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## TURBULENT HEATING OF COLLIDING STREAMS IN THE SOLAR WIND

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### ABSTRACT

Turbulent heating of colliding plasma streams has previously been observed in the solar wind. The original data were interpreted in terms of a fluid model. We argue that a plasma-kinetic description is the more appropriate theoretical approach and is necessary in order to better understand the microscopic physical phenomena that underlie all fluid models. We used microscopic solar-wind parameters characteristic of conditions during the observations, together with the quasi-linear plasma-kinetic theory, to compute the expected magnetic field and temperature enhancements in the interaction region between two counterstreaming plasma beams. The physical mechanism of excitation is the electromagnetic two-stream instability in which Alfvén waves are unstable. We compute a total field in the interaction region of  $B \sim 8 \gamma$  ( $10\text{--}12 \gamma$  is observed) and a change in temperature of  $\Delta T \sim 1 \times 10^5 \text{ }^\circ\text{K}$  ( $1\text{--}2 \times 10^5 \text{ }^\circ\text{K}$  is observed). Other features of the observations are discussed in terms of a plasma-kinetic theory.

*Subject headings:* plasmas — solar wind — turbulence — instabilities

### I. INTRODUCTION

Theoretical studies of the nature of the interplanetary medium traditionally have dealt with the hydrodynamic and thermodynamic behavior of the freely flowing uniform solar wind. However, experimental studies have often reported observations of strong gradients in the various physical parameters of the interplanetary plasma (for a review, see Hundhausen 1970). This is a natural result of the experimenter's desire to report interesting phenomena and the need of the theoretician to find a tractable problem. In addition, the solar-wind flow is rarely as quiet as it is described to be in general theoretical models.

In this paper, it is our purpose to apply the kinetic theory of plasmas to the elucidation of the experimental results of Burlaga *et al.* (1971). They used data from two satellites (*Explorer 34* and *Vela 4B*) taken in the interaction region of two solar-wind plasma streams, one of which was overtaking the other. Their findings may be summarized briefly as follows: (1) enhancements of number density, magnetic field and ion temperature in the interaction region; (2) conservation of electron temperature and ion anisotropies in the interaction region; (3) participation of the  $\alpha$ -particle component in the observed heating in the same ratio as the protons; and (4) low-frequency field fluctuations in the interaction regions or "hot spots," but not downstream of them.

These findings led Burlaga *et al.* (1971) to the conclusion that their experiment can be interpreted partially in the light of the nonsteady, nonlinear, adiabatic fluid model of Hundhausen and Gentry (1969). The failures of the model were attributed to its being a single-fluid model, and the belief was expressed that a hydromagnetic three-fluid model would provide an adequate description and explanation of the observed phenomena.

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The question of the relevance of fluid models to the hydrodynamics and thermodynamics of collisionless, finite-temperature plasmas has been discussed elsewhere (Eviatar and Schulz 1970; Schulz and Eviatar 1972). Historically, the theory of the solar wind has evolved from a basic hydrodynamical model (Parker 1958, 1963; Hartle and Sturrock 1968). Corrections and refinements have been imposed to explain the growing knowledge of the structure and behavior of the solar wind (Barnes 1968, 1969; Hartle and Barnes 1970, Forslund 1970). The exospheric approach of Brandt and Cassinelli (1966) has also required modification (Kennel and Scarf 1968; Eviatar and Schulz 1970; Schulz and Eviatar 1972) to describe adequately the observed state of the interplanetary plasma. In general, the two approaches, hydrodynamic and exospheric, are complementary in that the true solar wind requires a kinetic theory, which occupies the gray area between them. To the extent that each approach is used judiciously and that they converge upon the same results, they are equally valid. (Certainly neither should be applied uncritically to a given situation. If neither limiting approach is valid, plasma-kinetic theory is required.)

It is our contention that a kinetic theory is necessary to describe the phenomena reported by Burlaga *et al.* (1971). It is only through such a theory that one can hope to understand why fluid models yield good results when describing the behavior of the collisionless plasma of the solar wind. The colliding-stream problem appears to us to be the classic two-stream configuration, and calls for an application of quasi-linear plasma-kinetic theory to derive the observed phenomena.

## II. TWO-STREAM INSTABILITY

In the situation described above, the "driver" gas overtakes the "driven" gas at a relative velocity  $u$ , on the order of the Alfvén velocity. This is a counterstreaming plasma configuration. The linear stability theory has been treated by many investigators. (Kahn 1957; Harris 1961; Stix 1962 and references cited there.) More recently, the problem has been treated at the quasi-linear level by Rowlands, Shapiro, and Shevchenko (1966, hereafter referred to as RSS). Their results, with some modification, are applicable to the phenomena described here.

## III. LINEAR THEORY

The linear growth rate can be found as the imaginary part of the solution of the dispersion relation obtained from the linearized Vlasov-Maxwell equations (Montgomery and Tidman 1964). For gyrotropic plasmas in which the phase velocities of the waves greatly exceed the thermal velocities the growth rate is

$$\gamma = -\pi^2 \sum_{\nu} \frac{\omega_{p\nu}^2 \omega}{N_{\nu} \partial D / \partial \omega} \frac{1}{|k|} \int_0^{\infty} dv_{\perp} v_{\perp}^2 \left[ \frac{v_{\perp}}{v_{\parallel}} \frac{\partial f_{0\nu}}{\partial v_{\parallel}} - \frac{\Omega_{\nu}}{\omega} \left( \frac{\partial f_{0\nu}}{\partial v_{\perp}} - \frac{v_{\perp}}{v_{\parallel}} \frac{\partial f_{0\nu}}{\partial v_{\parallel}} \right) \right] \Big|_{v_{\parallel} = \omega \pm \Omega_{\nu}/k = v_{res}}, \quad (1)$$

where it is assumed that  $\gamma \ll \omega$ , the real part of the frequency. The following notation is used:

$$\omega_{p\nu} = \left( \frac{4\pi q_{\nu}^2 N_{\nu}}{M_{\nu}} \right)^{1/2} \quad \text{is the plasma frequency of species } \nu;$$

$$\Omega_{\nu} = \frac{q_{\nu} B}{M_{\nu} c} \quad \text{is the respective gyrofrequency } (q_e = -q_i);$$

$N_\nu$  and  $M_\nu$  are number density and mass, respectively;  $B$  is the ambient magnetic field; and  $f_{0\nu}(v_\perp^2, v_\parallel)$  is the single-particle velocity distribution function of species  $\nu$ . Also  $k = k_\parallel$  is the wave normal of circularly polarized electromagnetic waves propagating along  $B$ . The function  $D(\omega, k)$  is the zero-temperature dielectric function of the plasma (Stix 1962). In our discussion, we will ignore the electron dynamics and consider only left-hand polarized waves. (The right-hand ion-cyclotron waves are strongly damped.) In addition, we assume that  $|\omega/\Omega_e| \ll 1$  and  $\omega/\Omega_i \lesssim 1$ . Consequently, we will be describing excitation of Alfvén waves. Finally, we assume that the plasma has zero temperature in the *real* part of the dispersion relation, but finite temperature  $T$  in the imaginary part (e.g., Montgomery and Tidman 1964; Kennel 1966).

With these assumptions, the zero-temperature dielectric function is given by

$$D(\omega, k) = k^2 c^2 - \omega^2 \left[ 1 + \frac{\omega_{pi}^2}{\Omega_i(\omega + \Omega_i)} \right]. \quad (2)$$

Strictly  $D(\omega, k)$  should describe two zero-temperature counterstreaming plasma beams, rather than the stationary situation described by equation (2). For two zero-temperature streams  $D(\omega, k)$  is given by

$$D(\omega, k) = k^2 c^2 - \omega^2 - \frac{\omega^2 \omega_{pi}^2}{2\Omega_i(\omega + \Omega_i)} - \frac{(\omega - ku)^2 \omega_{pi}^2}{2\Omega_i(\omega - ku + \Omega_i)}. \quad (3)$$

Use of equation (3) would introduce considerable mathematical complexity into the quasi-linear analysis without altering the basic physics. In fact, one can show that when equation (3) is used, the instability is excited over a broader range of frequencies and grows with a larger linear growth rate than is predicted from equation (2). Consequently, because we have used  $D(\omega, k)$  given by equation (2) in the following discussion, our results should be considered an underestimate of the magnitude of the magnetic-field oscillations and temperature enhancement excited by the two-stream instability. Furthermore, the frequency bandwidth of the power spectrum will be only qualitatively described by our analysis.

Equating  $D(\omega, k)$  to zero leads to a cubic equation for the phase velocity,  $v_{ph} = \omega/k$ , as a function of the resonant velocity,  $v_{res} = (\omega + \Omega_i)/k$  (see Appendix). From the appendix we see that Alfvén waves can be excited if

$$v_{res} \geq (27/4)^{1/2} v_A, \quad (4)$$

where

$$v_A = \frac{B}{(4\pi N_i M_i)^{1/2}}$$

is the Alfvén velocity.

Because the solar-wind velocity is super-Alfvénic, Alfvén waves excited in the interaction region of the two plasma streams cannot propagate downstream into the slower-moving plasma. However, one would expect to observe enhanced magnetic-field fluctuations for some distance upstream in the fast plasma stream. This is in fact consistent with the observations of Burlaga *et al.* (1971).

We describe the contrastreaming plasma in terms of the following proton distribution function:

$$f_0(v_\perp, v_\parallel) = \frac{1}{2} N_i \left( \frac{M_i}{2\pi K T} \right)^{3/2} \exp(-v_\perp^2/2V^2) \{ \exp[-(v_\parallel - u)^2/2V^2] + \exp(-v_\parallel^2/2V^2) \}, \quad (5)$$

where  $V = (KT/M_i)^{1/2}$  is the proton thermal speed. The plasma density  $N_i$  is assumed to be equal in the two streams. In figure 1 we have plotted  $f_0(v_{\parallel})$  for  $v_{\parallel} \geq 0$ . Using equations (2) and (5), we can write the growth rate as (RSS)

$$\gamma = -\frac{1}{2V} (1/2\pi)^{1/2} \frac{\omega_{pi}^2}{\partial D/\partial \omega} \{ (u - v_{ph}) \exp [-(v_{\parallel} - u)^2/2V^2] - v_{ph} \exp (-v_{\parallel}^2/2V^2) \} |_{v_{\parallel}=v_{res}} \quad (6)$$

In the analysis below we used  $N_i = 5 \text{ cm}^{-3}$ ,  $u = 100 \text{ km s}^{-1}$ ,  $B = 4\gamma$ , and  $T = 10^5 \text{ }^\circ \text{K}$ . From equations (2) and (6), and with these values for the parameters, one can easily show that the situation we are describing is strongly two-stream unstable. The growing waves will reach amplitudes larger than the zero-order field magnitude in a much shorter time than that necessary for an Alfvén wave to propagate across the interaction-region. Consequently, linear theory rapidly breaks down, and a quasi-linear treatment is necessary.

IV. QUASI-LINEAR THEORY

The phenomena under discussion are amenable to treatment by quasi-linear plasma-kinetic theory. This follows from the relatively short timescale which prevents mode-coupling from becoming important. This minimizes the secularities by which all finite-order perturbation theories are plagued.

The quasi-linear theory of electromagnetic waves of finite frequency was developed formally by RSS. They calculated the asymptotic behavior of wave amplitude and zero-order distribution function for an Alfvén wave spectrum. The formalism that they developed is directly applicable to this problem and requires slight modification only because we are interested in a situation which is at the threshold of instability

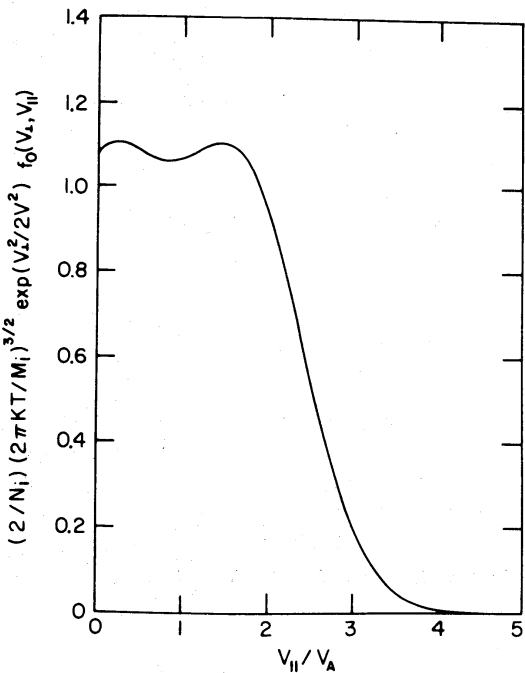


FIG. 1.—The single-particle velocity distribution function  $f_0(v_{\parallel})$  plotted against  $v_{\parallel}/v_A$ . The values of the parameters used are  $u = 100 \text{ km s}^{-1}$ ,  $T_i = 1 \times 10^5 \text{ }^\circ \text{K}$ , and  $B = 4 \gamma$ .

(i.e.,  $v_{\text{res}} \gtrsim (27/4)^{1/2} v_A$ , while RSS were interested in  $v_{\text{res}} \gg (27/4)^{1/2} v_A$ ). Because the basic theory is well known (RSS), we shall only outline the calculation here. Following RSS, we reduce the kinetic equation for the time evolution of the zero-order velocity distribution function to the following one-dimensional form:

$$\frac{\partial f_0}{\partial t} = \left( \frac{q}{2Mc} \right)^2 \frac{\partial}{\partial v} \left[ v_{\perp}^2(w, v) \frac{|b|^2}{|v - d\omega/dk|} \frac{\partial f_0}{\partial v} \right], \quad (7)$$

where

$$w = v_{\perp}^2 + v_{\parallel}^2 - 2 \int_{v_{\min}}^v v_{\text{ph}}(v') dv', \quad v = v_{\parallel},$$

and  $|b|^2$  is the square of the magnetic amplitude of the excited wave. We have dropped the subscript  $i$  in the remainder of our discussion.

Equation (7) describes the diffusion of resonant particles along the level curves of  $w$  in the  $(v_{\perp}, v)$ -plane. The diffusion occurs as the distribution function,  $f_0(t, w, v)$ , evolves in time from  $f_0^0(0, w, v)$  to its asymptotic form  $f_0^{\infty}(w)$  given by

$$f_0^{\infty, \text{I}}(w) = \frac{\pi^{1/2} N \exp [-(w + u^2)/2V^2]}{2(v_{\max}^{\text{I}}(w) - v_{\text{lim}})(2\pi V^2)^{3/2} G} \exp \left\{ \left( \frac{-1}{2V^2} \right) [-2uv_m + 2 \times 3^{1/2} v_A (v_m - v_{\text{lim}}) + 2 \times 6^{1/2} v_A^2 v_{\text{lim}}^{-3/2} (v_m - v_{\text{lim}})^{3/2}] \right\} \\ \times \{ \text{erf} [(v_{\max}^{\text{I}}(w) - v_m)G] + \text{erf} [(v_m - v_{\text{lim}})G] \} \quad (8a)$$

for  $v < y = \frac{9}{2} v_A (\sqrt{3} - 1)$ , and

$$f_0^{\infty, \text{II}}(w) = \frac{N \exp [-(w + u^2)/2V^2]}{(2\pi V^2)^{3/2} (v_{\max}^{\text{II}}(w) - v_{\text{lim}})} \\ \times \left\{ \frac{\sqrt{\pi}}{2G} \exp \left\{ \left( \frac{-1}{2V^2} \right) [-2uv_m + 2 \times 3^{1/2} v_A (v_m - v_{\text{lim}}) + 2 \times 6^{1/2} v_A^2 v_{\text{lim}}^{-3/2} (v_m - v_{\text{lim}})^{3/2}] \right\} \right. \\ \left. + 2 \times 6^{1/2} v_A^2 v_{\text{lim}}^{-3/2} (v_m - v_{\text{lim}})^{3/2} \right\} \left[ \text{erf} [(y - v_m)G] + \text{erf} [(v_m - v_{\text{lim}})G] \right] \\ + \left( \frac{V^2}{u - v_A} \right) \exp \left( \frac{2.43 v_A^2}{2V^2} \right) \left\{ \exp \left[ \frac{u - v_A}{V^2} v_{\max}^{\text{II}}(w) \right] - \exp \left( \frac{u - v_A}{V^2} y \right) \right\} \quad (8b)$$

for  $v > y$ , where  $\text{erf}(x)$  is the error function; and

$$v_m = \frac{3\sqrt{3}}{4} \left( \frac{u^2}{v_A} + 5v_A - 2 \times 3^{1/2} u \right), \\ G = \frac{v_A}{V} \left[ \frac{3\sqrt{6}}{8} v_{\text{lim}}^{-3/2} (v_m - v_{\text{lim}})^{-1/2} \right]^{1/2}.$$

The derivation of equations (8) and the definitions of  $v_{\max}^{\text{I,II}}(w)$  are given in the Appendix. Qualitatively, equations (8) are a Maxwellian with a high-energy tail.

We use this asymptotic distribution function to calculate the spectrum of magnetic field oscillations and to estimate the temperature enhancement in the interaction region. These two quantities can then be directly compared with the satellite measurements (Burlaga *et al.* 1971).

The two-stream instability causes Alfvén waves to grow at the expense of the kinetic energy of the counterstreaming plasma. Wave-particle interactions limit the wave growth and heat the plasma. The equations of quasi-linear theory enable one to compute the asymptotic energy density of the magnetic-field oscillations, and the temperature enhancement of the plasma.

The power spectrum of the excited Alfvén waves is given by (RSS)

$$\frac{1}{8\pi} \frac{\partial}{\partial \nu} |b|^2 = \frac{-4\pi^3 M c^2 |1 - v/(d\omega/dk)| \times 10^{10}}{\partial D/\partial \omega} \times \int_{v_{lim}}^v dv' \int_{w_{min}(v')}^{\infty} dw [f_0^\infty(w, v_m) - f_0^0(w, v')] \gamma^2 \text{ Hz}^{-1}, \tag{9}$$

where  $\nu = \omega/2\pi$  and  $w_{min}(v)$  is defined in the Appendix. Using equations (2), (5), (8) and parameters from the Appendix, one can estimate equation (9) analytically. However, we have found it more accurate to evaluate equation (9) numerically. The resulting power spectrum is shown in figure 2. The area under the curve gives the field enhancement to be  $\Delta B \simeq 4 \gamma$ . This means that the total magnetic field in the interaction region is  $B \sim 8 \gamma$ , which is very close to the observed value of  $B \sim 10 \gamma$ . Detailed comparison of the observed and computed power spectrum cannot be made for two reasons. First, experimenters used 20-second averages of the magnetic-field data in presenting the results. Therefore, this experiment is sensitive only to frequencies  $\nu < 0.05 \text{ Hz}$ , which is the low-frequency tail of the computed wave spectrum. Observations with an instrument sensitive to higher frequencies would be necessary to compare

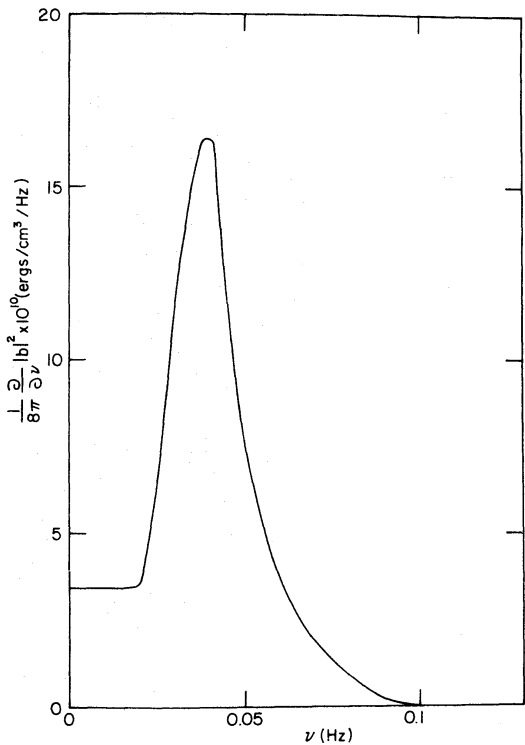


FIG. 2.—The power spectrum of the magnetic field,  $\frac{1}{8\pi} \pi^{-1} \partial |b|^2 / \partial \nu$ , in  $\gamma^2$  per Hz versus frequency,  $\nu$ . The total field enhancement is  $\Delta B \sim 3.8 \gamma$ .



with the power spectrum computed from quasi-linear theory. Second, in addition to underestimating the magnetic oscillations by using equation (2) rather than equation (3) in our analysis, the frequency bandwidth of the power spectrum would be broadened to include higher and lower frequencies. It is not clear where the peak in the spectrum would lie in a more detailed analysis. We have shown that the two-stream instability can lead to sufficient magnetic-field enhancement in a hot spot. However, the approximations that we have made in order to arrive at a tractable problem preclude a more quantitative comparison of the observed and computed power spectra.

One can also estimate the temperature enhancement in the interaction region. The change in energy of the resonant particles is given by (RSS)

$$\begin{aligned}\Delta\epsilon_{\text{res}} &= (4\pi c)^{-2} \int_0^\infty dv \left( \frac{\partial}{\partial v} |b|^2 \right) \frac{\partial D}{\partial v} v_{\text{ph}}(v - v_{\text{ph}})/\Omega \\ &= -\frac{1}{2}M \int dv (v_\perp^2 + v_\parallel^2)(f_0^\infty - f_0^0). \end{aligned} \quad (10)$$

The energy lost by the resonant particles goes into increasing the magnetic turbulence and the transverse energy of the nonresonant particles. One can estimate the resulting increase in temperature of the plasma by noting that the change in temperature is approximately given by

$$\frac{\Delta T}{T} \simeq \frac{M}{3NKT} \int dv (f_0^\infty - f_0^0)(v_\perp^2 + v_\parallel^2) = -\frac{2}{2NKT} \Delta\epsilon_{\text{res}}, \quad (11)$$

where we have assumed that as  $t \rightarrow \infty$  the interaction region moves with the fast plasma stream.

The integrations in equation (10) were done numerically. The result is

$$\Delta T/T \simeq 1.1.$$

The range of observed values was  $\Delta T/T \simeq 1-2$ .

The strength of the effect we have described is dependent on the values chosen for the interplanetary parameters. If  $u$  were much greater than the Alfvén speed, one would expect to observe a shock wave rather than the unstable two-stream configuration. If  $B$  is much larger than  $\sim 4\gamma$ , or  $u$  much smaller than  $100 \text{ km s}^{-1}$ , then the condition for the growth of Alfvén waves,  $v_{\text{res}} > v_A$ , would not be satisfied.

## V. ALPHA PARTICLES AND ELECTRONS

Our calculations have all assumed a pure hydrogen composition of the solar wind. The  $\alpha$ -particle component is observed to be heated to the same extent proportionally as the protons, but apparently no electron heating takes place. The thermal anisotropies appear to be unaffected by the heating. These phenomena can be understood readily in the framework of a kinetic model.

The intercomponent friction that keeps the  $\alpha$ -particles streaming at the proton bulk velocity can be attributed to transit-time damping of magnetosonic waves (Eviatar and Schulz 1970) which tends to enhance anisotropy of both components (Barnes 1968, 1969). This enhancement must be nullified by the isotropizing effect of right-hand polarized cyclotron waves (Kennel and Scarf 1968; Eviatar and Schulz 1970). The same mechanism can serve as a means of equipartition between the protons and  $\alpha$ -particles. Thus we should expect both ion components to participate in the same proportion in the macroscopic heating, which is the observed manifestation of the microscopic instability. An  $\alpha$ -particle Alfvén wave growing at the expense of helium



contraststreaming would be expected to have a slower growth rate. This results from the greater ion inertia and the reduced number density (5 percent of the proton concentration).

The electrons, the thermodynamics of which is monitored by magnetosonic waves (Schulz and Eviatar 1972), would not be expected to be heated by this mechanism because their thermal velocities greatly exceed the solar-wind bulk velocity (Kellogg and Liehmohn 1960). Electron contraststreaming involves much smaller space scales than those associated with ion streaming (Ek, Kahalas, and Tidman 1962), and their interaction region would not be resolvable by these experiments.

## VI. DENSITY ENHANCEMENT

The fluid-like behavior of the plasma in the interaction region is a consequence of the wave-particle interactions which to a certain extent play the role of Coulomb collisions in a dense plasma. It should be kept in mind that in general the gross fluid picture of the solar wind is meaningful only over distance scales of the order of a sizable fraction of an astronomical unit, unless some phenomenon, such as wave-particle interactions, introduces an effective mean free path to replace the role of Coulomb collisions. The quasi-linear theory that we have used cannot provide us with the variation of particle concentration because the asymptotic distribution function was derived with the aid of the conservation of particles in  $w$ -space. In addition, the plasma was assumed to be homogeneous. A full nonlinear inhomogeneous treatment of the density variation lies beyond the scope of this paper.

Despite these limitations, certain qualitative conclusions can be drawn from the fluid equations. The equation of continuity can describe the variation of density and velocity between the fast and slow streams. The equation

$$\rho^{-1}d\rho/dt = -\nabla \cdot \mathbf{v} \quad (10)$$

can, for the one-dimensional steady state case, be integrated to give

$$\rho v = \text{const.}; \quad (11)$$

i.e., the decrease in velocity in the region of overtaking plasma is compensated by an increase in density. This behavior is, qualitatively, what is observed.

We conclude that a quasi-linear treatment of the two-stream instability can, at least qualitatively, predict the magnetic-field and temperature enhancements observed in colliding streams in the solar wind. Furthermore, wave-particle interactions are strong enough to introduce an effective mean free path much smaller than that from Coulomb scattering. Consequently, it is easier to understand the success of fluid models in describing certain aspects of small scale phenomena observed in the solar wind.

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## APPENDIX

We outline here the calculation of the asymptotic single-particle velocity distribution function in the quasi-linear regime. We are concerned only with the excitation of Alfvén waves because ion cyclotron waves will be strongly damped. Also, one can show that the high-frequency helical wave is unimportant when  $D(\omega, k)$  is given by equation (2). However, if  $D(\omega, k)$  is given by equation (3), then the high-frequency

branch of the dispersion relation becomes more important and a more detailed investigation of the role of mode coupling would be necessary. As we discussed above, use of equation (2) tends to underestimate the strength of the instability, and at the same time leaves us with a tractable mathematical problem.

Equation (2), when set equal to zero, can be written in terms of the velocity of the resonant particles that amplify the waves. The result is (RSS)

$$v \equiv v_{\text{res}} = \frac{\omega + \Omega}{k} = \frac{v_{\text{ph}}^3}{(v_{\text{ph}}^2 - v_A^2)}, \quad (\text{A1})$$

where  $v_{\text{ph}} = \omega/k$  and we have used the fact that  $v_A \ll c$ .

Alfvén waves will be excited if  $v \geq v_{\text{min}} = (27/4)^{1/2}v_A$ . This follows directly from equation (A1). The cubic equation has real roots for Alfvén waves only if  $v \geq v_{\text{min}}$ . It is usually assumed (RSS) that  $v \gg v_{\text{min}}$ , so that  $v_{\text{ph}} \simeq v_A$ . This introduces considerable simplification into the analysis. However, for the solar-wind problem we are describing, one finds  $v \sim v_{\text{min}}$ . From the observations reported by Burlaga *et al.* (1971),  $v \gtrsim (27/4)^{1/2}v_A$  when a “hot spot” is seen. Presumably, when this condition is not satisfied, there will be no enhancement of magnetic field or temperature.

We should emphasize again that the approximation made in using equation (2) in place of equation (3) tends to overestimate  $v_{\text{min}}$  by almost a factor of 2.

Consequently we cannot use the approximation  $v_{\text{ph}} \simeq v_A$ . For  $v \simeq v_{\text{min}}$ , equation (A1) can be expanded in Taylor series, and one finds that the Alfvén branch is approximated by

$$v_{\text{ph}} \simeq 3^{1/2}v_A - 2^{1/2}v_A \left( \frac{v}{v_{\text{min}}} - 1 \right)^{1/2}; \quad (\text{A2})$$

when  $v \simeq \frac{9}{2}(\sqrt{3} - 1)v_A$ , one can write

$$v_{\text{ph}} \simeq v_A. \quad (\text{A3})$$

Using equations (A2) and (A3), we can now evaluate the variable  $w$  defined in equation (7). In addition to  $w$ , we will need two other related quantities,  $v_{\text{max}}(w)$  and  $w_{\text{min}}(v)$ . As particles diffuse so that  $w$  is constant, the resonant velocity varies between  $v_{\text{min}}$  and  $v_{\text{max}}(w)$ , which is defined by setting  $v_{\perp} = 0$  in equation (7), so that

$$w = v_{\text{max}}^2(w) - 2 \int_{v_{\text{min}}}^{v_{\text{max}}} v_{\text{ph}}(v) dv. \quad (\text{A4})$$

Conversely, the minimum value of  $w$  is defined as

$$w_{\text{min}}(v) = v^2 - 2 \int_{v_{\text{min}}}^v v_{\text{ph}}(v') dv'. \quad (\text{A5})$$

We find that for  $v < \frac{9}{2}(\sqrt{3} - 1)v_A$

$$w = v_{\perp}^2 + v^2 - 2 \times 3^{1/2}v_A(v - v_{\text{min}})[1 - 4(27^{1/2}v_A)^{-3/2}(v - v_{\text{min}})^{1/2}v_A]$$

and

$$v_{\text{max}}^{\text{I}}(w) \simeq \left\{ w + 2 \times 3^{1/2}v_A^2 \left( \frac{w^{1/2}}{v_{\text{min}}} - 1 \right) \left[ \frac{3}{2} \sqrt{3} - 2^{1/2} \left( \frac{w^{1/2}}{v_{\text{min}}} - 1 \right)^{1/2} \right] \right\}^{1/2} \quad (\text{A6})$$

when

$$v \geq \frac{9}{2}(\sqrt{3} - 1)v_A, \quad w = v_{\perp}^2 + v^2 - 2v_A v + 4.86v_A^2,$$

and

$$v_{\max}^{\text{II}}(w) \simeq v_A[1 + (w/v_A^2 - 3.86)^{1/2}]. \quad (\text{A7})$$

One should note that the approximate expressions for  $v_{\text{ph}}$  (eqs. [A2] and [A3]) are used only to enable us to compute  $w$  and  $v_{\max}(w)$ . Whenever  $v_{\text{ph}}$  is explicitly needed (e.g., in eq. [6]), it is found from the exact solution of equation (A1).

The asymptotic form of the distribution function can now be calculated from its definition (RSS):

$$f_0^{\infty}(w) \simeq \frac{1}{(v_{\max}(w) - v_{\text{lim}})} \int_{v_{\text{lim}}}^{v_{\max}(w)} dv f_0^0(w, v), \quad (\text{A8})$$

where  $f_0^0(w, v) = f_0(v_{\perp}, v_{\parallel})$ .

The integration in equation (A8) can be performed using the method of steepest descent. The result is equation (8).

#### REFERENCES

- Barnes, A. 1968, *Ap. J.*, **154**, 751.  
 ———. 1969, *ibid.*, **155**, 311.  
 Brandt, J. C., and Cassinelli, J. P. 1966, *Icarus*, **5**, 47.  
 Burlaga, L. F., Ogilvie, K. W., Fairfield, D. H., Montgomery, M. D., and Bame, S. J. 1971, *Ap. J.*, **164**, 137.  
 Ek, F., Kahalas, S. L., and Tidman, D. A. 1962, *Phys. Fluids*, **5**, 328.  
 Eviatar, A., and Schulz, M. 1970, *Planet and Space Sci.*, **18**, 321.  
 Forslund, D. W. 1971, *J. Geophys. Res.*, **75**, 17.  
 Harris, E. G. 1967, *J. Nucl. Energy, Part C*, **2**, 138.  
 Hartle, R. E., and Barnes, A. 1970, *J. Geophys. Res.*, **75**, 6915.  
 Hartle, R. E., and Sturrock, P. A. 1968, *Ap. J.* **151**, 1155.  
 Hundhausen, A. J. 1970, *Rev. Geophys.*, **8**, 729.  
 Hundhausen, A. J., and Gentry, R. A. 1969, *J. Geophys. Res.*, **74**, 2908.  
 Kahn, F. D. 1957, *J. Fluid Mech.*, **2**, 601.  
 Kellogg, P. J., and Liehmohn, H. 1960, *Phys. Fluids*, **3**, 40.  
 Kennel, C. F. 1966, *Phys. Fluids*, **9**, 2190.  
 Kennel, C. F., and Scarf, F. C. 1968, *J. Geophys. Res.*, **73**, 6149.  
 Lerche, I. 1966, *J. Geophys. Res.*, **71**, 2365.  
 Montgomery, D. C., and Tidman, D. A. 1964, *Plasma Kinetic Theory* (New York: McGraw-Hill Book Co.).  
 Parker, E. N. 1958, *Ap. J.*, **128**, 664.  
 ———. 1963, *Interplanetary Dynamical Processes* (New York: Interscience).  
 Rowlands, J., Shapiro, V. D., and Shevchenko, V. I. 1966, *J. Exper. and Theoret. Phys. (USSR)*, **50**, 979 (English transl. 1966, *Soviet Phys.—JETP*, **23**, 651).  
 Schulz, M., and Eviatar, A. 1972, *Cosmic Electrodynamics*, **2**, 402.  
 Stix, T. H. 1962, *The Theory of Plasma Waves* (New York: McGraw-Hill Book Co.).