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Shortcuts to adiabaticity: suppression of pair production in driven Dirac dynamics

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E-mail: sebastian.deffner@gmail.com**Keywords:** shortcuts to adiabaticity, Dirac dynamics, fast-forward technique

Abstract

Achieving effectively adiabatic dynamics in finite time is a ubiquitous goal in virtually all areas of modern physics. So-called shortcuts to adiabaticity refer to a set of methods and techniques that allow us to produce in a short time the same final state that would result from an adiabatic, infinitely slow process. In this paper we generalize one of these methods—the fast-forward technique—to driven Dirac dynamics. As our main result we find that shortcuts to adiabaticity for the $(1 + 1)$ -dimensional Dirac equation are facilitated by a combination of both scalar and pseudoscalar potentials. Our findings are illustrated for two analytically solvable examples, namely charged particles driven in spatially homogeneous and linear vector fields.

1. Introduction

Although initially it was not even fully appreciated by its discoverer, the ‘Dirac equation’ [1] has turned out to be one of the most important and versatile results of theoretical physics. In particle physics, the Dirac equation is a relativistic wave equation, which describes massive spin- $1/2$ particles, such as electrons and quarks [2, 3]. Among its achievements are the explanation of spin as a consequence of special relativity and the discovery of the existence of antimatter. Even almost a century after its inception the analysis of Dirac dynamics and the enhancement of pair production—the creation of antimatter—still attract vigorous research activity [4–7].

However, the study and applications of the Dirac equation are no longer restricted to problems in particle physics. For instance, Dirac dynamics were recently analyzed in the context of the quantum speed limit [8], in optomechanics [9], and in quantum thermodynamics [10]. More importantly, however, in a wide range of materials, such as d -wave superconductors and graphene, low-energy excitations behave as massless Dirac particles rather than fermionic Schrödinger particles. So-called ‘Dirac materials’ share many universal properties and are the subject of intense experimental and theoretical research [11]. Similarly to electron–positron pair production in particle physics, it has been studied how pair creation can be enhanced in Dirac materials [12].

All mechanisms for the enhancement of pair production, in particle physics as well as condensed matter physics, have to overcome the limitations rooted in the quantum adiabatic theorem for Dirac dynamics [13–16]. In this analysis we are interested in the reverse problem. Imagine, e.g., a situation in which a Dirac material is subjected to an externally controlled electric field. For finite-time variation of such a field, electron–hole pairs are created. For instance, in superconductors these electron–hole pairs can be described as quasiparticles, which are normal conducting and experience Joule heating. Such nonequilibrium excitations are often undesirable, but they are an inevitable consequence of finite-time driving. In this case one is interested in particular finite-time processes that mimic adiabatic evolution—shortcuts to adiabaticity. In recent years a great deal of theoretical and experimental research has been dedicated to the design of shortcuts to adiabaticity for Schrödinger dynamics [17]. To this end a variety of techniques have been developed: the use of dynamical invariants [18], the inversion of scaling laws [19], the fast-forward technique [20–25], transitionless quantum driving [26–29], optimal protocols from optimal control theory [30–32], optimal driving from properties of the

quantum work statistics [33], ‘environment’-assisted methods [34], use of the properties of Lie algebras [35], and approximate methods, such as linear response theory [36] and fast quasistatic dynamics [37].

Among these methods, the fast-forward technique stands out. Within this approach one determines an auxiliary potential such that the time-dependent wave function becomes identical to a ‘target state’ at the end of the finite-time process [23]. For intermediate times, however, the time-dependent state is allowed to stray arbitrarily far from the adiabatic manifold. This is in stark contrast to, for instance, counterdiabatic driving, for which the time-dependent solution remains on the adiabatic manifold of the ‘unperturbed’ Hamiltonian at all times [28]. However, the counterdiabatic fields tend to be highly complicated and highly non-local [29, 32], whereas the auxiliary fast-forward potential is more tractable [24]. Therefore, generalizing the fast-forward technique to facilitate shortcuts to adiabaticity for Dirac dynamics appears to be more promising than, for instance, counterdiabatic driving. Generally, one would expect optimal control techniques to require even fewer resources than the fast-forward technique. However, solving optimal control problems can become rather mathematically involved [38]. Thus, this work focuses on a mathematically rather ‘simple’, yet effective method—the fast forward technique.

In the following we briefly review the main properties of Dirac dynamics in section 2, before we develop the fast-forward technique for Dirac particles in time-dependent fields in section 3. We will see that shortcuts to adiabaticity for Dirac dynamics are facilitated by pseudoscalar potentials, which is illustrated for two analytically solvable examples in section 4.

2. Relativistic quantum mechanics: the Dirac equation

We start by briefly reviewing the main properties of the Dirac dynamics. In its original formulation for free particles, the Dirac equation reads [1, 2]

$$i\hbar \dot{\Psi}(\mathbf{x}, t) = (-i\hbar c \boldsymbol{\alpha} \cdot \nabla + \alpha_0 mc^2) \Psi(\mathbf{x}, t). \quad (1)$$

Here, $\Psi(\mathbf{x}, t)$ is the wave function of an electron with rest mass m at position $\mathbf{x} = (x_1, x_2, x_3)$, and c is the speed of light. In covariant form the matrices $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and α_0 can be expressed as [2, 3],

$$\alpha^0 = \gamma^0 \quad \text{and} \quad \gamma^0 \alpha^k = \gamma^k. \quad (2)$$

The γ -matrices are commonly written in terms of 2×2 sub-matrices with the Pauli-matrices $\sigma_x, \sigma_y, \sigma_z$ and the identity \mathbb{I}_2 as,

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}. \quad (3)$$

The solution of equation (1), the Dirac wave function $\Psi(\mathbf{p}, t)$, is a bispinor, which can be interpreted as a superposition of a spin-up electron, a spin-down electron, a spin-up positron, and a spin-down positron [2, 3].¹

We are now interested in situations in which Dirac particles are driven by a time-dependent vector field $\mathbf{A}(\mathbf{x}, t)$. Note that only considering situations with a time-dependent vector field but no scalar potential is not a loss of generality, since electric and magnetic fields, $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$, are gauge invariant. In particular, the physical fields are given by

$$\mathbf{E}(\mathbf{x}, t) = -\nabla V(\mathbf{x}, t) - \partial_t \mathbf{A}(\mathbf{x}, t) \quad \text{and} \quad \mathbf{B}(\mathbf{x}, t) = \nabla \times \mathbf{A}(\mathbf{x}, t). \quad (4)$$

Furthermore, for the sake of simplicity we will restrict ourselves in the following analysis to a one-dimensional system in the x -direction. In this case the four-component Dirac spinor can be separated into two identical two-component bispinors, and we choose a representation in which the $(1 + 1)$ -dimensional Dirac equation reads [39, 40],

$$i\hbar \dot{\Psi}(x, t) = \left[(-i\hbar c \partial_x + A(x, t)) \sigma_x + mc^2 \sigma_z \right] \Psi(x, t). \quad (5)$$

For time-dependent but spatially homogeneous vector fields, this situation was solved analytically recently for oscillating $A(t)$ in [5] and for linear and exponential parametrization in [10]. In addition, equation (5) describes particle–antiparticle production in counterpropagating laser light, which was proposed to be observable in an experiment [5].

¹ The momentum of Dirac particles is confined by the light cone, whereas Schrödinger particles can travel with arbitrary velocities. This has interesting consequences for the quantum work statistics [10].

3. Fast-forward technique

In the following we generalize the fast-forward technique to facilitate shortcuts to adiabaticity for the time-dependent Dirac equation (5). We begin by reviewing concepts and establishing notation for the corresponding Schrödinger case.

3.1. Schrödinger dynamics

In the paradigm of the fast-forward technique [20, 21, 23–25] we are interested in finding a shortcut for the unperturbed time-dependent dynamics,

$$i\hbar \partial_t \psi_0(x, t) = \frac{1}{2m} \left(-i\hbar \partial_x + \frac{1}{c} A(x, \alpha_t) \right)^2 \psi_0(x, t), \quad (6)$$

where $\psi_0(x, t)$ denotes the unperturbed, non-adiabatic solution and the vector field is controlled externally according to some protocol α_t . Now, we consider an ansatz of a time-dependent wave function,

$$\psi(x, t) = \exp\left(-\frac{i}{\hbar} \int_0^t ds \epsilon(\alpha_s)\right) \exp(i f(x, t)) \phi(x, \alpha_t), \quad (7)$$

where $\epsilon(\alpha_t)$ is the instantaneous eigenenergy corresponding to an instantaneous eigenstate $\phi(x, \alpha_t)$,

$$\epsilon(\alpha_t) \phi(x, \alpha_t) = \frac{1}{2m} \left(-i\hbar \partial_x + \frac{1}{c} A(x, \alpha_t) \right)^2 \phi(x, \alpha_t). \quad (8)$$

The task is to determine the phase $f(x, t)$ and the auxiliary scalar potential $V(x, t)$ such that $\psi(x, t)$ (7) is a solution of

$$i\hbar \partial_t \psi(x, t) = \frac{1}{2m} \left(-i\hbar \partial_x + \frac{1}{c} A(x, \alpha_t) \right)^2 \psi(x, t) + V(x, t) \psi(x, t). \quad (9)$$

A shortcut to adiabaticity during finite time τ is achieved if we additionally have $\psi(x, 0) = \phi(x, \alpha_0)$ and $\psi(x, \tau) = \exp\left(-i/\hbar \int_0^\tau ds \epsilon(\alpha_s)\right) \phi(x, \alpha_\tau)$, that is $f(x, 0) = 0$ and $f(x, \tau) = 0$. Therefore, $\psi(x, \tau)$ becomes identical to the final state of the adiabatically driving equation (6), which means that $\psi(x, \tau)$ is given by the instantaneous eigenstate times a space-independent phase.

For the following analysis it will prove convenient to express the instantaneous eigenstate in polar representation,

$$\phi(x, \alpha_t) = \beta(x, \alpha_t) \exp(i \gamma(x, \alpha_t)) \quad (10)$$

with real amplitude $\beta(x, \alpha_t)$ and phase $\gamma(x, \alpha_t)$. Substituting the ansatz (7) into the modified Schrödinger equation (9) we obtain after a few lines

$$\begin{aligned} V(x, t) = & -\hbar \partial_t f(x, t) - \hbar \partial_t \gamma(x, \alpha_t) + \frac{\hbar}{2mc} A(x, \alpha_t) \partial_x f(x, t) \\ & + \frac{\hbar^2}{2m} [\partial_x f(x, t)]^2 + \frac{\hbar^2}{m} \partial_x f(x, t) \partial_x \gamma(x, \alpha_t). \end{aligned} \quad (11)$$

The time-dependent phase $f(x, t)$ is determined by the second-order differential equation,

$$\hbar \beta(x, \alpha_t) \partial_x^2 f(x, t) + 2\hbar \partial_x \beta(x, \alpha_t) \partial_x f(x, t) - 2m \partial_t \beta(x, \alpha_t) = 0. \quad (12)$$

It is then easy to see that a shortcut to adiabaticity is obtained for all parametrizations with $\dot{\alpha}_0 = 0$ and $\dot{\alpha}_\tau = 0$ [24], where the dot is a short notation for a derivative with respect to time.

It is interesting to note that the auxiliary potential $V(x, t)$ explicitly depends on the eigenstate $\phi(x, \alpha_t)$. This is in stark contrast to counterdiabatic driving, for which the counterdiabatic field constitutes a shortcut to adiabaticity for all eigenstates [29].

3.2. Dirac dynamics

In complete analogy to the previous case of Schrödinger dynamics (6), we are now interested in constructing a shortcut to adiabaticity for the time-dependent Dirac problem,

$$i\hbar \partial_t \Psi_0(x, t) = \left[\left(-i\hbar c \partial_x + A(x, \alpha_t) \right) \sigma_x + mc^2 \sigma_z \right] \Psi_0(x, t), \quad (13)$$

where the vector field, $A(x, \alpha_t)$, is again parametrized by a control parameter α_t . As before (7), we consider an ansatz

$$\Psi(x, t) = \exp\left(-\frac{i}{\hbar} \int_0^t ds \epsilon(\alpha_s)\right) \exp(i f(x, t)) \Phi(x, \alpha_t), \quad (14)$$

where $\Phi(x, \alpha_t) = (\phi_1(x, \alpha_t), \phi_2(x, \alpha_t))$ is an eigenspinor with

$$\epsilon(\alpha_t) \Phi(x, \alpha_t) = \left[(-i\hbar \partial_x + A(x, \alpha_t)) \sigma_x + mc^2 \sigma_z \right] \Phi(x, \alpha_t). \quad (15)$$

In complete analogy to the Schrödinger case, the task is now to determine phase, $f(x, t)$, and an auxiliary potential matrix, $\mathbb{V}(x, t)$, such that $\Psi(x, t)$ (14) is a solution of

$$i\hbar \partial_t \Psi(x, t) = \left[(-i\hbar \partial_x + A(x, \alpha_t)) \sigma_x + mc^2 \sigma_z + \mathbb{V}(x, t) \right] \Psi(x, t). \quad (16)$$

In the most general case the Lorentz invariant, Hermitian potential matrix, $\mathbb{V}(x, t)$, can be written as [2, 40],

$$\mathbb{V}(x, t) = \mathbb{I}_2 V_t(x, t) + \sigma_x V_e(x, t) + \sigma_y V_p(x, t) + \sigma_z V_s(x, t). \quad (17)$$

Here, V_t and V_e are the time and space components of the two-vector potential, V_s is the scalar potential, and V_p denotes the pseudoscalar potential [2, 40–43]. The existence of pseudoscalar potentials is an important consequence of Dirac dynamics [2], and such potentials can be obtained by means of the ‘inverse scattering method’ [44]. Physically, pseudoscalar potentials are induced by circularly polarized fields [45].

As before, we work in a gauge such that $V_e(x, t) \equiv 0$, i.e., the space component of the vector potential is fully determined by the unperturbed problem (13). Substituting the ansatz (14) into the modified evolution equation (16) and rearranging the terms we obtain from the first component,

$$\begin{aligned} V_t(x, t) \phi_1(x, \alpha_t) = & -V_s(x, t) \phi_1(x, \alpha_t) + iV_p(x, t) \phi_2(x, \alpha_t) - \hbar \phi_1(x, \alpha_t) \partial_t f(x, t) \\ & - c\hbar \phi_2(x, \alpha_t) \partial_x f(x, t) + i\hbar \partial_t \phi_1(x, \alpha_t), \end{aligned} \quad (18)$$

and from the second component

$$\begin{aligned} V_t(x, t) \phi_2(x, \alpha_t) = & V_s(x, t) \phi_2(x, \alpha_t) - iV_p(x, t) \phi_1(x, \alpha_t) - \hbar \phi_2(x, \alpha_t) \partial_t f(x, t) \\ & - c\hbar \phi_1(x, \alpha_t) \partial_x f(x, t) + i\hbar \partial_t \phi_2(x, \alpha_t). \end{aligned} \quad (19)$$

Multiplying equation (18) by $\phi_2(x, \alpha_t)$ and equation (19) by $\phi_1(x, \alpha_t)$, and subtracting the resulting equations, we have,

$$\begin{aligned} c\hbar \left(\phi_1^2(x, \alpha_t) - \phi_2^2(x, \alpha_t) \right) \partial_x f(x, t) = & 2V_s(x, t) \phi_1(x, \alpha_t) \phi_2(x, \alpha_t) \\ & - iV_p(x, t) \left(\phi_1^2(x, \alpha_t) + \phi_2^2(x, \alpha_t) \right) \\ & - i\hbar \left(\phi_2(x, \alpha_t) \partial_t \phi_1(x, \alpha_t) - \phi_1(x, \alpha_t) \partial_t \phi_2(x, \alpha_t) \right) \end{aligned} \quad (20)$$

Finally, noting that $V_s(x, t), V_p(x, t) \in \mathbb{R}$ and $f(x, t) \in \mathbb{R}$, equations (18)–(20) allow us to fully determine the auxiliary potential matrix $\mathbb{V}(x, t)$ and the phase $f(x, t)$.

Separating the real and imaginary parts in equation (20) is a simple exercise, but the resulting expressions are rather lengthy. Therefore, we will continue in the next section with two illustrative and analytically solvable examples.

4. Illustrative examples

4.1. Spatially homogeneous electric field

We start with the simplest possible example. Imagine charged particles driven by a time-dependent but spatially homogeneous vector field and we simply have

$$A(x, \alpha_t) = \alpha_t. \quad (21)$$

This situation was analyzed in detail recently in the context of pair production [5] and the quantum work distribution [10].

4.1.1. Schrödinger dynamics

To build intuition and as a point of reference we treat the corresponding Schrödinger case first. The unperturbed but driven dynamics (6) becomes,

$$i\hbar \partial_t \psi_0(x, t) = \frac{1}{2m} \left(-i\hbar \partial_x + \frac{\alpha_t}{c} \right)^2 \psi_0(x, t). \quad (22)$$

It is then easy to see that the instantaneous eigenstates are given by [10]

$$\phi(x, \alpha_t) = \exp \left(\frac{i}{\hbar} \left(\hbar\kappa + \frac{\alpha_t}{c} \right) x \right), \quad (23)$$

with the corresponding eigenenergies $\epsilon_\kappa(\alpha) = (\hbar\kappa + \alpha/c)^2/2m$, while κ denotes the quantum number. Comparing equation (23) with equation (10) we identify $\beta(x, \alpha_t) = 1$ and $\gamma(x, \alpha_t) = (c\hbar\kappa + \alpha_t)x/\hbar c$. Then, the determining equation for the phase $f(x, t)$ becomes, $\partial_x^2 f(x, t) = 0$, for which a solution is given by

$$f(x, t) = a(t) + b(t)x. \quad (24)$$

Substituting the instantaneous eigenstate (23) and equation (24) into equation (11), we obtain

$$V(x, t) = -\hbar \left(\dot{a}(t) + \dot{b}(t)x \right) - \frac{\dot{\alpha}_t}{c} x + \frac{\hbar \alpha_t b(t)}{2mc} + \frac{\hbar^2 b(t)^2}{2m} + \frac{\hbar b(t)}{m} \left(\hbar\kappa + \frac{\alpha_t}{c} \right), \quad (25)$$

where the dot is again a short notation for a derivative with respect to time. Choosing

$$a(t) = 0 \quad \text{and} \quad b(t) = 0 \quad (26)$$

the phase and auxiliary potential simplify to read,

$$f(x, t) = 0 \quad \text{and} \quad V(x, t) = -\frac{\dot{\alpha}_t}{c} x. \quad (27)$$

From equation (27) it is obvious that for $\dot{\alpha}_0 = 0$ and $\dot{\alpha}_\tau = 0$ the auxiliary potential $V(x, t)$ realizes the desired shortcut. It is interesting to note that the phase $f(x, t)$ vanishes. This, however, is to be expected as the solution of the unperturbed problem (22) is fully determined by a time-dependent phase [10].

4.1.2. Dirac dynamics

The solution of the analogous, time-dependent Dirac problem is mathematically more involved [10]. In general, a solution to the unperturbed equation,

$$i\hbar \partial_t \Psi_0(x, t) = \left[(-i\hbar c \partial_x + \alpha_t) \sigma_x + mc^2 \sigma_z \right] \Psi_0(x, t), \quad (28)$$

can only be written in terms of special functions [5, 10]. Nevertheless, we will see shortly that finding a shortcut to adiabaticity is a straightforward exercise.

The instantaneous eigenspinor reads [10]

$$\Phi(x, \alpha_t) = \frac{\exp \left(i/\hbar \left(\hbar\kappa + \alpha_t/c \right) x \right)}{\sqrt{1 + \left(\sqrt{\Pi_t^2 + 1} - \Pi_t \right)^2}} \left(\sqrt{\Pi_t^2 + 1} - \Pi_t \right), \quad (29)$$

with $\Pi_t = (c\hbar\kappa + \alpha_t)/mc^2$, and the eigenenergies are $\epsilon_\kappa(\alpha) = \pm \sqrt{(c\hbar\kappa + \alpha_t)^2 + (mc^2)^2}$. Therefore, equation (20) becomes

$$\frac{2}{\sqrt{1 + \Pi_t^2}} \left(c\hbar \Pi_t \partial_x f(x, t) - V_s(x, t) \right) + i \left(2 V_p(x, t) + \frac{\hbar \dot{\Pi}_t}{1 + \Pi_t^2} \right) = 0 \quad (30)$$

which is already separated into real and imaginary parts. Hence, the pseudoscalar potential is determined to read

$$V_p(x, t) = -\frac{\hbar \dot{\Pi}_t}{2 + 2 \Pi_t^2} \quad (31)$$

which vanishes for all parametrizations with $\dot{\alpha}_0 = 0$ and $\dot{\alpha}_\tau = 0$, whereas the scalar potential is given by

$$V_s(x, t) = c\hbar \Pi_t \partial_x f(x, t). \quad (32)$$

Substituting equations (29), (31), and (32) into equation (18), we obtain

$$V_t(x, t) = -mc \dot{\Pi}_t x - \hbar \partial_t f(x, t) - \hbar c \sqrt{1 + \Pi_t^2} \partial_x f(x, t). \quad (33)$$

Now, choosing in complete analogy to the Schrödinger case the space-dependent phase to vanish, $f(x, t) \equiv 0$, we have $f(x, 0) = 0$ and $f(x, \tau) = 0$, and $V_s(x, t) \equiv 0$. In conclusion, a shortcut to adiabaticity for the driven Dirac equation (28) is facilitated by the auxiliary potential matrix,

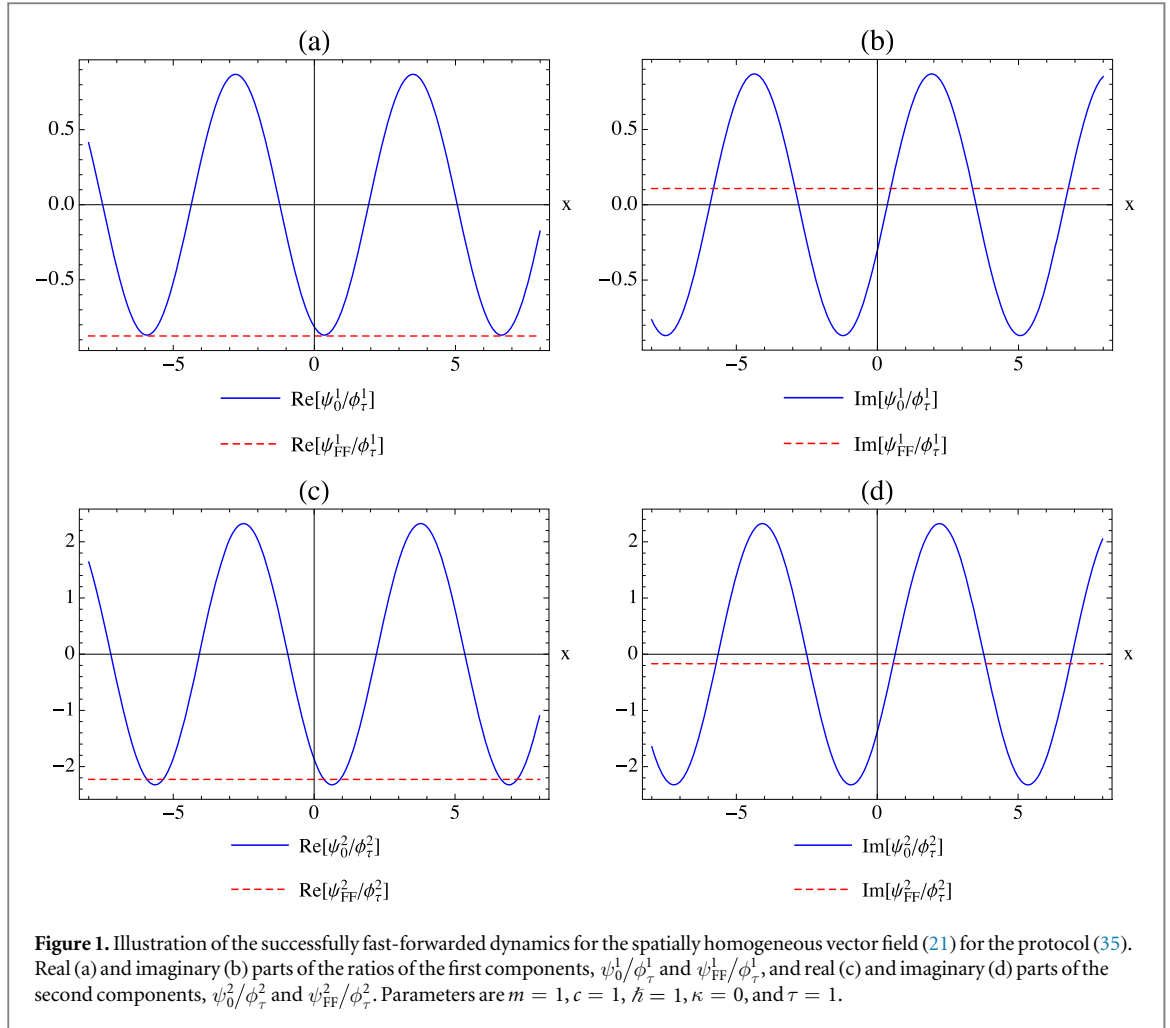


Figure 1. Illustration of the successfully fast-forwarded dynamics for the spatially homogeneous vector field (21) for the protocol (35). Real (a) and imaginary (b) parts of the ratios of the first components, ψ_0^1/ϕ_τ^1 and ψ_{FF}^1/ϕ_τ^1 , and real (c) and imaginary (d) parts of the second components, ψ_0^2/ϕ_τ^2 and ψ_{FF}^2/ϕ_τ^2 . Parameters are $m = 1$, $c = 1$, $\hbar = 1$, $\kappa = 0$, and $\tau = 1$.

$$\mathbb{V}(x, t) = -mc \dot{\Pi}_t x \mathbb{I}_2 - \frac{\hbar \dot{\Pi}_t}{2 + 2 \Pi_t^2} \sigma_y = -\frac{\dot{\alpha}_t}{c} x \mathbb{I}_2 - \frac{\hbar \dot{\alpha}_t}{2(\alpha_t + c \hbar \kappa + mc^2)} \sigma_y \quad (34)$$

for all parametrizations with $\dot{\alpha}_0 = 0$ and $\dot{\alpha}_\tau = 0$. It is interesting to note that the $V_t(x, t)$ -component is identical to the auxiliary potential of the Schrödinger case (27), and the pseudoscalar potential $V_p(x, t)$ ensures the suppression of pair production at the final state.

Numerical verification and illustration. Our analytical results (34) can be easily verified. To this end, we solve the unperturbed but driven Dirac equation (28), as well as the fast-forwarded dynamics (16) with the auxiliary potential (34) numerically. The vector field (21) is parametrized by the protocol,

$$\alpha_t = [\sin(\pi t/2\tau)]^2 + 1 \quad (35)$$

which fulfills $\dot{\alpha}_0 = 0$ and $\dot{\alpha}_\tau = 0$. We further introduce the notation for the final state of the unperturbed dynamics (28), $\Psi_0(x, \tau) = (\psi_0^1, \psi_0^2)$, for the fast-forwarded state (16), $\Psi(x, \tau) = (\psi_{FF}^1, \psi_{FF}^2)$, and for the instantaneous eigenstate, $\Phi(x, \alpha_\tau) = (\phi_\tau^1, \phi_\tau^2)$.

In figure 1 we plot the real and imaginary parts of the ratios ψ_0^1/ϕ_τ^1 , ψ_0^2/ϕ_τ^2 , ψ_{FF}^1/ϕ_τ^1 , and ψ_{FF}^2/ϕ_τ^2 . We observe that these ratios are independent of position x for the fast-forwarded state, whereas the solution of the unperturbed dynamics (28) yields a strong dependence on x . This is in full agreement with the analytical treatment, for which we expect the fast-forwarded state $\Psi(x, \tau)$ to be identical to an eigenspinor up to a space-independent phase.

4.2. Linear electric field

As a second example we consider charged particles driven by a linear electric field. The electromagnetic vector field is then given by,

$$A(x, \alpha_t) = \alpha_t x. \quad (36)$$

As before, we will first solve the Schrödinger case and then analyze the fast-forwarded Dirac dynamics.

4.2.1. Schrödinger dynamics

The unperturbed but driven Schrödinger equation (6) becomes for equation (36),

$$i\hbar \partial_t \psi_0(x, t) = \frac{1}{2m} \left(-i\hbar \partial_x + \frac{\alpha_t}{c} x \right)^2 \psi_0(x, t). \quad (37)$$

It is easy to see that an instantaneous solution of equation (37) can be written as,

$$\phi(x, \alpha_t) = \exp \left(-\frac{i}{\hbar} \left(\frac{\alpha_t}{2c} x^2 - \hbar \kappa x \right) \right) \quad (38)$$

with the instantaneous eigenenergies $\epsilon_\kappa = (\hbar \kappa)^2 / 2m$. Accordingly, we identify $\beta(x, \alpha_t) = 1$ and $\gamma(x, \alpha_t) = -\alpha_t x^2 / 2\hbar c + \kappa x$. Therefore, the differential equation for the phase $f(x, t)$ again simplifies to $\partial_x^2 f(x, t) = 0$, for which we write the solution as

$$f(x, t) = a(t) + b(t)x. \quad (39)$$

Substituting equations (37) and (27) into equation (11) we have,

$$V(x, t) = \frac{\dot{\alpha}_t}{2c} x^2 - \left(\frac{\hbar}{2mc} b(t) \alpha_t + \hbar \dot{b}(t) \right) x + \frac{\hbar^2 \kappa}{m} b(t) + \frac{\hbar^2}{2m} b(t)^2 - \hbar \dot{a}(t). \quad (40)$$

In order to have a shortcut, we have to demand that phase, $f(x, t)$ and auxiliary potential $V(x, t)$ vanish for $t = 0$ and $t = \tau$. This requirement is met for the particular choice $a(t) = 0$ and $b(t) = 0$, and we have

$$V(x, t) = \frac{\dot{\alpha}_t}{2c} x^2 \quad (41)$$

which facilitates a shortcut to adiabaticity for all protocols with $\dot{\alpha}_0 = 0$ and $\dot{\alpha}_\tau = 0$.

4.2.2. Dirac dynamics

As a final example we now consider the corresponding Dirac problem. The driven Dirac equation (13) becomes

$$i\hbar \partial_t \Psi_0(x, t) = \left[(-i\hbar c \partial_x + \alpha_t x) \sigma_x + mc^2 \sigma_z \right] \Psi_0(x, t), \quad (42)$$

for which the instantaneous eigenstates are given by

$$\Phi(x, \alpha_t) = \frac{\exp \left(-i/\hbar \left(\alpha_t x^2 / 2c - \hbar \kappa x \right) \right)}{\sqrt{1 + \left(\sqrt{\Lambda^2 + 1} - \Lambda \right)^2}} \left(\frac{1}{\sqrt{\Lambda^2 + 1} - \Lambda} \right), \quad (43)$$

where $\Lambda = \hbar \kappa / mc$. In appendix A we summarize how to determine $\Phi(x, \alpha_t)$, and for the three-dimensional case the derivation can be found in [46]. Accordingly, equation (20) simplifies to

$$\frac{1}{\sqrt{\Lambda^2 + 1}} \left(\hbar c \Lambda \partial_x f(x, t) - V_s(x, t) \right) + iV_p(x, t) = 0, \quad (44)$$

from which we determine $V_p(x, t) \equiv 0$ and $V_s(x, t) = \hbar c \Lambda \partial_x f(x, t)$. Thus, $V_t(x, t)$ can be written as

$$V_t(x, t) = \frac{\dot{\alpha}_t}{2c} x^2 - \hbar \partial_t f(x, t) - \hbar c \sqrt{1 + \Lambda^2} \partial_x f(x, t). \quad (45)$$

Again choosing $f(x, t) \equiv 0$, a shortcut to adiabaticity is induced by the auxiliary potential matrix

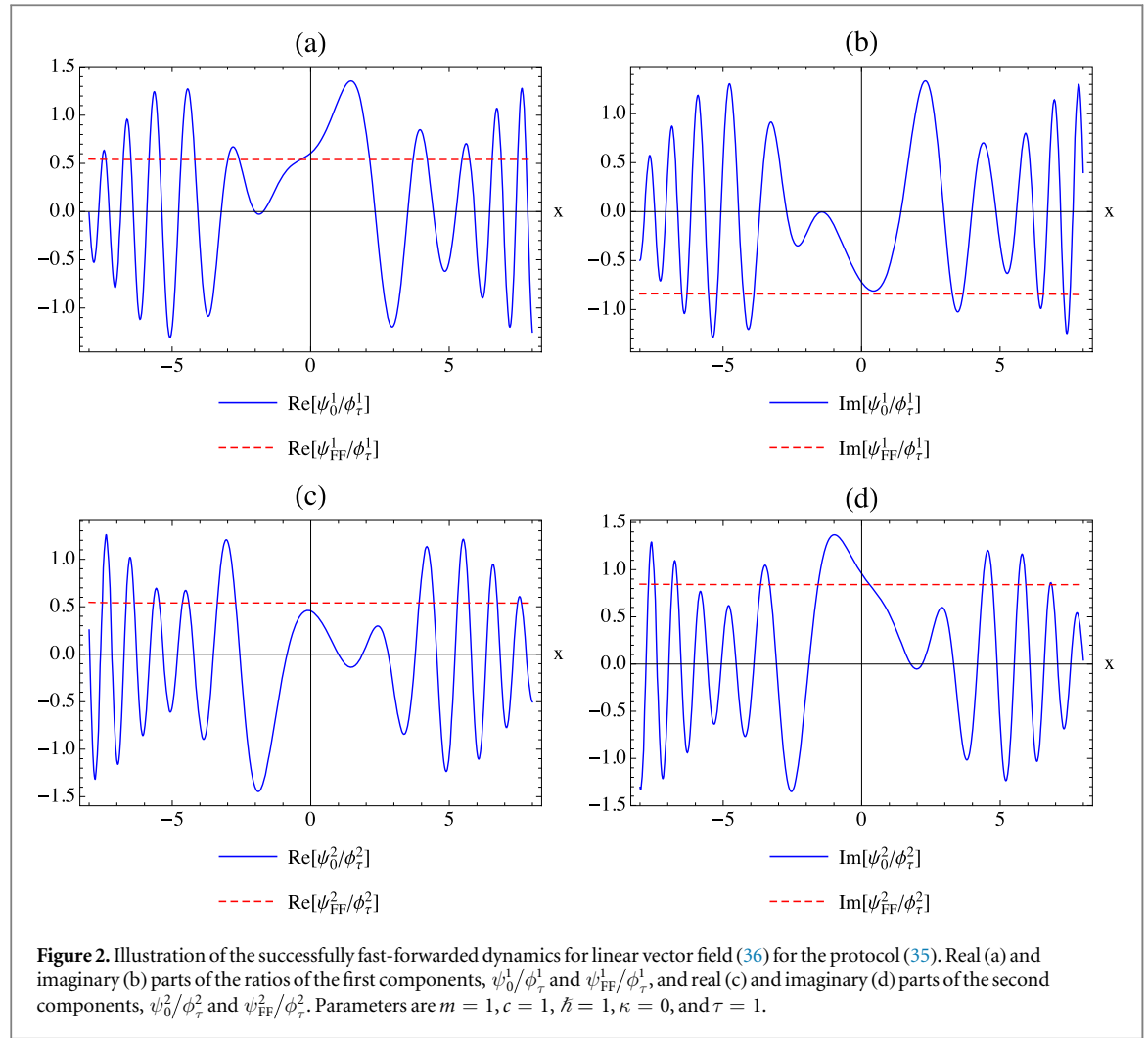
$$\mathbb{V}(x, t) = \frac{\dot{\alpha}_t}{2c} x^2 \mathbb{I}_2 \quad (46)$$

for all protocols with $\dot{\alpha}_0 = 0$ and $\dot{\alpha}_\tau = 0$. Note that in the case of a linear electric field (36) the auxiliary potentials for Schrödinger dynamics (41) and for Dirac dynamics (45) are identical and no pseudoscalar potential is necessary to suppress pair creation at the final state.

Numerical verification and illustration. In figure 2 we again plot the real and imaginary parts of the ratios ψ_0^1/ϕ_τ^1 , ψ_0^2/ϕ_τ^2 , $\psi_{\text{FF}}^1/\phi_\tau^1$, and $\psi_{\text{FF}}^2/\phi_\tau^2$ for the sinusoidal protocol (35). As in figure 1, we observe that the fast-forwarded dynamics only result in a space-independent phase, whereas the final state for unperturbed equation (42) lies far from the instantaneous eigenspinor.

5. Concluding remarks

In this work we have generalized the fast-forward technique to shortcuts to adiabaticity for Dirac dynamics. For the Dirac equation such a shortcut not only suppresses parasitic nonequilibrium excitations, but also hinders the production of pairs. Therefore, our results could be applied in particle physics for problems where one is



interested in the loss-free transport of charged particles, as well as in condensed matter physics where one is interested in the dissipationless control of Dirac materials. As an illustration of the technique we worked out two analytically solvable examples, namely charged particles in spatially homogeneous and linear fields. We found that the scalar component of the auxiliary potential matrix is identical to the auxiliary potential in the analogous Schrödinger problem. However, we have also seen that in order to facilitate a shortcut in Dirac dynamics pseudoscalar potentials are important.

As a final observation, we remark that in all the examples we were able to choose the fast-forward phase to vanish at all times. Hence, for the examples the auxiliary potential matrix was identical to the counterdiabatic field [29], and the fast-forwarded Dirac spinor did not stray from the adiabatic manifold of the unperturbed Hamiltonian.

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Appendix A. Dirac particles in linear electric field

Separated into its components, the instantaneous eigenvalue problem for the driven Dirac equation (42) reads

$$\begin{aligned} (\epsilon - mc^2)\phi_1(x, \alpha) &= -i\hbar c \partial_x \phi_2(x, \alpha) + \alpha x \phi_2(x, \alpha) \\ (\epsilon + mc^2)\phi_2(x, \alpha) &= -i\hbar c \partial_x \phi_1(x, \alpha) + \alpha x \phi_1(x, \alpha). \end{aligned} \quad (\text{A.1})$$

Multiplying the first equation by $(\epsilon + mc^2)$ and substituting the second one into the first, we obtain

$$(\epsilon^2 - m^2c^4)\phi_1(x, \alpha) = (-i\hbar c \partial_x + \alpha x)^2 \phi_1(x, \alpha), \quad (\text{A.2})$$

which is similar to the corresponding Schrödinger equation (37). Accordingly, a solution is given by

$$\phi_1(x, \alpha) = \exp\left(-\frac{i}{\hbar}\left(\frac{\alpha_t}{2c}x^2 - \hbar\kappa x\right)\right) \quad (\text{A.3})$$

for which the eigenenergies read

$$\epsilon_\kappa(\alpha) = \pm \sqrt{(\hbar\kappa c)^2 + (mc^2)^2}. \quad (\text{A.4})$$

Re-substituting equation (A.4) into equation (A.1), we obtain the second component as

$$\phi_2(x, \alpha) = \frac{\hbar\kappa c}{\sqrt{(\hbar\kappa c)^2 + (mc^2)^2} + mc^2} \exp\left(-\frac{i}{\hbar}\left(\frac{\alpha_t}{2c}x^2 - \hbar\kappa x\right)\right). \quad (\text{A.5})$$

Combining $\phi_1(x, \alpha)$ and $\phi_2(x, \alpha)$, and normalizing in momentum space we have the instantaneous eigenspinors (43) (see also [10, 46]).

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