\omega < \omega_{mp}$ and $\omega > \omega_{ep}$; note also that V_g and n reach values close to zero near the edges of the propagative regions. The two band edges are defined, respectively, by the conditions $\omega = \omega_{ep}$ and ω_{mp}. It is indeed near the band edges of the propagative regions that ultraslow GV temporal solitons make their appearance. In Fig. 1(b) we show the so-called GVD parameter

$$\beta_2 \equiv \left(\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{1}{V_g}\right)\right).$$

The zones at the edges of the propagative region where low GV is achieved are also characterized by a strong dispersion that, as we will see later, substantially increases the threshold intensity necessary to launch the fundamental soliton. We also note that in the case of an impedance matched metamaterial, i.e., $\omega_{mp} = \omega_{ep}$, the opaque region disappears and, as a consequence, the GV remains substantially close to c over the entire spectrum.

The paper is organized as follows: in Section 2 we derive the basic equations and arrive at a system of two NLSEs that couple together the electric and magnetic field; in Section 3 we discuss the analytical solutions and, in particular, we focus on the ultraslow soliton solutions near the band edges of the metamaterial; and in Section 4 we go to the conclusions.

2. BASIC EQUATIONS

Before proceeding with our technical analysis we would like to emphasize that our approach follows a path similar to the one that was first proposed in [6] by Scalora *et al.* The main differences between our current work and [6] are that (a) in [6] they arrive at a single NLSE with only the effect of the electric nonlinearity taken into account, while here we derive a set of two coupled NLSEs in

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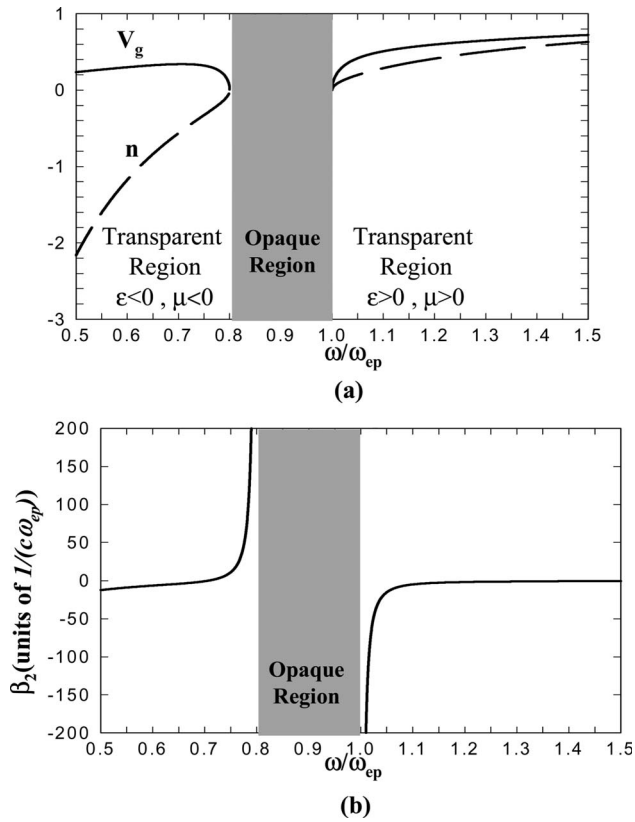

Fig. 1. (a) n (dashed curve) and V_g in units of c (solid curve) versus ω/ω_{ep} for a metamaterial described by a Drude model with $\omega_{mp}/\omega_{ep}=0.8$ as outlined in the main text. (b) GVD parameter β_2 versus ω/ω_{ep}.

the electric and magnetic field and demonstrate that they can be decoupled using a suitable field transformation; and (b) in [6] they analyze the NLSE in a region away from the band edges of the metamaterial, while here we concentrate in a region close to the band edges of the metamaterial where ultraslow solitons make their appearance.

Now, let us start our analysis with the Maxwell equations in the (1+1) dimension for linearly polarized fields $\partial E/\partial z=-(\partial B/\partial t)$, $-(\partial H/\partial z)=\partial D/\partial t$ where E, B, H, and D are, respectively, the electric field, the magnetic induction, the magnetic field, and the electric displacement or electric induction. In what follows we use nondimensional units, i.e., $\varepsilon_0=\mu_0=c=1$, where ε_0 and μ_0 are, respectively, the permittivity and permeability *in vacuo* and c is the speed of light *in vacuo*. We write the electric and magnetic field as the product of an envelope function multiplied by a harmonic oscillation at frequency ω_0: $E(z,t)=(1/2)\times[\tilde{E}(z,t)e^{-i\omega_0 t}+\text{c.c.}]$ and $H(z,t)=(1/2)[\tilde{H}(z,t)e^{-i\omega_0 t}+\text{c.c.}]$. In the hypothesis that the envelope of the fields is slowly varying in time with respect to the oscillation associated with the carrier frequency ω_0, the time derivatives over B and D, respectively, can be approximated as follows:

$$\frac{\partial D}{\partial t}\cong\frac{1}{2}e^{-i\omega_0 t}\left[-i\omega_0\varepsilon\tilde{E}+\left(\frac{\partial(\varepsilon\omega)}{\partial\omega}\right)_{\omega=\omega_0}\frac{\partial\tilde{E}}{\partial t}+\frac{i}{2}\left(\frac{\partial^2(\varepsilon\omega)}{\partial\omega^2}\right)_{\omega=\omega_0}\frac{\partial^2\tilde{E}}{\partial t^2}-i\omega_0\varepsilon\tilde{\chi}_\varepsilon^{(3)}|\tilde{E}|^2\tilde{E}\right]+\text{c.c.},$$

$$\frac{\partial B}{\partial t}\cong\frac{1}{2}e^{-i\omega_0 t}\left[-i\omega_0\mu\tilde{H}+\left(\frac{\partial(\mu\omega)}{\partial\omega}\right)_{\omega=\omega_0}\frac{\partial\tilde{H}}{\partial t}+\frac{i}{2}\left(\frac{\partial^2(\mu\omega)}{\partial\omega^2}\right)_{\omega=\omega_0}\frac{\partial^2\tilde{H}}{\partial t^2}-i\omega_0\mu\tilde{\chi}_\mu^{(3)}|\tilde{H}|^2\tilde{H}\right]+\text{c.c.},$$

where $\tilde{E}$ and $\tilde{H}$ are the complex envelopes for the electric and magnetic fields, respectively; ε and μ are, respectively, the relative permittivity and the permeability of the material, permittivity and permeability that we suppose real quantities; and $\tilde{\chi}_\varepsilon^{(3)}=(3/4)\chi_\varepsilon^{(3)}$ and $\tilde{\chi}_\mu^{(3)}=(3/4)\chi_\mu^{(3)}$ are, respectively, the cubic electric and magnetic nonlinearity that we suppose nondispersive for the sake of simplicity. In the derivation of the approximated expressions for $\partial D/\partial t$ and $\partial B/\partial t$ we have neglected (a) the third-order time derivative over the field envelopes; (b) the first-order time derivative over the nonlinear terms, i.e., the self-steepening terms such as $\partial(|\tilde{E}|^2\tilde{E})/\partial t$ and $\partial(|\tilde{H}|^2\tilde{H})/\partial t$; and (c) the nonlinear terms whose temporal oscillations are different from the oscillation at the carrier frequency (rotating wave approximation), i.e., we retain only the self-phase modulation terms. Now, if we put into evidence the fundamental spatial oscillation by writing $\tilde{E}=\hat{E}e^{ikz}$ and $\tilde{H}=\hat{H}e^{ikz}$, where $k=n\omega_0$ is the fundamental wave vector, and substitute the above expressions into Maxwell equations, we arrive at the following system of coupled equations:

$$\frac{\partial\hat{E}}{\partial z}+ik\hat{E}=i\omega_0(\mu+\tilde{\chi}_\mu^{(3)}|\hat{H}|^2)\hat{H}-\beta_{1,\mu}\frac{\partial\hat{H}}{\partial t}-\frac{i}{2}\beta_{2,\mu}\frac{\partial^2\hat{H}}{\partial t^2},\tag{1.1}$$

$$\frac{\partial\hat{H}}{\partial z}+ik\hat{H}=i\omega_0(\varepsilon+\tilde{\chi}_\varepsilon^{(3)}|\hat{E}|^2)\hat{E}-\beta_{1,\varepsilon}\frac{\partial\hat{E}}{\partial t}-\frac{i}{2}\beta_{2,\varepsilon}\frac{\partial^2\hat{E}}{\partial t^2},\tag{1.2}$$

where

$$\beta_{m,\varepsilon}\equiv\left(\frac{\mathrm{d}^m(\omega\varepsilon)}{d\omega^m}\right)_{\omega=\omega_0},$$

$$\beta_{m,\mu}\equiv\left(\frac{d^m(\omega\mu)}{\mathrm{d}\omega^m}\right)_{\omega=\omega_0}\quad(m=1,2,\ldots).$$

Equations (1) have an evident symmetry. In fact Eq. (1.2) can be obtained from Eq. (1.1) and vice versa with the formal substitutions $\varepsilon\to\mu$, $\mu\to\varepsilon$, $\hat{E}\to\hat{H}$, and $\hat{H}\to\hat{E}$. We now concentrate on the manipulation of only one of the two equations. Let us concentrate on Eq. (1.1). By deriving Eq. (1.1) with respect to z, using Eq. (1.2) and neglecting the third-order spatiotemporal derivatives over the envelopes and the first-order derivatives over the nonlinear terms, we arrive at the following equation:

$$\frac{1}{2n\omega_0}\frac{\partial^2\hat{E}}{\partial z^2}+i\frac{\partial\hat{E}}{\partial z}+\frac{i(\varepsilon\beta_{1,\mu}+\mu\beta_{1,\varepsilon})}{2n}\frac{\partial\hat{E}}{\partial t}-\left(\frac{\beta_{1,\varepsilon}\beta_{1,\mu}}{2n\omega_0}+\frac{\beta_{2,\varepsilon}\mu}{4n}+\frac{\beta_{2,\varepsilon}n}{4\mu}\right)\frac{\partial^2\hat{E}}{\partial t^2}+\frac{\omega_0}{2}\left(\frac{\mu}{n}\tilde{\chi}_\varepsilon^{(3)}|\hat{E}|^2\hat{E}+\tilde{\chi}_\mu^{(3)}|\hat{H}|^2\hat{H}\right)=0. \quad (2)$$

Note that by neglecting the third-order spatiotemporal derivatives we are implicitly assuming that the electromagnetic pulse is slowly varying both in time and space. While in standard positive index materials ($n>1$, $\mu=1$) the condition that the pulse is slowly varying in time generally implies that the pulse is also slowly varying in space, in our case the situation is not so simple because of the high dispersion present in the propagative regions near the electric or the magnetic plasma frequency of the metamaterial (see Fig. 1), i.e., where slow GV solitons are present. In general the condition of the slowly varying envelope in space is broken anytime the pulse extension in the material $\Delta z=V_gT_0$ (T_0 is the temporal pulse duration) becomes comparable with the wavelength of the pulse $\lambda=2\pi c/(n\omega_0)$. In other words Eq. (2) retains its validity when $T_0\gg 2\pi c/(V_g n\omega_0)$. It is clear that this condition put a lower limit on how short in time a low GV soliton can be. In [12] Brabec and Krausz have demonstrated that the slowly varying envelope can be applied even to single-cycle pulses if the group velocity (V_g) and the phase velocity $V_p=(c/n)$ satisfy the following inequality: $|1-V_p/V_g|\ll 1$, i.e., the material dispersion is small. Here we are exactly in the opposite regime, i.e., extremely high material dispersion near the band edges of the structure where the phase velocity tends to infinity and the GV tends to zero and the following condition is valid: $|1-V_p/V_g|\gg 1$. We will come later to a quantitative analysis of the condition $T_0\gg 2\pi c/(V_g n\omega_0)$. For the time being, let us continue with the manipulation of Eq. (2). By defining

$$\beta_m\equiv\left(\frac{d^m(n\omega)}{\mathrm{d}\omega^m}\right)_{\omega=\omega_0}\quad (m=1,2\ldots),$$

by using the typical coordinate transformation $\xi=z$, $\tau=t-\beta_1 z$ and, finally, by neglecting the terms that are of the same order as the third-order spatiotemporal derivatives over the envelopes and the first-order derivatives over the nonlinear terms, Eq. (2) can be put in the following form:

$$i\frac{\partial\hat{E}}{\partial\xi}-\frac{\beta_2}{2}\frac{\partial^2\hat{E}}{\partial\tau^2}+\frac{\omega_0}{2}\left(\frac{\mu}{n}\tilde{\chi}_\varepsilon^{(3)}|\hat{E}|^2\hat{E}+\tilde{\chi}_\mu^{(3)}|\hat{H}|^2\hat{H}\right)=0. \quad (3.1)$$

In arriving at Eq. (3.1) we have also used the following equalities:

$$\beta_1=(\varepsilon\beta_{1,\mu}+\mu\beta_{1,\varepsilon})/2n,$$

$$\beta_2=-\frac{1}{n\omega_0}\left(\frac{1}{V_g^2}-\beta_{1,\varepsilon}\beta_{1,\mu}-\frac{\beta_{2,\varepsilon}\omega_0\mu}{2}-\frac{\beta_{2,\mu}\omega_0\varepsilon}{2}\right),$$

which can be demonstrated. We note that both β_1 and β_2 are invariant under the transformations $\varepsilon\to\mu$, $\mu\to\varepsilon$. Physically this means that the electric and magnetic fields obviously must have the same GV and GVD. Equation (3.1) is the first of the two equations we were looking for. It becomes the standard NLSE [1,2] for $\tilde{\chi}_\mu^{(3)}=0$. Now, using the same procedure outlined above, but starting by deriving Eq. (1.2) with respect to z, we can arrive at the second equation,

$$i\frac{\partial\hat{H}}{\partial\xi}-\frac{\beta_2}{2}\frac{\partial^2\hat{H}}{\partial\tau^2}+\frac{\omega_0}{2}\left(\frac{\varepsilon}{n}\tilde{\chi}_\mu^{(3)}|\hat{H}|^2\hat{H}+\tilde{\chi}_\varepsilon^{(3)}|\hat{E}|^2\hat{E}\right)=0. \quad (3.2)$$

Equations (3.1) and (3.2) form a system of two coupled NLSEs that preserve the same symmetry of Eqs. (1), in fact we can formally obtain Eq. (3.2) from Eq. (3.1) and vice versa with the formal substitutions $\varepsilon\to\mu$, $\mu\to\varepsilon$, $\hat{E}\to\hat{H}$, and $\hat{H}\to\hat{E}$. Before going to analyze the analytical solutions of Eqs. (3.1) and (3.2) we make some cosmetic manipulations by introducing the following new variables and parameters: $\tilde{z}=\xi/L_D$, $L_D=T_0^2/|\beta_2|$, $\tilde{t}=\tau/T_0$, $u_1=\sqrt{L_D\omega_0/2}\hat{E}$, $u_2=\sqrt{L_D\omega_0/2}\hat{H}$, and $Z=\sqrt{\mu/\varepsilon}$ where T_0 is the pulse duration; L_D is the so-called second-order dispersion length [1,2]; and Z is the *impedance* of the medium, which obviously is always positive. In the new variables Eqs. (3.1) and (3.2) become

$$i\frac{\partial u_1}{\partial\tilde{z}}-\frac{\mathrm{sgn}(\beta_2)}{2}\frac{\partial^2u_1}{\partial\tilde{t}^2}+(Z\tilde{\chi}_\varepsilon^{(3)}|u_1|^2u_1+\tilde{\chi}_\mu^{(3)}|u_2|^2u_2)=0, \quad (4.1)$$

$$i\frac{\partial u_2}{\partial\tilde{z}}-\frac{\mathrm{sgn}(\beta_2)}{2}\frac{\partial^2u_2}{\partial\tilde{t}^2}+\left(\frac{1}{Z}\tilde{\chi}_\mu^{(3)}|u_2|^2u_2+\tilde{\chi}_\varepsilon^{(3)}|u_1|^2u_1\right)=0, \quad (4.2)$$

where $\mathrm{sgn}(\beta_2)$ stands for the sign of the GVD parameter.

3. ULTRASLOW SOLITONS

The fundamental soliton solutions of Eqs. (4) can be easily found by noting that *this system of two coupled NLSEs can be decoupled into a single NLSE by using the following transformations:* $u_1=Zu$ and $u_2=u$. The transformations used to decouple the system of Eqs. (4) have an obvious physical meaning: the electric and the magnetic field must be proportional to each other through the impedance Z of the medium, as one may expect. The single NLSE in the u variable takes the following form:

$$i\frac{\partial u}{\partial\tilde{z}}-\frac{\mathrm{sgn}(\beta_2)}{2}\frac{\partial^2u}{\partial\tilde{t}^2}+\Gamma|u|^2u=0, \quad (5)$$

where $\Gamma=(Z^3\tilde{\chi}_\varepsilon^{(3)}+(1/Z)\tilde{\chi}_\mu^{(3)})$. The fundamental soliton solutions of Eq. (5) are well known [1,2]. Bright soliton: $u(\tilde{z},\tilde{t})=(\sec h(\tilde{t})/\sqrt{|\Gamma|})\exp(\pm i\tilde{z}/2)$ for $\mathrm{sgn}(\beta_2)=-1$ and $\Gamma>0$ or $\mathrm{sgn}(\beta_2)=1$ and $\Gamma<0$. Dark soliton: $u(\tilde{z},\tilde{t})=(\tanh(\tilde{t})/\sqrt{|\Gamma|})\exp(\pm i\tilde{z})$ for $\mathrm{sgn}(\beta_2)=1$ and $\Gamma>0$ or $\mathrm{sgn}(\beta_2)=-1$ and $\Gamma<0$. The threshold intensity to launch the fundamental solitons can be calculated from the maximum of the Poynting vector, $S_{thr}=(1/2)\mathrm{Re}[\tilde{E}\tilde{H}^*]_{\mathrm{Max}}$, which in our case gives

$$S_{thr} = \frac{Z}{L_D \omega_0 |\Gamma|} = \frac{|\beta_2| \varepsilon \mu}{T_0^2 \omega_0 |\mu^2 \tilde{\chi}_\varepsilon^{(3)} + \varepsilon^2 \tilde{\chi}_\mu^{(3)}|}. \quad (6)$$

Equation (6) is the analytical expression for the threshold intensity necessary to launch the fundamental solitons in a generic metamaterial having both electric and magnetic cubic nonlinearity. To analyze quantitatively the situation for simplicity we assume that the electric and magnetic nonlinearity are of the same order of magnitude. By tuning the carrier frequency of the pulse near the electric plasma frequency, i.e., $\omega_0 \approx \omega_{ep}$ and $\varepsilon \sim 0$, we obtain from Eq. (6) the following expression: $S_{thr} \approx |\beta_2|(\varepsilon/\mu)/(T_0^2 \omega_0 |\tilde{\chi}_\varepsilon^{(3)}|)$, which means that the ultraslow GV soliton is of an "electrical" nature; namely, it is the electric nonlinearity that plays the dominant role for its formation while the magnetic nonlinearity is quenched. Vice versa, near the magnetic plasma frequency, i.e., $\omega_0 \approx \omega_{mp}$ and $\mu \sim 0$, we obtain from Eq. (6) $S_{thr} \approx |\beta_2|(\mu/\varepsilon)/(T_0^2 \omega_0 |\tilde{\chi}_\mu^{(3)}|)$. That is, this time it is the magnetic nonlinearity that plays a dominant role in the formation of the ultraslow GV soliton. It is worthwhile to remark that in the above discussion we are operating *in regions near the band edges, but not exactly at the band edges;* in other words, the carrier frequency of the pulse is always slightly detuned with respect to the electric or magnetic plasma frequency of the metamaterial. This means that the refractive index and the permittivity or permeability are very small but not quite zero, therefore avoiding any intrinsic singularity in Eq. (2) or in any other equation that derives from Eq. (2). Finally in the case of an impedance matched metamaterial, i.e., $\omega_{mp} = \omega_{ep}$, we obtain $S_{thr} = |\beta_2|/(T_0^2 \omega_0 |\tilde{\chi}_\varepsilon^{(3)} + \tilde{\chi}_\mu^{(3)}|)$, which means that the electric and magnetic nonlinearity play an equally important role in the soliton formation, although in this case no ultraslow GV soliton is excited. To have an idea of how much intensity is necessary to launch these ultraslow GV light pulses we draw a comparison with the intensity necessary to launch a soliton in a standard fiber. For simplicity we fix the electric plasma wavelength of the metamaterial at $\lambda_{ep} = 1\ \mu$m ($1/\omega_{ep} \cong 0.55$ fs). In a standard single-mode fiber the dispersion $D = -(2\pi c/\lambda^2)\beta_2$ is approximately $D \sim 15$ ps/(Km nm) at $\sim 1.55\ \mu$m [1,2]. The GVD parameter of the standard fiber at $1.55\ \mu$m expressed in units of $(c\omega_{ep})^{-1}$ is approximately $\beta_2^{\text{fiber}} \approx -10^{-2}(c\omega_{ep})^{-1}$ and the threshold intensity is $S_{thr}^{\text{fiber}} \approx 3 \times 10^{-2}/(T_0^2 \omega_{ep}^2 c |\tilde{\chi}_{\text{fiber}}^{(3)}|)$ where we take $n_{\text{fiber}} \sim 1.4$. In Fig. 2 we plot $(S_{thr}|\tilde{\chi}_{\text{fiber}}^{(3)}|)/(S_{thr}^{\text{fiber}}|\tilde{\chi}_\varepsilon^{(3)}|)$ and V_g near the electric plasma frequency of the metamaterial. In this case β_2 is negative [see Fig. 1(b)] and bright solitons can be launched when $\tilde{\chi}_\varepsilon^{(3)} > 0$. For $\omega_0 = 1.00002\omega_{ep}$ (i.e., a detuning from the high frequency band edge $\delta\omega = \omega_0 - \omega_{ep} = 2 * 10^{-5}\omega_{ep}$) the GV will be $\sim c/100$ and $(S_{thr}|\tilde{\chi}_{\text{fiber}}^{(3)}|)/(S_{thr}^{\text{fiber}}|\tilde{\chi}_\varepsilon^{(3)}|) \sim 10^4$. Of course, the actual value of the intensity necessary to launch the soliton will depend on the value of the cubic nonlinearity in the metamaterial. It has been noted [13] that microscopic fields can be dramatically enhanced in a composite structure such as a metamaterial therefore giving an enhancement of nonlinear phenomena. If we suppose $\tilde{\chi}^{(3)} \sim (10^3 - 10^4)\tilde{\chi}_{\text{fiber}}^{(3)}$, the intensity necessary to launch a $c/100$ GV soliton will be comparable with the intensity necessary to launch a soliton in a standard fiber. The intensity grows exponentially if slower pulses are to be launched. As we have already mentioned there is a limitation to how short a pulse can be. In the case of our example we have the following values at $\omega_0 = 1.00002\omega_{ep}$: $n \sim 4 * 10^{-3}$ and $V_g \sim 10^{-2}c$, therefore the temporal duration of the pulse must be $T_0 \gg 2\pi c/(V_g n \omega_0) \sim 100$ ps, which means at least ~ 1 ns pulses. There is one last issue that deserves to be discussed in some detail and that is the effect of higher-order dispersion terms. As we have already remarked in this work we are operating near the band edges of the structure, i.e., in regions of extremely high dispersion, as is evident, for example, from Fig. 1(b) where the GVD parameter β_2 is represented. On the other hand, in order to arrive at analytical solutions we have neglected the third- and higher-order dispersion terms. Now, in conventional fibers third- and higher-order dispersion terms are generally small and can be treated by perturbation approaches for a pulse duration $T_0 \gg 1$ ps [1,2]; in our case, given the remarkably high dispersion present near the band edges we may expect that much longer pulses than those of a conventional fiber are needed in order to reduce the effects of the third- and higher-order dispersion. We would like to remind the reader that we have already set up a lower limit to the pulse duration, i.e., $T_0 \gg 2\pi c/(V_g n \omega_0)$, based on the requirement that the pulse must be slowly varying in space; in our case, for a soliton with a group velocity $10^{-2}c$, this condition calls for pulses that are at least 1 ns in duration. Let us here concentrate on the third-order dispersion in particular. To consider the third-order dispersion as a small perturbation we need to impose the condition that $L_D^{(3)} \gg L_D$ where $L_D^{(3)} = T_0^3/|\beta_3|$ is the so-called third-order dispersion length [1,2,14] and $L_D = T_0^2/|\beta_2|$ is the second-order dispersion length. This means that the pulse duration must satisfy the following condition: $T_0 \gg |\beta_3|/|\beta_2|$. In Fig. 3 we plot $|\beta_3|/|\beta_2|$ for the metamaterial described in Fig. 1 when the carrier frequency of the pulse is tuned near the high frequency band edge, i.e., $\omega_0 \sim \omega_{ep}$. In par-

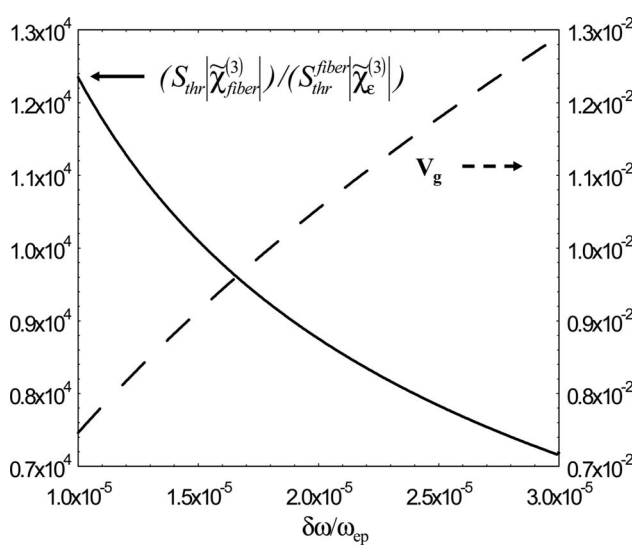

Fig. 2. $(S_{thr}|\tilde{\chi}_{fiber}^{(3)}|)/(S_{thr}^{fiber}|\tilde{\chi}_\varepsilon^{(3)}|)$ (left y axis) and V_g in units of c (right y axis) versus $\delta\omega/\omega_{ep}$ where $\delta\omega = \omega_0 - \omega_{ep}$ is the detuning from the high frequency band edge.

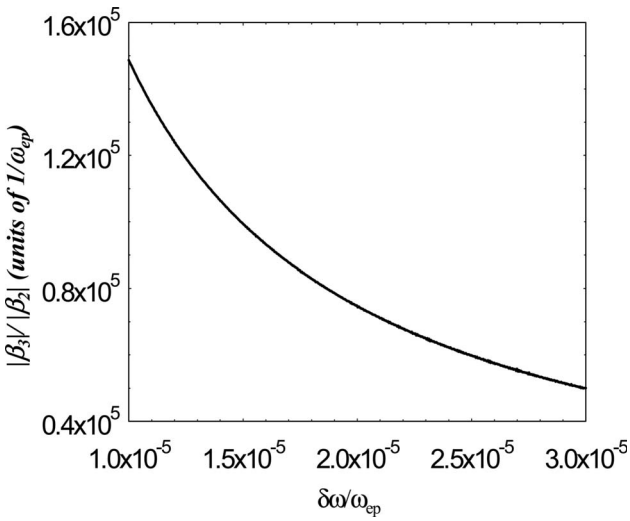

Fig. 3. $|\beta_3|/|\beta_2|$ (units of $1/\omega_{ep}$) versus $\delta\omega/\omega_{ep}$.

ticular we plot $|\beta_3|/|\beta_2|$ as a function of the detuning $\delta\omega = \omega_0 - \omega_{ep}$. In the case of a detuning $\delta\omega/\omega_{ep} \approx 2*10^{-5}$ the GV of the soliton is $c/100$ and $T_0 \gg |\beta_3|/|\beta_2| \approx 7.4*10^4/\omega_{ep} \sim 40$ ps. If we now recall that for the same tuning condition the pulse duration necessary to maintain the validity of the slowly varying envelope approximation is $T_0 \gg 2\pi c/(V_g n \omega_0) \sim 100$ ps it should appear evident that nanosecond pulses not only ensure the validity of the slowly varying envelope approximation in space, but also allow us to treat the third-order dispersion as a small perturbation. The generalization of Eq. (5), which includes the third-order dispersion, can be written as

$$i\frac{\partial u}{\partial \tilde{z}} - \frac{\mathrm{sgn}(\beta_2)}{2}\frac{\partial^2 u}{\partial \tilde{t}^2} + \Gamma|u|^2 u = i\delta_3 \frac{\partial^3 u}{\partial \tilde{t}^3}, \tag{7}$$

where δ_3 is the third-order dispersion parameter $\delta_3 = \beta_3/(6T_0|\beta_2|)$ and $|\delta_3| \ll 1$ in our case. Here we consider for simplicity the case that the pulse is tuned near the high frequency band edge and the electric nonlinearity is positive. It can be demonstrated by using a perturbation approach [2,15] that the fundamental bright soliton solution modified by the third-order dispersion can be written in the following form:

$$u(\tilde{z},\tilde{t}) = (\mathrm{sech}(\tilde{t} - \delta_3\tilde{z})/\sqrt{\Gamma})\exp(i\tilde{z}/2). \tag{8}$$

In other words, what the third-order dispersion does is to modify the actual GV in the following form:

$$V_g^{(1)} = \frac{V_g}{1 + \dfrac{V_g \beta_3}{6T_0^2}}, \tag{9}$$

where clearly $V_g = 1/\beta_1$ is the usual GV. In Fig. 4 we calculate the correction brought to the usual GV by the third-order dispersion for a 1 ns pulse tuned near the high frequency band edge. The figure shows that the GV is corrected for less than one part over 1000 and therefore, for all intents and purposes, this correction can be considered negligible. Similar results to those exposed above can be expected if ω_0 is tuned near ω_{mp}, except that in this latter case bright solitons can be launched when

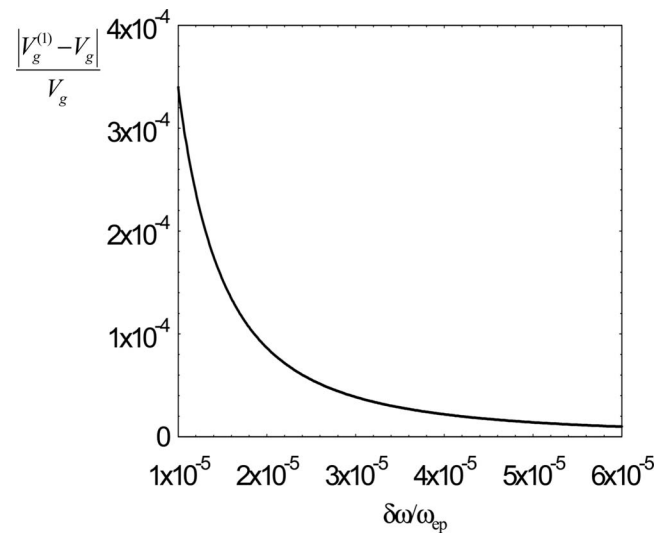

Fig. 4. $|V_g^{(1)} - V_g|/V_g$ versus $\delta\omega/\omega_{ep}$.

$\tilde{\chi}_\mu^{(3)} < 0$. Here we have explicitly analyzed the case $\omega_{mp} < \omega_{ep}$. In the opposite situation, $\omega_{mp} > \omega_{ep}$, we would have bright solitons at $\omega_0 \approx \omega_{mp}$ for $\tilde{\chi}_\mu^{(3)} > 0$ and at $\omega_0 \approx \omega_{ep}$ for $\tilde{\chi}_\varepsilon^{(3)} < 0$.

Once again, we would like to caution the reader that our approach is valid for sufficient long pulses because they allow us to treat the higher-order dispersion terms perturbatively. For shorter pulses the situation is much more complicated, because in that case the strong dispersion present near the band edges may prevent its expansion into the various terms (first order, second order, etc.), but the entire dispersion of the material might have to be considered all together; a task that can be undertaken only numerically by an *ab initio* integration of Maxwell equations. While, of course, the dynamics of short pulses near the band edges of a metamaterial may represent a fertile ground for future research, it is clearly outside the scope of the present work where, on the contrary, we have focused our attention on longer pulses, i.e., on cases in which the expansion of the dispersion remains valid and analytical solutions are available.

A final note regarding the absorption of the metamaterial. In this work we have neglected the absorption of the metamaterial. In currently available metamaterials in the near infrared [16] the absorption is still so high that it is premature to think about any practical soliton application. Nevertheless, in principle, there is nothing that prevents the possible availability of low-loss metamaterials in the near future.

4. CONCLUSIONS

In conclusion, we have performed an analytical study on the possibility to excite ultraslow solitons near the band edges of a metamaterial. We have investigated the roles played by the permittivity, the permeability, the electric cubic nonlinearity, the magnetic cubic nonlinearity, and the pulse duration. We hope that our results may stimulate further research aiming at the study of this new class of materials in applications that involve slow light, such as all-optical buffering and switching, for example.

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