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Semigroup Well-posedness of A Linearized, Compressible Fluid with An Elastic Boundary ^{*}

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Abstract

We address semigroup well-posedness of the fluid-structure interaction of a linearized compressible, viscous fluid and an elastic plate (in the absence of rotational inertia). Unlike existing work in the literature, we linearize the compressible Navier-Stokes equations about an arbitrary state (assuming the fluid is barotropic), and so the fluid PDE component of the interaction will generally include a nontrivial ambient flow profile \mathbf{U} . The appearance of this term introduces new challenges at the level of the stationary problem. In addition, the boundary of the fluid domain is unavoidably Lipschitz, and so the well-posedness argument takes into account the technical issues associated with obtaining necessary boundary trace and elliptic regularity estimates. Much of the previous work on flow-plate models was done via Galerkin-type constructions after obtaining good a priori estimates on solutions (specifically [18]—the work most pertinent to ours here); in contrast, we adopt here a Lumer-Phillips approach, with a view of associating solutions of the fluid-structure dynamics with a C_0 -semigroup $\{e^{\mathcal{A}t}\}_{t \geq 0}$ on the natural finite energy space of initial data. So, given this approach, the major challenge in our work becomes establishing of the maximality of the operator \mathcal{A} which models the fluid-structure dynamics. In sum: our main result is semigroup well-posedness for the fully coupled fluid-structure dynamics, under the assumption that the ambient flow field $\mathbf{U} \in \mathbf{H}^3(\mathcal{O})$ has zero normal component trace on the boundary (a standard assumption with respect to the literature). In the final sections we address well-posedness of the system in the presence of the von Karman plate nonlinearity, as well as the stationary problem associated with the dynamics.

Keywords: fluid-structure interaction, compressible fluid, well-posedness, semigroup

AMS Mathematics Subject Classification 2010: 34A12, 74F10, 35Q35, 76N10

In memory of Igor D. Chueshov

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1 Introduction

In this work, we consider a linearized fluid-structure model with respect to some reference state, including an arbitrary spatial vector field. The coupled system here describes the interaction between a plate and a flow of *compressible* barotropic, viscous fluid. Such interactive dynamics are crucially considered in the design of many engineering systems (e.g., aircraft, engines, and bridges). The study of compressible flows (gas dynamics) itself has implications to high-speed aircraft, jet engines, rocket motors, hyperloops, high-speed entry into a planetary atmosphere, gas pipelines, commercial applications (such as abrasive blasting), and many other fields (see [10, 11, 31], for instance). In these applications, the density of a gas may change significantly along a streamline. *Compressibility*—i.e., the fractional change in volume of the fluid element per unit change in pressure—becomes important, for instance, in flows for which

$$\text{Mach Number} = M \equiv \frac{\text{velocity}}{\text{local speed of sound}} > 0.3.$$

The cases $M < 0.3$ and $0.3 < M < 0.8$ are subsonic/incompressible and subsonic/compressible regimes, respectively. Compressible flows can be either transonic ($0.8 < M < 1.2$) or supersonic ($1.2 < M < 3.0$). In supersonic flows, pressure effects are only transported downstream; the upstream flow is not affected by conditions downstream.

In the study of incompressible flows, the associated analysis typically involves only two unknowns: pressure and velocity. These are usually found by solving two equations that describe conservation of mass and linear momentum, with the fluid density presumed to be constant. By contrast, in compressible flow, the gas density and temperature are variables. Consequently, the solution of compressible flow problems will require two more equations: namely, an equation of state for the gas, and a conservation of energy equation.¹ Moreover, the imposition of external forces to the governing equations may not immediately result in a uniform flow throughout the system. In particular, the fluid may compress in the vicinity of the applied force; that is to say, the density may increase locally in response to the given force.

The effects due to compressibility and viscosity on an (uncoupled) fluid dynamics will have to be taken in account when subsequently considering the mathematical properties of PDE's describing interactions of said fluid dynamics with some given structure. In aeroelasticity, the compressible gas is often assumed to be inviscid—i.e., viscosity-free—and the flow irrotational (potential flow). These assumptions are often invoked in practice, as they reduce the flow dynamics to a wave equation [25, 31] (and see Section 4 below). However, there are situations where viscous effects cannot be neglected, e.g., in the transonic region [31]. The *mathematical* literature—especially in the last 20 years—on fluid-structure interactions across each of these fluid regimes is quite vast. We will certainly not attempt here a general overview of this literature, but in Section 4 we will provide an in-depth discussion of those key modern references that pertain to the present work. At this point, we mention the primary motivating reference, [18]: in this work, the author considers the dynamics of a nonlinear plate, located on a flat portion of the boundary of a three dimensional cavity, as it interacts with a compressible, barotropic (linearized) fluid that fills the cavity. In the present work, we will analyze a comparable setup, but with additional physical terms in the equations; the focus here will be on establishing and describing the essential semigroup dynamics which drives the coupled PDE model.

¹Throughout, we will assume the fluid is barotropic—the pressure depends only on the density.

Remark 1.1. *The accommodation of physically relevant nonlinearities—i.e., those seen in [18, 26, 23]—can be readily made subsequent to the present analysis, which develops a “good” linear theory. Linearities that are amenable to such treatment include those of Berger, Kirchhoff, or von Karman type, inasmuch as they are locally Lipschitz [22, 41] on the plate’s natural energy space. As a key illustrative example, we include a discussion of the well-posedness of this fluid-structure model in the presence of the von Karman plate nonlinearity in Section 7.*

In many cases, it is the addition of structural nonlinearity which ultimately leads to global-in-time boundedness of corresponding trajectories [36, 42]. Accordingly, long-time behavior of nonlinear dynamics will be considered in a forthcoming work.

We also consider a Lipschitz geometry, as opposed to the common assumption [18, 30] that the domain is smooth. Given the transition between the elastic and inelastic components of the boundary, a Lipschitz boundary is surely more natural and physically relevant—see Figure 1 below—and also more amenable to pertinent generalizations, e.g., tubular domains (finite or infinite) [21, 27]. Distinguishing our work from [18], we take the linearization of the compressible Navier-Stokes equations about a rest state which has a *nonzero ambient flow component*. Since this linearization process produces some additional terms that depend on the ambient flow, previous techniques to obtain the well-posedness result cannot be directly applied². We note that the resultant terms, due to the presence of the ambient flow, **do not represent bounded perturbations of principal spatial operators**. To obtain the primary result we utilize a semigroup approach, invoking the well known Lumer-Phillips theorem [41, p.13].

We believe that the present treatment fits nicely within the context of the recent work of I. Chueshov, where the interactive dynamics between fluid and a plate (or shell) are considered from various points of view [18, 19, 21, 26, 27, 28]. Moreover, one can draw comparisons and contrasts between the well-posedness analysis here and that in [5] and [3], which deal with *incompressible* fluid-structure interactions: in [5] and [3], there is also a two dimensional elastic structure existing on the boundary of a three dimensional domain, in which a fluid evolves. However, the earlier well-posedness work [5] requires an appropriate *mixed variational, Babuška-Brezzi* formulation, which is nonstandard and intrinsic to the particular dynamics under consideration (see e.g., Theorem 3.1.5 of [35]); whereas the present effort combines the Lax-Milgram Theorem with a critical well-posedness result for (static versions of) the pressure PDE component of the fluid-structure system (the first equation in (2.4) and Theorem 9.1 of the Appendix.) For fluid-structure well-posedness studies that involve a three dimensional solid immersed in a three dimensional fluid, and which utilize semigroup techniques, see [8],[9],[6].)

Eventually, we are interested in learning if and how the presence of the dissipating fluid dynamics affects the stability of the structure (as in, e.g., [24, 18, 28, 27],[4]). In particular, for the *linear* compressible fluid-structure interaction PDE model, we are interested in strong/uniform stability properties of the associated C_0 -semigroup. On the other hand, if one inserts nonlinearity into the structural component of the interaction, the existence and nature of global attractors become the primary objects of interest for the associated PDE dynamical system. Qualitative properties of fluid-structure models

²In fact, the late author of [18]—to whom this work is dedicated—suggested in personal correspondence the precise model (2.4)–(2.6); [17]. In this communication, he remarked that the approaches utilized in [18] with $\mathbf{U} \equiv 0$ were not amenable to the problem studied with $\mathbf{U} \neq 0$. He noted that a semigroup approach might be fruitful, and this comment provided an impetus for the present work.

(such as well-posedness and stability of solutions, and the existence of compact global attractors) have been intensely investigated by many authors over the past 30 years. For the PDE model under consideration, (2.4)–(2.6) below, issues of long-time behavior of solutions are addressed in the forthcoming work [7].

The paper is organized as follows: In Section 2 we describe the PDE model and discuss our standing hypotheses. In Section 3 we provide a discussion of the principal dynamics operator (on the natural space of finite energy), as well its domain; we then formally state the semigroup generation result which immediately yields well-posedness of fluid-structure model, in the sense of Hadamard. We also include in this section the notion of *energy balance* for semigroup solutions. Section 4 provides an in-depth discussion of the key pertinent references, and their relationship to the result presented here. Section 5 gives the proof of the main result via the Lumer-Phillips theorem: namely, we establish dissipativity and maximality of a certain bounded perturbation of the modeling fluid-structure operator. Section 6 gives a description of stationary solutions to the dynamics at hand, and proves that the PDE system can be recovered (in some sense) from the stationary variational problem. Section 7 discusses the von Karman plate nonlinearities and the associated nonlinear dynamic well-posedness and stationary results, treating the nonlinearity as a locally-Lipschitz perturbation of the linear dynamics. Lastly, the Appendix provides a proof of a key technical lemma on the well-posedness of solutions to the stationary version of the decoupled pressure PDE component in (2.4)–(2.6) below.

1.1 Notation

For the remainder of the text we write \mathbf{x} for $(x_1, x_2, x_3) \in \mathbb{R}_+^3$ or $(x_1, x_2) \in \Omega \subset \mathbb{R}_{\{(x_1, x_2)\}}^2$, as dictated by context. For a given domain D , its associated $L^2(D)$ will be denoted as $\|\cdot\|_D$ (or simply $\|\cdot\|$ when the context is clear). The symbols \mathbf{n} and $\boldsymbol{\tau}$ will be used to denote, respectively, the unit external normal and tangent vectors to \mathcal{O} . Inner products in $L^2(\mathcal{O})$ or $\mathbf{L}^2(\mathcal{O})$ are written $(\cdot, \cdot)_{\mathcal{O}}$ (or simply (\cdot, \cdot) when the context is clear), while inner products $L^2(\partial\mathcal{O})$ are written $\langle \cdot, \cdot \rangle$. We will also denote pertinent duality pairings as $\langle \cdot, \cdot \rangle_{X \times X'}$, for a given Hilbert space X . The space $H^s(D)$ will denote the Sobolev space of order s , defined on a domain D , and $H_0^s(D)$ denotes the closure of $C_0^\infty(D)$ in the $H^s(D)$ -norm $\|\cdot\|_{H^s(D)}$ or $\|\cdot\|_{s,D}$. We make use of the standard notation for the boundary trace of functions defined on \mathcal{O} , which are sufficiently smooth: i.e., for a scalar function $\phi \in H^s(\mathcal{O})$, $\frac{1}{2} < s < \frac{3}{2}$, $\gamma(\phi) = \phi|_{\partial\mathcal{O}}$, a well-defined and surjective mapping on this range of s , owing to the Sobolev Trace Theorem on Lipschitz domains (see e.g., [40], or Theorem 3.38 of [38]).

2 PDE Model

Let $\mathcal{O} \subset \mathbb{R}^3$ be a *bounded* and *convex* fluid domain (and so has Lipschitz boundary $\partial\mathcal{O}$; see e.g., Corollary 1.2.2.3 of [33]). The boundary decomposes into two pieces \bar{S} and $\bar{\Omega}$ where $\partial\mathcal{O} = \bar{S} \cup \bar{\Omega}$, with $S \cap \Omega = \emptyset$. We consider S to be the solid boundary, with no interactive dynamics, and Ω to be the equilibrium position of the elastic domain, upon which the interactive dynamics takes place. We also assume that: (i) the active component $\Omega \subset \mathbb{R}^2$ is flat, with Lipschitz boundary, and embedded in the

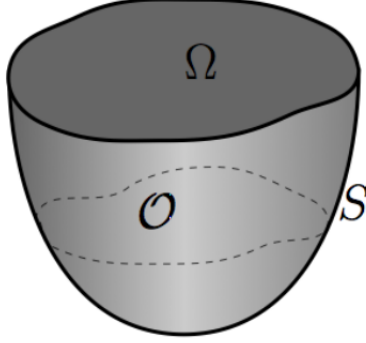


Figure 1: The Fluid-Structure Geometry

$x_1 - x_2$ plane; (ii) the inactive component S lies below the $x_1 - x_2$ plane. This is to say,

$$\Omega \subset \{\mathbf{x} = (x_1, x_2, 0)\} \quad (2.1)$$

$$S \subset \{\mathbf{x} = (x_1, x_2, x_3) : x_3 \leq 0\}. \quad (2.2)$$

Letting $\mathbf{n}(\mathbf{x})$ denote the unit outward normal vector to $\partial\mathcal{O}$, we have $\mathbf{n}|_{\Omega} = (0, 0, 1)$. (See Figure 1.)

We consider the compressible Navier-Stokes system [15], assuming the fluid is barotropic, and linearize the system with respect to some reference rest state of the form $\{p_*, \mathbf{U}, \varrho_*\}$. The pressure and density components p_*, ϱ_* are assumed to be scalar constants, and the arbitrary ambient flow field $\mathbf{U} : \mathcal{O} \rightarrow \mathbb{R}^3$ is given by:

$$\mathbf{U}(x_1, x_2, x_3) = [U_1(x_1, x_2, x_3), U_2(x_1, x_2, x_3), U_3(x_1, x_2, x_3)]. \quad (2.3)$$

Deleting non-critical lower order terms (see Remark 2.4 below), and setting the pressure and density reference constants equal to unity, we obtain the following *perturbation equations*:

$$\begin{cases} p_t + \mathbf{U} \cdot \nabla p + \operatorname{div} \mathbf{u} = 0 & \text{in } \mathcal{O} \times (0, \infty) \\ \mathbf{u}_t + \mathbf{U} \cdot \nabla \mathbf{u} - \operatorname{div} \sigma(\mathbf{u}) + \eta \mathbf{u} + \nabla p = 0 & \text{in } \mathcal{O} \times (0, \infty) \\ (\sigma(\mathbf{u})\mathbf{n} - p\mathbf{n}) \cdot \boldsymbol{\tau} = 0 & \text{on } \partial\mathcal{O} \times (0, \infty) \\ \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } S \times (0, \infty) \\ \mathbf{u} \cdot \mathbf{n} = w_t & \text{on } \Omega \times (0, \infty) \end{cases} \quad (2.4)$$

$$\begin{cases} w_{tt} + \Delta^2 w + [2\nu \partial_{x_3}(\mathbf{u})_3 + \lambda \operatorname{div}(\mathbf{u}) - p]_{\Omega} = 0 & \text{on } \Omega \times (0, \infty) \\ w = \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty) \end{cases} \quad (2.5)$$

$$[p(0), \mathbf{u}(0), w(0), w_t(0)] = [p_0, \mathbf{u}_0, w_0, w_1]. \quad (2.6)$$

Here, $p(t) : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $\mathbf{u}(t) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ (pointwise in time) are given as the pressure and the fluid velocity field, respectively. The quantity $\eta > 0$ represents a drag force of the domain on the viscous

fluid. In addition, the quantity $\boldsymbol{\tau}$ in (2.4) is in the space $TH^{1/2}(\partial\mathcal{O})$ of tangential vector fields of Sobolev index $1/2$; that is,

$$\boldsymbol{\tau} \in TH^{1/2}(\partial\mathcal{O}) = \{\mathbf{v} \in \mathbf{H}^{\frac{1}{2}}(\partial\mathcal{O}) : \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{O}\}^3. \quad (2.7)$$

With respect to the “ambient flow” field \mathbf{U} appearing in (2.4), we define the space

$$\mathbf{V}_0 = \{\mathbf{v} \in \mathbf{H}^1(\mathcal{O}) : \mathbf{v}|_{\partial\mathcal{O}} \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{O}\}; \quad (2.8)$$

and subsequently impose the standard assumption that

$$\mathbf{U} \in \mathbf{V}_0 \cap \mathbf{H}^3(\mathcal{O}) \quad (2.9)$$

(see the analogous—and actually slightly stronger—specifications made on ambient fields on p.529 of [30] and pp.102–103 of [44]).

Remark 2.1. *As mentioned above, the presence of \mathbf{U} in the modeling introduces the term $\mathbf{U} \cdot \nabla p$ into the pressure equation, which **does not** represent a bounded perturbation of the dynamics.*

Given the *Lamé Coefficients* $\lambda \geq 0$ and $\nu > 0$, the *stress tensor* σ of the fluid is defined as

$$\sigma(\boldsymbol{\mu}) = 2\nu\epsilon(\boldsymbol{\mu}) + \lambda[I_3 \cdot \epsilon(\boldsymbol{\mu})]I_3,$$

where the *strain tensor* ϵ is given by

$$\epsilon_{ij}(\boldsymbol{\mu}) = \frac{1}{2} \left(\frac{\partial \mu_j}{\partial x_i} + \frac{\partial \mu_i}{\partial x_j} \right), \quad 1 \leq i, j \leq 3$$

(see [35, p.129]). With this notation it is easy to see that

$$\operatorname{div} \sigma(\boldsymbol{\mu}) = \nu \Delta \boldsymbol{\mu} + (\nu + \lambda) \nabla \operatorname{div}(\boldsymbol{\mu}),$$

where λ and ν are the non-negative viscosity coefficients.

The boundary conditions that are invoked in (2.4) for the fluid PDE component are the so-called *impermeability-slip* conditions [11, 15]. Their physical interpretation is that no fluid passes through the boundary (the normal component of the fluid field \mathbf{u} on the active boundary portion Ω matches the plate velocity w_t), and that there is no stress in the tangential direction τ .

Remark 2.2. *Other possible physically relevant boundary conditions have appeared in the literature. We mention the Kutta-Joukowski type condition (2.10) [25], as well as the adherence condition (2.11).*

$$\sigma(\mathbf{u})\mathbf{n} - p\mathbf{n} = \mathbf{0} \text{ on } S; \quad \mathbf{u} \cdot \mathbf{n} = w_t \text{ on } \Omega; \quad (2.10)$$

$$\mathbf{u} = \mathbf{0} \text{ on } S \quad \mathbf{u} \cdot \mathbf{n} = w_t \text{ on } \Omega. \quad (2.11)$$

Though the focus of this treatment is on the *linear dynamics* of the fluid-plate interaction, we do provide a brief discussion of nonlinearity in the model in Section 7. We now mention the principal nonlinear plate model of interest: the scalar von Karman plate. Writing the plate equation in (2.5) as

$$w_{tt} + \Delta^2 w + [2\nu \partial_{x_3}(\mathbf{u})_3 + \lambda \operatorname{div}(\mathbf{u}) - p]_{\Omega} = f(w) \text{ on } \Omega \times (0, \infty) \quad (2.12)$$

³See e.g., p.846 of [13].

where we have

$$f(w) = [w, v(w) + F_0],$$

where F_0 is a given function from $H^4(\Omega)$ and the von Karman bracket $[u, v]$ is given by

$$[u, w] = \partial_{xx}u \cdot \partial_{yy}w + \partial_{yy}u \cdot \partial_{xx}w - 2 \cdot \partial_{xy}u \cdot \partial_{xy}w,$$

and the Airy stress function $v(u, w)$ solves the following elliptic problem

$$\Delta^2 v(u, w) + [u, w] = 0 \quad \text{in } \Omega, \quad \partial_\nu v(u, w) = v(u, w) = 0 \quad \text{on } \partial\Omega. \quad (2.13)$$

Von Karman equations are well known in nonlinear elasticity and constitute a basic model describing nonlinear oscillations of a plate accounting for large deflections, see [22] and references therein.

Remark 2.3. *In this paper we provide a discussion of the most physically relevant large deflection plate model. We do not fully discuss the breadth of nonlinear plate dynamics, as is done in [18]. However, the discussion we provide here is easily adapted to the other common plate nonlinearities of Berger or Kirchhoff type (see, for instance, [23] and [16, 32]).*

Remark 2.4. *The above fluid equations in (2.4) might be referred to as the Oseen equations for viscous compressible barotropic fluids. In the linearization procedure, without making additional assumptions on \mathbf{U} , we obtain:*

$$(\partial_t + \mathbf{U} \cdot \nabla)p + \operatorname{div} \mathbf{u} + \operatorname{div}(\mathbf{U})p = F_1(\mathbf{x}) \quad \text{in } \mathcal{O} \times \mathbb{R}_+, \quad (2.14)$$

$$(\partial_t + \mathbf{U} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} - (\nu + \lambda) \nabla \operatorname{div} \mathbf{u} + \nabla p + \nabla \mathbf{U} \cdot \mathbf{u} + (\mathbf{U} \cdot \nabla \mathbf{U})p = \mathbf{F}_2(\mathbf{x}) \quad \text{in } \mathcal{O} \times \mathbb{R}_+, \quad (2.15)$$

for a prescribed scalar function F_1 and vector field \mathbf{F}_2 . In our analysis we retain only the principal mathematical terms in (2.4)–(2.6), as the others may be viewed as zeroth order perturbations, and handled in a standard fashion.

3 Main Results

We are primarily interested in Hadamard well-posedness of the linearized coupled system given in (2.4)–(2.6). Specifically, we will ascertain well-posedness of the PDE model (2.4)–(2.6) for arbitrary initial data in the natural space of finite energy. To accomplish this, we will adopt a semigroup approach; namely, we will pose and validate an explicit semigroup generator representation for the fluid-structure dynamics (2.4)–(2.6).

With respect to the coupled PDE system (2.4)–(2.6), the associated space of well-posedness will be

$$\mathcal{H} \equiv L^2(\mathcal{O}) \times \mathbf{L}^2(\mathcal{O}) \times H_0^2(\Omega) \times L^2(\Omega). \quad (3.1)$$

\mathcal{H} is a Hilbert space, topologized by the following inner-product:

$$(\mathbf{y}_1, \mathbf{y}_2)_{\mathcal{H}} = (p_1, p_2)_{L^2(\mathcal{O})} + (\mathbf{u}_1, \mathbf{u}_2)_{\mathbf{L}^2(\mathcal{O})} + (\Delta w_1, \Delta w_2)_{L^2(\Omega)} + (v_1, v_2)_{L^2(\Omega)} \quad (3.2)$$

for any $\mathbf{y}_i = (p_i, \mathbf{u}_i, w_i, v_i) \in \mathcal{H}$, $i = 1, 2$.

In what follows, we consider the linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$, which expresses the compressible fluid-structure PDE system (2.4)–(2.6) as the abstract ODE:

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} p \\ \mathbf{u} \\ w \\ w_t \end{bmatrix} &= \mathcal{A} \begin{bmatrix} p \\ \mathbf{u} \\ w \\ w_t \end{bmatrix}; \\ [p(0), \mathbf{u}(0), w(0), w_t(0)] &= [p_0, \mathbf{u}_0, w_0, w_1]. \end{aligned} \quad (3.3)$$

To wit,

$$\mathcal{A} = \begin{bmatrix} -\mathbf{U} \cdot \nabla(\cdot) & -\operatorname{div}(\cdot) & 0 & 0 \\ -\nabla(\cdot) & \operatorname{div} \sigma(\cdot) - \eta I - \mathbf{U} \cdot \nabla(\cdot) & 0 & 0 \\ 0 & 0 & 0 & I \\ [\cdot]_{\Omega} & -[2\nu \partial_{x_3}(\cdot)_3 + \lambda \operatorname{div}(\cdot)]_{\Omega} & -\Delta^2 & 0 \end{bmatrix}. \quad (3.4)$$

Here, the domain $D(\mathcal{A})$ is given as

$$D(\mathcal{A}) = \{(p_0, \mathbf{u}_0, w_1, w_2) \in L^2(\mathcal{O}) \times \mathbf{H}^1(\mathcal{O}) \times H_0^2(\Omega) \times H_0^2(\Omega) : (i)-(v) \text{ hold below}\},$$

where

$$(A.i) \quad \mathbf{U} \cdot \nabla p_0 \in L^2(\mathcal{O})$$

$$(A.ii) \quad \operatorname{div} \sigma(\mathbf{u}_0) - \nabla p_0 \in L^2(\mathcal{O})$$

$$(A.iii) \quad -\Delta^2 w_0 - [2\nu \partial_{x_3}(\mathbf{u}_0)_3 + \lambda \operatorname{div}(\mathbf{u}_0)]_{\Omega} + p_0|_{\Omega} \in L^2(\Omega)$$

$$(A.iv) \quad (\sigma(\mathbf{u}_0)\mathbf{n} - p_0\mathbf{n}) \perp TH^{1/2}(\partial\mathcal{O}). \text{ That is,}$$

$$\langle \sigma(\mathbf{u}_0)\mathbf{n} - p_0\mathbf{n}, \boldsymbol{\tau} \rangle_{\mathbf{H}^{-\frac{1}{2}}(\partial\mathcal{O}) \times \mathbf{H}^{\frac{1}{2}}(\partial\mathcal{O})} = 0 \text{ for all } \boldsymbol{\tau} \in TH^{1/2}(\partial\mathcal{O}).$$

$$(A.v) \quad \mathbf{u}_0 = \boldsymbol{\mu}_0 + \tilde{\boldsymbol{\mu}}_0, \text{ where } \boldsymbol{\mu}_0 \in \mathbf{V}_0 \text{ and } \tilde{\boldsymbol{\mu}}_0 \in \mathbf{H}^1(\mathcal{O}) \text{ satisfies}^4$$

$$\tilde{\boldsymbol{\mu}}_0|_{\partial\mathcal{O}} = \begin{cases} 0 & \text{on } S \\ w_2\mathbf{n} & \text{on } \Omega \end{cases}$$

(and so $\boldsymbol{\mu}_0|_{\partial\mathcal{O}} \in TH^{1/2}(\partial\mathcal{O})$).

In the following theorem, we provide semigroup well-posedness for $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$, the proof of which is based on the well known Lumer-Phillips Theorem.

Theorem 3.1. *The map $\{p_0, \mathbf{u}_0; w_0, w_1\} \rightarrow \{p(t), \mathbf{u}(t); w(t), w_t(t)\}$ defines a strongly continuous semigroup $\{e^{At}\}$ on the space \mathcal{H} , and hence the system (2.4)–(2.6) is well-posed (in the sense of mild solutions—see Remark 3.2 below). In addition, the semigroup enjoys the following estimate:*

$$\|e^{At}\|_{\mathcal{L}(\mathcal{H})} \leq \exp\left(\frac{t}{2}\|\operatorname{div}(\mathbf{U})\|_{\infty}\right), \quad \forall t > 0. \quad (3.5)$$

⁴The existence of an $\mathbf{H}^1(\mathcal{O})$ -function $\tilde{\boldsymbol{\mu}}_0$ with such a boundary trace on Lipschitz domain \mathcal{O} is assured; see e.g., Theorem 3.33 of [38], or see also the proof of Lemma 5.1 below.

Remark 3.1. Given the existence of a semigroup $\{e^{At}\}$ for the fluid-structure generator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$: if initial data $[p_0, \mathbf{u}_0; w_0, w_1] \in D(\mathcal{A})$, the corresponding solution $[p(t), \mathbf{u}(t); w(t), w_t(t)] \in C([0, \infty), D(\mathcal{A}))$. In particular, the solution satisfies the condition (A.iv) in the definition of the generator. This means that one has that the tangential boundary condition

$$[\sigma(\mathbf{u}_0)\mathbf{n} - p_0\mathbf{n}] \cdot \boldsymbol{\tau} = 0 \quad \text{for all } \boldsymbol{\tau} \in TH^{1/2}(\partial\mathcal{O}),$$

satisfied in the sense of distributions. That is to say, $\forall \boldsymbol{\tau} \in TH^{1/2}(\partial\mathcal{O})$ and $\forall \phi \in \mathcal{D}(\partial\mathcal{O})$,

$$\langle \sigma(\mathbf{u}_0)\mathbf{n} - p_0\mathbf{n}, \phi\boldsymbol{\tau} \rangle_{\partial\mathcal{O}} = 0. \quad (3.6)$$

Remark 3.2 (Notions of Solution). We note that semigroup solutions, as arrived at in Theorem 3.1 for initial data $\mathbf{y} \in \mathcal{H}$, correspond to so called mild solutions (satisfying an integral form of (2.4)–(2.6)) in the sense of [41, Section 4.2]. Moreover, for initial data $\mathbf{y} \in D(\mathcal{A})$, we obtain so called strong solutions, which satisfy the PDE in a pointwise sense.

In [18], semigroup techniques are not used in demonstrating well-posedness. As such, the author takes care to define an appropriate notion of weak solution corresponding to a Galerkin construction (see [18, pp.653–654]). Such a notion of weak solution is relevant here, and can be obtained by making minor modifications that take into account the vector field \mathbf{U} . Here we assert that mild solutions (obtained via our semigroup) are in fact weak solutions as in [18]. In this way, we recover the well-posedness result of [18] (in the linear and nonlinear cases, with \mathcal{O} bounded) by simply letting $\mathbf{U} \equiv \mathbf{0}$. (Note that in Section 6 we discuss the relation between the weak and strong forms of the stationary problem associated with (2.4)–(2.5), and in Section 7 we discuss the presence of plate nonlinearity, in line with [18].)

Finally, we describe the energy balance equation for semigroup solutions to (2.4)–(2.6). We introduce the natural notion of *energy* into the analysis. Semigroup solutions obtained on the finite energy space \mathcal{H} are measured in the finite energy norm, which provides us with the energy functional: for $\mathbf{y}_0 = (p_0, \mathbf{u}_0, w_0, w_1) \in \mathcal{H}$, we have

$$\mathcal{E}(\mathbf{y}_0) = \frac{1}{2} \|\mathbf{y}_0\|_{\mathcal{H}}^2 = \frac{1}{2} \left\{ \|p_0\|_{\Omega}^2 + \|\mathbf{u}_0\|_{\Omega}^2 + \|\Delta w_0\|_{\Omega}^2 + \|w_1\|_{\Omega}^2 \right\}.$$

Let us also introduce the convenient notation:

$$a_{\mathcal{O}}(\mathbf{u}, \boldsymbol{\psi}) = (\sigma(\mathbf{u}), \epsilon(\boldsymbol{\psi}))_{\mathcal{O}} + \eta(\mathbf{u}, \boldsymbol{\psi})_{\mathcal{O}}. \quad (3.7)$$

With strong solutions in hand (corresponding to smooth data in $D(\mathcal{A})$), we may test (2.4)–(2.6) with p, \mathbf{u} , and w_t (respectively) to obtain the energy balance. The energy balance is then obtained for semigroup (mild) solutions through the standard limiting process. Equivalently, it is admissible to test with semigroup solutions (for $\mathbf{y}_0 \in \mathcal{H}$, $p \in L^2(0, t; L^2(\mathcal{O}))$, $\mathbf{u} \in L^2(0, t; \mathbf{L}^2(\mathcal{O}))$, and $w_t \in L^2(0, t; L^2(\Omega))$) in the weak form of the problem in [18]. This also yields the energy balance below.

Lemma 3.2. Consider $\mathbf{y}_0 = (p_0, \mathbf{u}_0, w_0, w_1) \in \mathcal{H}$ and $\mathbf{U} \in \mathbf{V}_0$. Any mild solution $y(t) = e^{At}\mathbf{y}_0 = (p(t), \mathbf{u}(t), w(t), w_t(t))$ to (2.4)–(2.6) satisfies for $t > 0$:

$$\begin{aligned} \mathcal{E}(p(t), \mathbf{u}(t), w(t), w_t(t)) + \int_0^t a_{\mathcal{O}}(\mathbf{u}(\tau), \mathbf{u}(\tau)) d\tau &= \mathcal{E}(p_0, \mathbf{u}_0, w_0, w_1) \\ &+ \frac{1}{2} \int_0^t \int_{\mathcal{O}} \operatorname{div}(\mathbf{U}) [|p(\tau)|^2 + |\mathbf{u}(\tau)|^2] dx d\tau. \end{aligned} \quad (3.8)$$

Remark 3.3. We note two features of the energy identity: first, when the field $\mathbf{U} \in \mathbf{V}_0$ is also divergence free, the energy identity remains the same as in the case where $\mathbf{U} \equiv 0$ (like [18]). Secondly, the dissipation integral $\int_0^\infty a_{\mathcal{O}}(\mathbf{u}, \mathbf{u}) d\tau$ depends on the quantity $\operatorname{div} \mathbf{U}$ as well: again, with $\operatorname{div} \mathbf{U} \equiv 0$, we see that

$$\int_0^\infty a_{\mathcal{O}}(\mathbf{u}(\tau), \mathbf{u}(\tau)) d\tau < +\infty,$$

with a bound that depends only on the initial data.

We conclude this section by noting that we provide a discussion of solutions in the presence of nonlinear (von Karman) plate dynamics, including well-posedness, energy-balance, and stationary solutions, but we relegate this discussion to Section 7.

4 Discussion of Main Results in Relation to the Literature

The model under consideration describes the case of a (possibly viscous) *compressible* gas/fluid flow and was recently studied in [18] in the case with *zero* speed ($\mathbf{U} = 0$) of the unperturbed flow. Beginning with compressible Navier-Stokes, one can obtain several fluid-plate cases which are important from an applied point of view:

- **Incompressible Fluid**, i.e., $\operatorname{div} \mathbf{u} = 0$ and density constant: In the *viscous* case, the standard linearized Navier-Stokes equations arise; fluid-plate interactions in this case were studied in [28, 26, 27, 29]. Results on well-posedness and attractors for different elastic descriptions and domains were obtained. In this case, we also mention the work [5, 3, 4] which addresses semigroup well-posedness of a related linear fluid-plate model, and decay rates via *frequency domain* techniques. The *inviscid* case was studied in [19] in the same context.
- **Compressible Fluid**: In the inviscid case we can obtain wave-type dynamics for the (perturbed) velocity potential ($\mathbf{u} = \nabla \phi$, potential flow) of the form (see also [10, 11, 31]):

$$\begin{cases} (\partial_t + \mathbf{U} \cdot \nabla)^2 \phi = \Delta \phi & \text{in } \mathcal{O} \times (0, T), \\ \partial_z \phi = L(w_t, \nabla w) & \text{on } \Omega \times (0, T) \\ \partial_z \phi = 0 & \text{on } \partial \mathcal{O} \setminus \Omega \times (0, T). \end{cases} \quad (4.1)$$

In these variables, the pressure/density of the fluid has the form $p = (\partial_t + \mathbf{U} \cdot \nabla)\phi$. Due to the impermeability assumption, in the case of the perfect fluid, we have only one Neumann-type boundary condition given above via the operator L . The (semigroup) well-posedness [23, 42] and stability properties [22, 24, 36] of this model have been intensively studied.

The *viscous* case was studied in [18], and is the motivation of the current work.

In all the papers cited above, the interactive dynamics between fluid and a plate (or shell) are considered. These analyses are distinguished from those for other fluid-structure interactive PDE models in that the elastic structure is two dimensional, and evolves on the boundary of the three dimensional fluid domain. One of the key issues for the present configuration—and indeed, one of the main points in the bulk of the literature above—is the determination of how, and to what extent, the fluid (de)stabilizes the structure. In [18, 28, 27], after obtaining well-posedness of the models

(with structural nonlinearity), the existence of compact global attractors for the dynamics is shown; in some cases the existence of this invariant set is due strictly to the presence of the fluid, rather than some underlying structural phenomenon.⁵ In addition, it is sometimes possible (perhaps under additional assumptions) to show strong stabilization to equilibrium for the fluid-structure dynamics (e.g., [36, 22]). In all cases where the ambient flow field $\mathbf{U} \neq \mathbf{0}$, the stability properties of the model depend greatly on the structure and magnitude of the flow field \mathbf{U} [1]. This will certainly be the case for the dynamics considered here, as one can see from (3.8).

Remark 4.1. *The survey papers [29, 25] provides a nice overview of the modeling, well-posedness, and long-time behavior results for the family of dynamics described above.*

We emphasize that in any study involving *compressible fluids*, the enforced compressibility produces additional density/pressure variables, and, as a result, well-posedness cannot be obtained in a straightforward way. In fact, the primary difficulty lies in showing the maximality (range) condition of the generator, since one has to address this density/pressure component. This variable cannot be readily eliminated, and therefore accounts for an elliptic equation which must be solved. To overcome this, we develop a methodology based on the application of a static well-posedness result given in the Appendix of [30] (see also [37]). That paper, as well as [44], deals with the stationary compressible Navier-Stokes equations. Their principal result (obtained independently, through different methodologies) is a small data well-posedness for the fully nonlinear fluid problem. However, both approaches first necessarily provide a framework for the linearized problem; in particular, [30] provides a strategy for our analysis of the stationary compressible fluid-structure PDE which is associated with maximality of the generator.

The pioneering work [18], which we cite as the primary motivating reference, considers the model presented here with $\mathbf{U} \equiv 0$. In this paper, solvability and dynamical properties of the model are considered in the case of a general (possibly unbounded) smooth domain and in the presence of plate nonlinearity. Along with the well-posedness result, the existence of a finite dimensional compact global attractor is proved when the domain is bounded. The techniques used are consistent with those in [28, 26, 19], namely, Galerkin-type procedures are implemented, along with good a priori estimates, in order to produce solutions. As with many fluid-structure interactions, the critical issue in [18] is the appearance of ill-defined traces at the interface. In the incompressible case, one can recover negative Sobolev trace regularity of the pressure p at the interface via properties of the Stokes' operator. However, in the viscous compressible case this is no longer true. Our semigroup approach does not require the use of approximate solutions. Indeed, we overcome the key difficulty of trace regularity issues by exploiting cancellations at the level of solutions with data in the generator. In this way we do not have to work component-wise on the dynamic equations, though we must work carefully (and component-wise) on the static problem associated with maximality of the generator. We remark that, despite these trace regularity issues, when $\mathbf{U} \equiv 0$, uniform decay of finite energy solutions is obtained in [18] through a clever Lyapunov approach that makes use of a Neumann lifting map with associated estimates; this construction is fundamentally obstructed by the addition of the $\mathbf{U} \cdot \nabla p$ term in the pressure equation here. See the forthcoming work [7] on the decay properties of the model considered here.

⁵We mention that one of the prominent tools utilized in these fluid-structure interactions—with nonlinearity present in the structure—is the recently developed *quasi-stability* theory for dissipative dynamical systems (see [20, 22]).

5 The Proof of Theorem 3.1

Our proof of well-posedness hinges on showing that the matrix $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ generates a C_0 -semigroup. At this point, we should note that due to the existence of the generally nonzero ambient vector field \mathbf{U} in the model, we have a lack of dissipativity of the operator \mathcal{A} . Accordingly, we introduce the following bounded perturbation $\hat{\mathcal{A}}$ of our generator \mathcal{A} :

$$\hat{\mathcal{A}} = \mathcal{A} - \frac{\operatorname{div}(\mathbf{U})}{2} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D(\hat{\mathcal{A}}) = D(\mathcal{A}). \quad (5.1)$$

Therewith, the proof of Theorem 3.1 is geared towards establishing the maximal dissipativity of the linear operator $\hat{\mathcal{A}}$; subsequently, an application of the Lumer-Phillips Theorem will yield that $\hat{\mathcal{A}}$ generates a C_0 semigroup of contractions on \mathcal{H} . In turn, applying the standard perturbation result [34] (given, for instance, in [41, Theorem 1.1, p.76]) yields semigroup generation for the original modeling fluid-structure operator \mathcal{A} of (3.4), via (5.1).

5.1 Dissipativity

Considering the inner-product for the state space \mathcal{H} given in (3.2), for any $\mathbf{y} = [p_0, \mathbf{u}_0, w_1, w_2]^T \in D(\mathcal{A})$ we have

$$\begin{aligned} \left(\hat{\mathcal{A}} \begin{bmatrix} p_0 \\ \mathbf{u}_0 \\ w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} p_0 \\ \mathbf{u}_0 \\ w_1 \\ w_2 \end{bmatrix} \right)_{\mathcal{H}} &= -(\mathbf{U} \cdot \nabla p_0, p_0)_{\mathcal{O}} - \frac{1}{2}(\operatorname{div}(\mathbf{U})p_0, p_0)_{\mathcal{O}} - (\operatorname{div}(\mathbf{u}_0), p_0)_{\mathcal{O}} - \eta \|\mathbf{u}_0\|_{\mathbf{L}^2(\mathcal{O})}^2 \\ &\quad + (\operatorname{div} \sigma(\mathbf{u}_0) - \nabla p_0, \mathbf{u}_0)_{\mathcal{O}} - (\mathbf{U} \cdot \nabla \mathbf{u}_0, \mathbf{u}_0)_{\mathcal{O}} - \frac{1}{2}(\operatorname{div}(\mathbf{U})\mathbf{u}_0, \mathbf{u}_0)_{\mathcal{O}} \\ &\quad + (\Delta w_2, \Delta w_1)_{\Omega} - (\Delta^2 w_1, w_2)_{\Omega} - ([2\nu \partial_{x_3}(\mathbf{u}_0)_3 + \lambda \operatorname{div}(\mathbf{u}_0)]_{\Omega} - p_0|_{\Omega}, w_2)_{\Omega}. \end{aligned} \quad (5.2)$$

Applying Green's Theorems to right hand side, we subsequently have

$$\begin{aligned} \left(\hat{\mathcal{A}} \begin{bmatrix} p_0 \\ \mathbf{u}_0 \\ w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} p_0 \\ \mathbf{u}_0 \\ w_1 \\ w_2 \end{bmatrix} \right)_{\mathcal{H}} &= -(\mathbf{U} \cdot \nabla p_0, p_0)_{\mathcal{O}} - \frac{1}{2}(\operatorname{div}(\mathbf{U})p_0, p_0)_{\mathcal{O}} - (\operatorname{div}(\mathbf{u}_0), p_0)_{\mathcal{O}} \\ &\quad - (\sigma(\mathbf{u}_0), \epsilon(\mathbf{u}_0))_{\mathcal{O}} - \eta \|\mathbf{u}_0\|_{\mathbf{L}^2(\mathcal{O})}^2 + (p_0, \operatorname{div}(\mathbf{u}_0))_{\mathcal{O}} + \langle \sigma(\mathbf{u}_0)\mathbf{n} - p_0\mathbf{n}, \mathbf{u}_0 \rangle_{\partial\mathcal{O}} \\ &\quad - (\mathbf{U} \cdot \nabla \mathbf{u}_0, \mathbf{u}_0)_{\mathcal{O}} - \frac{1}{2}(\operatorname{div}(\mathbf{U})\mathbf{u}_0, \mathbf{u}_0)_{\mathcal{O}} \\ &\quad + (\Delta w_2, \Delta w_1)_{\Omega} - (\Delta w_1, \Delta w_2)_{\Omega} - ([2\nu \partial_{x_3}(\mathbf{u}_0)_3 + \lambda \operatorname{div}(\mathbf{u}_0)]_{\Omega} - p_0|_{\Omega}, w_2)_{\Omega}. \end{aligned} \quad (5.3)$$

Invoking now the boundary conditions (A.iv) and (A.v), in the definition of the domain $D(\mathcal{A})$, there is then a cancellation of boundary terms so as to have

$$\begin{aligned} \left(\widehat{\mathcal{A}} \begin{bmatrix} p_0 \\ \mathbf{u}_0 \\ w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} p_0 \\ \mathbf{u}_0 \\ w_1 \\ w_2 \end{bmatrix} \right)_{\mathcal{H}} &= -(\mathbf{U} \cdot \nabla p_0, p_0)_{\mathcal{O}} - \frac{1}{2}(\operatorname{div}(\mathbf{U})p_0, p_0)_{\mathcal{O}} - 2i \operatorname{Im}(\operatorname{div}(\mathbf{u}_0), p_0)_{\mathcal{O}} \\ &\quad - (\sigma(\mathbf{u}_0), \epsilon(\mathbf{u}_0))_{\mathcal{O}} - \eta \|\mathbf{u}_0\|_{\mathbf{L}^2(\mathcal{O})}^2 - (\mathbf{U} \cdot \nabla \mathbf{u}_0, \mathbf{u}_0)_{\mathcal{O}} - \frac{1}{2}(\operatorname{div}(\mathbf{U})\mathbf{u}_0, \mathbf{u}_0)_{\mathcal{O}} \\ &\quad - 2i \operatorname{Im}(\Delta w_1, \Delta w_2)_{\Omega}. \end{aligned} \quad (5.4)$$

Moreover, via Green's Theorem, as well as the assumption that $\mathbf{U} \in \mathbf{V}_0$ (as defined in (2.8)), we obtain

$$2 \operatorname{Re}(\mathbf{U} \cdot \nabla p_0, p_0)_{\mathcal{O}} = - \int_{\mathcal{O}} \operatorname{div}(\mathbf{U}) |p_0|^2 d\mathcal{O}; \quad (5.5)$$

$$2 \operatorname{Re}(\mathbf{U} \cdot \nabla \mathbf{u}_0, \mathbf{u}_0)_{\mathcal{O}} = - \int_{\mathcal{O}} \operatorname{div}(\mathbf{U}) |\mathbf{u}_0|^2 d\mathcal{O}. \quad (5.6)$$

Applying these relations to the right hand of (5.4), we then have

$$\operatorname{Re} \left(\widehat{\mathcal{A}} \begin{bmatrix} p_0 \\ \mathbf{u}_0 \\ w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} p_0 \\ \mathbf{u}_0 \\ w_1 \\ w_2 \end{bmatrix} \right)_{\mathcal{H}} = -(\sigma(\mathbf{u}_0), \epsilon(\mathbf{u}_0))_{\mathcal{O}} - \eta \|\mathbf{u}_0\|_{\mathbf{L}^2(\mathcal{O})}^2 \leq 0,$$

which establishes the dissipativity of $\widehat{\mathcal{A}} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$.

5.2 Maximality

In this section we show the maximality property of the operator $\widehat{\mathcal{A}}$ on the space \mathcal{H} . To this end, we will need to establish the *range condition*, at least for parameter $\xi > 0$ sufficiently large. Namely, we must show

$$\operatorname{Range}(\xi I - \widehat{\mathcal{A}}) = \mathcal{H}, \quad \text{for some } \xi > 0. \quad (5.7)$$

This necessity is equivalent to finding $[p, \mathbf{v}, w_1, w_2] \in D(\mathcal{A})$ which satisfies, for given $[p^*, \mathbf{v}^*, w_1^*, w_2^*] \in \mathcal{H}$, the abstract equation

$$(\xi I - \widehat{\mathcal{A}}) \begin{bmatrix} p \\ \mathbf{v} \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} p^* \\ \mathbf{v}^* \\ w_1^* \\ w_2^* \end{bmatrix}. \quad (5.8)$$

Given the definition of \mathcal{A} in (3.4), then in PDE terms, solving the abstract equation (5.7) is equivalent to proving that the following system of equations, with given data $[p^*, \mathbf{v}^*, w_1^*, w_2^*] \in \mathcal{H}$, has a (unique)

solution $[p, \mathbf{v}, w_1, w_2] \in D(\mathcal{A})$:

$$\begin{cases} \xi p + \mathbf{U} \cdot \nabla p + \frac{1}{2} \operatorname{div}(\mathbf{U})p + \operatorname{div}(\mathbf{v}) = p^* & \text{in } \mathcal{O} \\ \xi \mathbf{v} + \mathbf{U} \cdot \nabla \mathbf{v} + \frac{1}{2} \operatorname{div}(\mathbf{U})\mathbf{v} - \operatorname{div} \sigma(\mathbf{v}) + \eta \mathbf{v} + \nabla p = \mathbf{v}^* & \text{in } \mathcal{O} \\ (\sigma(\mathbf{v})\mathbf{n} - p\mathbf{n}) \cdot \boldsymbol{\tau} = 0 & \text{on } \partial\mathcal{O} \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } S \\ \mathbf{v} \cdot \mathbf{n} = w_2 & \text{on } \Omega \end{cases} \quad (5.9)$$

$$\begin{cases} \xi w_1 - w_2 = w_1^* & \text{on } \Omega \\ \xi w_2 + \Delta^2 w_1 + [2\nu \partial_{x_3}(\mathbf{v})_3 + \lambda \operatorname{div}(\mathbf{v}) - p]_\Omega = w_2^* & \text{on } \Omega \\ w_1 = \frac{\partial w_1}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.10)$$

We will give our proof of maximality in two steps. In the first step, we will show the existence and uniqueness of “uncoupled” versions of the compressible fluid-structure PDE system (5.9) which is satisfied by variables $\{p, \mathbf{v}\}$. To this end, the key ingredient will be the well-posedness result Theorem 9.1, which is applicable to (uncoupled) equations of the type satisfied by the pressure variable. (See also [30] and [37].)

Subsequently, we proceed to establish the range condition (5.8), by sequentially proving the existence of the pressure-fluid-structure components $\{p, \mathbf{v}, w_1, w_2\}$ which solve the coupled system (5.9)–(5.10). This work for pressure-fluid-structure static well-posedness involves appropriate uses of the Lax-Milgram Theorem.

STEP 1:

Consider the following ξ -parameterized PDE system on the fluid domain \mathcal{O} , with given forcing terms $\{p^*, \mathbf{v}^*\} \in L^2(\mathcal{O}) \times [\mathbf{V}_0]'$ and boundary data $g \in H_0^{1/2+\epsilon}(\Omega)$, where $\epsilon > 0$.

$$\xi p + \mathbf{U} \cdot \nabla p + \frac{1}{2} \operatorname{div}(\mathbf{U})p + \operatorname{div}(\mathbf{v}) = p^* \quad \text{in } \mathcal{O} \quad (5.11)$$

$$\xi \mathbf{v} + \mathbf{U} \cdot \nabla \mathbf{v} + \frac{1}{2} \operatorname{div}(\mathbf{U})\mathbf{v} - \operatorname{div} \sigma(\mathbf{v}) + \eta \mathbf{v} + \nabla p = \mathbf{v}^* \quad \text{in } \mathcal{O} \quad (5.12)$$

$$(\sigma(\mathbf{v})\mathbf{n} - p\mathbf{n}) \cdot \boldsymbol{\tau} = 0 \quad \text{on } \partial\mathcal{O} \quad (5.13)$$

$$\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } S \quad (5.14)$$

$$\mathbf{v} \cdot \mathbf{n} = g \quad \text{on } \Omega \quad (5.15)$$

(and, where again, the ambient vector field $\mathbf{U} \in \mathbf{V}_0 \cap \mathbf{H}^3(\mathcal{O})$).

STEP 1 consists of proving the following (driving) lemma for the existence and uniqueness of the solution $\{p, \mathbf{v}\}$ of (5.11)–(5.15).

Lemma 5.1. *(i) With reference to problem (5.11)–(5.15): with given data*

$$[p^*, \mathbf{v}^*, g] \in L^2(\mathcal{O}) \times [\mathbf{V}_0]' \times H_0^{\frac{1}{2}+\epsilon}(\Omega),$$

and with $\xi > 0$ sufficiently large, there exists a unique solution $\{p, \mathbf{v}\} \in L^2(\mathcal{O}) \times \mathbf{H}^1(\mathcal{O})$ of (5.11)–(5.15).

(ii) The fluid solution component \mathbf{v} is of the form

$$\mathbf{v} = \mathbf{u} + \widetilde{\mathbf{v}}_0, \quad (5.16)$$

where $\mathbf{u} \in \mathbf{V}_0$, and $\widetilde{\mathbf{v}}_0 \in \mathbf{H}^1(\mathcal{O})$ satisfies

$$\widetilde{\mathbf{v}}_0|_{\partial\mathcal{O}} = \begin{cases} 0 & \text{on } S \\ g\mathbf{n} & \text{on } \Omega. \end{cases} \quad (5.17)$$

(iii) The trace term $[\sigma(\mathbf{v})\mathbf{n} - p\mathbf{n}]_{\partial\mathcal{O}} \in \mathbf{H}^{-\frac{1}{2}}(\partial\mathcal{O})$, and moreover satisfies

$$\langle \sigma(\mathbf{v})\mathbf{n} - p\mathbf{n}, \boldsymbol{\tau} \rangle_{\mathbf{H}^{-\frac{1}{2}}(\partial\mathcal{O}) \times \mathbf{H}^{\frac{1}{2}}(\partial\mathcal{O})} = 0 \quad \text{for all } \boldsymbol{\tau} \in TH^{1/2}(\partial\mathcal{O}), \quad (5.18)$$

and so the boundary condition (5.13) is satisfied in the sense of distributions; see (3.6) of Remark 3.1

(iv) The fluid and pressure solution components (p, \mathbf{v}) satisfies the following estimates, for $\xi = \xi(\mathbf{U})$ large enough:

$$\|p\|_{L^2(\mathcal{O})} \leq \frac{C}{\xi} \| [p^*, \mathbf{v}^*, g] \|_{\mathbf{L}^2(\mathcal{O}) \times [\mathbf{V}_0]' \times H_0^{\frac{1}{2}+\epsilon}(\Omega)}; \quad (5.19)$$

$$\|\mathbf{v}\|_{\mathbf{H}^1(\mathcal{O})} \leq C \| [p^*, \mathbf{v}^*, g] \|_{\mathbf{L}^2(\mathcal{O}) \times [\mathbf{V}_0]' \times H_0^{\frac{1}{2}+\epsilon}(\Omega)}. \quad (5.20)$$

Proof of Lemma 5.1. We give the proof in two parts. Our beginning point is to resolve the pressure term; this will be accomplished by applying Theorem 9.1 of the Appendix. To this end: If we initially consider the equation

$$\xi p + \mathbf{U} \cdot \nabla p + \frac{1}{2} \operatorname{div}(\mathbf{U})p = \sigma \quad \text{in } \mathcal{O}, \quad (5.21)$$

where $\sigma \in L^2(\mathcal{O})$ and $\mathbf{U} \in \mathbf{V}_0 \cap \mathbf{H}^3(\mathcal{O})$, we have by Theorem 9.1 the existence of a unique $L^2(\mathcal{O})$ -function p which is a weak solution of (5.21); namely, it satisfies the variational relation

$$\xi \int_{\mathcal{O}} p \phi d\mathbf{x} - \int_{\mathcal{O}} p \operatorname{div}(\phi \mathbf{U}) d\mathbf{x} + \frac{1}{2} \int_{\mathcal{O}} \operatorname{div}(\mathbf{U}) p \phi = \int_{\mathcal{O}} \sigma \phi d\mathbf{x}, \quad \text{for all } \phi \in H^1(\mathcal{O}) \quad (5.22)$$

(and in particular for $\phi \in \mathcal{D}(\mathcal{O})$; so we infer that for given L^2 -function σ , corresponding L^2 -solution p satisfies the PDE (5.21) pointwise a.e.)

Moreover, for $\xi = \xi(\mathbf{U})$ sufficiently large we have the estimate—see (9.4) of Theorem 9.1,

$$\|p\|_{L^2(\mathcal{O})} \leq \frac{1}{\xi} \|\sigma\|_{L^2(\mathcal{O})}. \quad (5.23)$$

With the well-posedness above, in order to find the existence and uniqueness of the fluid component \mathbf{v} , we now turn our attention to (5.11)–(5.15). In view of the well-posedness of (5.21), we decompose the fluid term \mathbf{v} and pressure term p as follows:

$$\mathbf{v} = \mathbf{u} + \widetilde{\mathbf{v}}_0; \quad (5.24)$$

$$p = p[\mathbf{u}] + p[\widetilde{\mathbf{v}}_0] + p[p^*], \quad (5.25)$$

where $\mathbf{u} \in \mathbf{V}_0$ is the new fluid solution variable, and $\widetilde{\mathbf{v}}_0 \in \mathbf{H}^1(\mathcal{O})$ is a vector field which is chosen to satisfy

$$\widetilde{\mathbf{v}}_0|_{\partial\mathcal{O}} = \begin{cases} 0 & \text{on } S \\ g\mathbf{n} & \text{on } \Omega. \end{cases} \quad (5.26)$$

To wit: since boundary data $g \in H_0^{\frac{1}{2}+\epsilon}(\Omega)$ – with necessarily $\epsilon > 0$ – we can extend by zero the function $g\mathbf{n}|_{\Omega} = g[1, 0, 0]$, so as to have a $\mathbf{H}^{\frac{1}{2}+\epsilon}$ function on all of $\partial\mathcal{O}$. (See e.g., Theorem 3.33, p.95 of [38].) In turn, since the Sobolev Dirichlet trace map from $H^s(\mathcal{O})$ into $H^{s-\frac{1}{2}}(\partial\mathcal{O})$ is surjective for $\frac{1}{2} < s < \frac{3}{2}$, then the existence of given $\widetilde{\mathbf{v}}_0 \in \mathbf{H}^{1+\epsilon}(\mathcal{O})$ is assured. (See e.g., Theorem 3.38 of [38], valid for Lipschitz domains.)

Moreover, the functions $p[\mathbf{u}]$, $p[\widetilde{\mathbf{v}}_0]$, and $p[p^*]$ solve the respective versions of (5.21):

$$\xi p[\mathbf{u}] + \mathbf{U} \cdot \nabla p[\mathbf{u}] + \frac{\operatorname{div}(\mathbf{U})}{2} p[\mathbf{u}] = -\operatorname{div}(\mathbf{u}) \quad (5.27)$$

$$\xi p[\widetilde{\mathbf{v}}_0] + \mathbf{U} \cdot \nabla p[\widetilde{\mathbf{v}}_0] + \frac{\operatorname{div}(\mathbf{U})}{2} p[\widetilde{\mathbf{v}}_0] = -\operatorname{div}(\widetilde{\mathbf{v}}_0) \quad (5.28)$$

$$\xi p[p^*] + \mathbf{U} \cdot \nabla p[p^*] + \frac{\operatorname{div}(\mathbf{U})}{2} p[p^*] = p^*, \quad (5.29)$$

with estimates—for $\xi = \xi(\mathbf{U})$ large enough, see (5.23) and (5.26) —

$$\|p[\mathbf{u}]\|_{L^2(\mathcal{O})} \leq \frac{C}{\xi} \|\mathbf{u}\|_{\mathbf{H}^1(\mathcal{O})} \quad (5.30)$$

$$\begin{aligned} \|p[\widetilde{\mathbf{v}}_0]\|_{L^2(\mathcal{O})} &\leq \frac{C}{\xi} \|\widetilde{\mathbf{v}}_0\|_{\mathbf{H}^1(\mathcal{O})} \\ &\leq \frac{C}{\xi} \|g\|_{H_0^{1/2+\epsilon}(\Omega)} \end{aligned} \quad (5.31)$$

$$\|p[p^*]\|_{L^2(\mathcal{O})} \leq \frac{C}{\xi} \|p^*\|_{\mathcal{O}}, \quad (5.32)$$

where again, fluid variable \mathbf{u} will be sought after.

Now, the rest part of the proof relies on the application of Lax-Milgram theorem, by way of solving for \mathbf{u} in (5.24). For this reason, we firstly define the operator $A \in \mathcal{L}(\mathbf{V}_0, [\mathbf{V}_0]')$ to be, for all $\boldsymbol{\psi} \in \mathbf{V}_0$,

$$\begin{aligned} \langle A\mathbf{u}, \boldsymbol{\psi} \rangle_{\mathbf{V}_0 \times [\mathbf{V}_0]'} &= (\xi \mathbf{u} + \mathbf{U} \cdot \nabla \mathbf{u} + \frac{1}{2} \operatorname{div}(\mathbf{U}) \mathbf{u}, \boldsymbol{\psi})_{\mathcal{O}} \\ &\quad + (\sigma(\mathbf{u}), \epsilon(\boldsymbol{\psi}))_{\mathcal{O}} + \eta(\mathbf{u}, \boldsymbol{\psi})_{\mathcal{O}} - (p[\mathbf{u}], \operatorname{div}(\boldsymbol{\psi}))_{\mathcal{O}}, \end{aligned} \quad (5.33)$$

where again, $p[\mathbf{u}]$ solves (5.27).

So, with a view of finding the solution pair (p, \mathbf{v}) , via the expressions (5.24) and (5.25), we are led to consider the following variational problem: Find $\mathbf{u} \in \mathbf{V}_0$ which solves, for every $\boldsymbol{\psi} \in \mathbf{V}_0$,

$$\langle A\mathbf{u}, \boldsymbol{\psi} \rangle_{\mathbf{V}_0 \times [\mathbf{V}_0]'} = \langle F, \boldsymbol{\psi} \rangle_{\mathbf{V}_0 \times [\mathbf{V}_0]'}; \quad (5.34)$$

where the forcing term $F \in [\mathbf{V}_0]'$ is given by

$$\begin{aligned} \langle F, \boldsymbol{\psi} \rangle_{\mathbf{V}_0 \times [\mathbf{V}_0]'} &= (\mathbf{v}^*, \boldsymbol{\psi})_{\mathcal{O}} - (\sigma(\widetilde{\mathbf{v}}_0), \epsilon(\boldsymbol{\psi}))_{\mathcal{O}} - \eta(\widetilde{\mathbf{v}}_0, \boldsymbol{\psi})_{\mathcal{O}} \\ &\quad - (\xi \widetilde{\mathbf{v}}_0 + \mathbf{U} \cdot \nabla \widetilde{\mathbf{v}}_0 + \frac{1}{2} \operatorname{div}(\mathbf{U}) \widetilde{\mathbf{v}}_0, \boldsymbol{\psi})_{\mathcal{O}} \\ &\quad + (p[\widetilde{\mathbf{v}}_0] + p[p^*], \operatorname{div}(\boldsymbol{\psi}))_{\mathcal{O}}, \end{aligned} \quad (5.35)$$

for given $\boldsymbol{\psi} \in \mathbf{V}_0$. After considering the definition of \mathbf{V}_0 and using the divergence theorem we note that

$$(\mathbf{U} \cdot \nabla \boldsymbol{\psi} + \frac{1}{2} \operatorname{div}(\mathbf{U}) \boldsymbol{\psi}, \boldsymbol{\psi})_{\mathcal{O}} = 0, \quad \forall \boldsymbol{\psi} \in \mathbf{V}_0. \quad (5.36)$$

Moreover, by estimate (5.30) we have for all $\boldsymbol{\psi} \in \mathbf{V}_0$,

$$\left| (p[\boldsymbol{\psi}], \operatorname{div}(\boldsymbol{\psi}))_{\mathcal{O}} \right| \leq \frac{C}{\xi} \|\boldsymbol{\psi}\|_{\mathbf{H}^1(\mathcal{O})}^2. \quad (5.37)$$

Combining (5.36) and (5.37) with Korn's inequality—see e.g., Theorem 2.6.5, p.93 of [35]—we then have, for $\xi > 0$ sufficiently large,

$$\langle A\boldsymbol{\psi}, \boldsymbol{\psi} \rangle_{\mathbf{V}_0 \times [\mathbf{V}_0]'} = (\xi + \eta) \|\boldsymbol{\psi}\|_{\mathcal{O}}^2 + \sigma(\boldsymbol{\psi}), \epsilon(\boldsymbol{\psi}) - (p[\boldsymbol{\psi}], \operatorname{div}(\boldsymbol{\psi}))_{\mathcal{O}} \geq c \|\boldsymbol{\psi}\|_{\mathbf{H}^1(\mathcal{O})}^2, \quad \forall \boldsymbol{\psi} \in \mathbf{V}_0, \quad (5.38)$$

where coercivity constant $c > 0$ is independent of sufficiently large $\xi > 0$. Therefore, $A \in \mathcal{L}(\mathbf{V}_0, [\mathbf{V}_0]')$ is \mathbf{V}_0 -elliptic for $\xi > 0$ large enough. Consequently, by the Lax-Milgram Theorem, the variational equation (5.34) has a unique solution $\mathbf{u} \in \mathbf{V}_0$, which will in turn yield the solution pair (\mathbf{v}, p) of (5.11)–(5.15 through the relations (5.24) and (5.25). In particular, from (5.24) and (5.26), $\mathbf{v} \in \mathbf{H}^1(\mathcal{O})$ admits of the decomposition (5.16); and since

$$\mathbf{v}|_{\partial\mathcal{O}} \cdot \mathbf{n} = \begin{cases} 0 & \text{on } S \\ g & \text{on } \Omega, \end{cases}$$

the obtained solution component \mathbf{v} satisfies the boundary conditions (5.14)–(5.15).

In addition, from (5.34), (5.24) and (5.25), (p, \mathbf{v}) satisfies the variational relation

$$\begin{aligned} (\xi \mathbf{v} + \mathbf{U} \cdot \nabla \mathbf{v} + \frac{1}{2} \operatorname{div}(\mathbf{U}) \mathbf{v}, \boldsymbol{\psi})_{\mathcal{O}} + (\sigma(\mathbf{v}), \epsilon(\boldsymbol{\psi}))_{\mathcal{O}} + \eta(\mathbf{v}, \boldsymbol{\psi}) - (p, \operatorname{div}(\boldsymbol{\psi}))_{\mathcal{O}} \\ = (\mathbf{v}^*, \boldsymbol{\psi})_{\mathcal{O}}, \quad \forall \boldsymbol{\psi} \in \mathbf{V}_0. \end{aligned} \quad (5.39)$$

In particular, if $\boldsymbol{\psi} \in [\mathcal{D}(\mathcal{O})]^3$, we then have

$$\begin{aligned} -(\operatorname{div} \sigma(\mathbf{v}), \boldsymbol{\psi})_{\mathcal{O}} + \eta(\mathbf{v}, \boldsymbol{\psi})_{\mathcal{O}} + (\nabla p, \boldsymbol{\psi})_{\mathcal{O}} \\ = -(\xi \mathbf{v} + \mathbf{U} \cdot \nabla \mathbf{v} + \frac{1}{2} \operatorname{div}(\mathbf{U}) \mathbf{v}, \boldsymbol{\psi})_{\mathcal{O}} + (\mathbf{v}^*, \boldsymbol{\psi})_{\mathcal{O}}. \end{aligned} \quad (5.40)$$

Since $\mathbf{v} \in \mathbf{H}^1(\mathcal{O})$, this relation and the density of $[\mathcal{D}(\mathcal{O})]^3$ in $\mathbf{L}^2(\mathcal{O})$, yield that

$$-\operatorname{div} \sigma(\mathbf{v}) + \eta \mathbf{v} + \nabla p = -(\xi \mathbf{v} + \mathbf{U} \cdot \nabla \mathbf{v} + \frac{1}{2} \operatorname{div}(\mathbf{U}) \mathbf{v}) + \mathbf{v}^* \quad (5.41)$$

in L^2 -sense. And so (p, \mathbf{v}) satisfies the coupled system (5.12) and (5.11) pointwise. (See the remark below (5.22)).

Finally, since $[-\operatorname{div} \sigma(\mathbf{v}) + \nabla p] \in \mathbf{L}^2(\mathcal{O})$, and $(p, \mathbf{v}) \in L^2(\mathcal{O}) \times \mathbf{H}^1(\mathcal{O})$, a classic integration by parts argument will yield the following trace regularity:

$$[\sigma(\mathbf{v}) \mathbf{n} - p \mathbf{n}] \in \mathbf{H}^{-1/2}(\partial\mathcal{O}) \quad (5.42)$$

(see e.g., Theorem 13.2.3, p.326, of [2]). Integrating by parts in (5.39), we consequently have for all $\psi \in \mathbf{V}_0$.

$$\begin{aligned} (\mathbf{v}^*, \psi)_{\mathcal{O}} &= (\xi \mathbf{v} + \mathbf{U} \cdot \nabla \mathbf{v} + \frac{1}{2} \operatorname{div}(\mathbf{U}) \mathbf{v}, \psi)_{\mathcal{O}} - (\operatorname{div} \sigma(\mathbf{v}) - \nabla p, \psi)_{\mathcal{O}} \\ &\quad + \eta(\mathbf{v}, \psi)_{\mathcal{O}} + \langle \sigma(\mathbf{v}) \mathbf{n} - p \mathbf{n}, \psi \rangle_{\partial \mathcal{O}}, \end{aligned}$$

or, after invoking (5.41):

$$\langle \sigma(\mathbf{v}) \mathbf{n} - p \mathbf{n}, \psi \rangle_{\partial \mathcal{O}} = 0 \text{ for every } \psi \in \mathbf{V}_0.$$

This orthogonality and the surjectivity of the trace mapping from $\mathbf{H}^1(\mathcal{O}) \rightarrow \mathbf{H}^{1/2}(\partial \mathcal{O})$, allow us to deduce that the obtained solution pair (p, \mathbf{v}) satisfies the boundary condition

$$[\sigma(\mathbf{v}) \mathbf{n} - p \mathbf{n}] \cdot \boldsymbol{\tau} = 0 \text{ for all } \boldsymbol{\tau} \in TH^{1/2}(\partial \mathcal{O}),$$

in *weak sense*; i.e., as in (3.6) of Remark 3.1.

Lastly, the necessary continuous dependence estimates (5.19)–(5.20) come from collecting (5.25) and (5.30)–(5.32) – for p in (5.19); and (5.24), (5.26), (5.38), and (5.35) – for \mathbf{v} in (5.20). This concludes the proof of Lemma 5.1, and so STEP 1 of the maximality argument. \square

STEP 2

With Lemma 5.1 in hand, we properly deal with the coupled fluid-structure PDE system (5.9)–(5.10). Our solution here will be predicated on finding the structural variable w_1 which solves the Ω -problem

$$\xi^2 w_1 + \Delta^2 w_1 + [2\nu \partial_{x_3}(\mathbf{v})_3]_{\Omega} + \lambda \operatorname{div}(\mathbf{v})|_{\Omega} - p|_{\Omega} = w_2^* + \xi w_1^* \quad \text{on } \Omega \quad (5.43)$$

$$w_1|_{\partial \Omega} = \partial_{\nu} w_1|_{\partial \Omega} = 0. \quad (5.44)$$

Let $(p^*, \mathbf{v}^*) \in L^2(\mathcal{O}) \times \mathbf{L}^2(\mathcal{O})$ be the pressure and fluid data from (5.9). Let $z \in H_0^2(\Omega)$ be given. Then from Lemma 5.1, we know that the following problem has a unique solution $\{p(z; p^*; \mathbf{v}^*), \mathbf{v}(z; p^*; \mathbf{v}^*)\}$:

$$\begin{aligned} \xi p(z; p^*; \mathbf{v}^*) + \mathbf{U} \cdot \nabla p(z; p^*; \mathbf{v}^*) + \frac{1}{2} \operatorname{div}(\mathbf{U}) p(z; p^*; \mathbf{v}^*) + \operatorname{div}(\mathbf{v}(z; p^*; \mathbf{v}^*)) &= p^* \quad \text{in } \mathcal{O} \\ \xi \mathbf{v}(z; p^*; \mathbf{v}^*) + \mathbf{U} \cdot \nabla \mathbf{v}(z; p^*; \mathbf{v}^*) + \frac{1}{2} \operatorname{div}(\mathbf{U}) \mathbf{v}(z; p^*; \mathbf{v}^*) \\ &\quad - \operatorname{div} \sigma(\mathbf{v}(z; p^*; \mathbf{v}^*)) + \eta \mathbf{v}(z; p^*; \mathbf{v}^*) + \nabla p(z; p^*; \mathbf{v}^*) = \mathbf{v}^* \quad \text{in } \mathcal{O} \\ (\sigma(\mathbf{v}(z; p^*; \mathbf{v}^*)) \mathbf{n} - p(z; p^*; \mathbf{v}^*) \mathbf{n}) \cdot \boldsymbol{\tau} &= 0 \quad \text{on } \partial \mathcal{O} \\ \mathbf{v}(z; p^*; \mathbf{v}^*) \cdot \mathbf{n} &= 0 \quad \text{on } S \\ \mathbf{v}(z; p^*; \mathbf{v}^*) \cdot \mathbf{n} &= z \quad \text{on } \Omega. \end{aligned} \quad (5.45)$$

Akin to what was done in STEP 1, we decompose the solution of the BVP (5.45) into two parts:

$$\mathbf{v}(z; p^*; \mathbf{v}^*) = \mathbf{v}(\mathbf{z}) + \mathbf{v}(p^*; \mathbf{v}^*); \quad (5.46)$$

$$p(z; p^*; \mathbf{v}^*) = p(z) + p(p^*; \mathbf{v}^*), \quad (5.47)$$

where $(p(z), \mathbf{v}(z)) \in \mathbf{H}^1(\mathcal{O}) \times L^2(\mathcal{O})$ is the solution of the problem

$$\xi p(z) + \mathbf{U} \cdot \nabla p(z) + \frac{1}{2} \operatorname{div}(\mathbf{U})p(z) + \operatorname{div}(\mathbf{v}(z)) = 0 \quad \text{in } \mathcal{O} \quad (5.48)$$

$$\xi \mathbf{v}(z) + \mathbf{U} \cdot \nabla \mathbf{v}(z) + \frac{1}{2} \operatorname{div}(\mathbf{U})\mathbf{v}(z) - \operatorname{div} \sigma(\mathbf{v}(z)) + \eta \mathbf{v}(z) + \nabla p(z) = 0 \quad \text{in } \mathcal{O} \quad (5.49)$$

$$(\sigma(\mathbf{v}(z))\mathbf{n} - p(z)\mathbf{n}) \cdot \boldsymbol{\tau} = 0 \quad \text{on } \partial\mathcal{O} \quad (5.50)$$

$$\mathbf{v}(z) \cdot \mathbf{n} = 0 \quad \text{on } S \quad (5.51)$$

$$\mathbf{v}(z) \cdot \mathbf{n} = z \quad \text{on } \Omega, \quad (5.52)$$

and $(p(p^*, \mathbf{v}^*), \mathbf{v}(p^*; \mathbf{v}^*)) \equiv (\bar{p}, \bar{\mathbf{v}}) \in L^2(\mathcal{O}) \times \mathbf{V}_0$ is the solution of the problem

$$\xi \bar{p} + \mathbf{U} \cdot \nabla \bar{p} + \frac{1}{2} \operatorname{div}(\mathbf{U})\bar{p} + \operatorname{div} \bar{\mathbf{v}} = p^* \quad \text{in } \mathcal{O} \quad (5.53)$$

$$\xi \bar{\mathbf{v}} + \mathbf{U} \cdot \nabla \bar{\mathbf{v}} + \frac{1}{2} \operatorname{div}(\mathbf{U})\bar{\mathbf{v}} - \operatorname{div} \sigma(\bar{\mathbf{v}}) + \eta \bar{\mathbf{v}} + \nabla \bar{p} = \mathbf{v}^* \quad \text{in } \mathcal{O} \quad (5.54)$$

$$(\sigma(\bar{\mathbf{v}})\mathbf{n} - p\mathbf{n}) \cdot \boldsymbol{\tau} = 0 \quad \text{on } \partial\mathcal{O} \quad (5.55)$$

$$\bar{\mathbf{v}} \cdot \mathbf{n} = 0 \quad \text{on } S \quad (5.56)$$

$$\bar{\mathbf{v}} \cdot \mathbf{n} = 0 \quad \text{on } \Omega. \quad (5.57)$$

Therewith: if we multiply the structural PDE component (5.43)—in solution variables $(p, \mathbf{v}, w_1, w_2)$ —by given $z \in H_0^2(\Omega)$, with associated fluid-pressure solution $(p(z), \mathbf{v}(z))$ of (5.48)–(5.52), integrate by parts, and utilize the boundary conditions in the BVP (5.48)–(5.52), we then have:

$$\begin{aligned} \langle w_2^* + \xi w_1^*, z \rangle_\Omega &= \xi^2 \langle w_1, z \rangle_\Omega + \langle \Delta w_1, \Delta z \rangle_\Omega \\ &\quad + \langle \sigma(\mathbf{v})\mathbf{n} - p\mathbf{n}, \mathbf{v}(z) \rangle_{\partial\mathcal{O}} \\ &\quad (\text{after using (5.16)–(5.18) of Lemma 5.1}) \\ &= \xi^2 \langle w_1, z \rangle_\Omega + \langle \Delta w_1, \Delta z \rangle_\Omega + (\operatorname{div}(\sigma(\mathbf{v})) - \nabla p, \mathbf{v}(z))_\mathcal{O} \\ &\quad + (\sigma(\mathbf{v}), \epsilon(\mathbf{v}(z)))_\mathcal{O} - (p, \operatorname{div}(\mathbf{v}(z)))_\mathcal{O} \\ &= \xi^2 \langle w_1, z \rangle_\Omega + \langle \Delta w_1, \Delta z \rangle_\Omega + \xi(\mathbf{v}, \mathbf{v}(z))_\mathcal{O} + (\mathbf{U} \cdot \nabla \mathbf{v}, \mathbf{v}(z))_\mathcal{O} \\ &\quad + (\sigma(\mathbf{v}), \epsilon(\mathbf{v}(z)))_\mathcal{O} + \frac{1}{2}(\operatorname{div}(\mathbf{U})\mathbf{v}, \mathbf{v}(z))_\mathcal{O} \\ &\quad + \eta(\mathbf{v}, \mathbf{v}(z))_\mathcal{O} - (p, \operatorname{div}(\mathbf{v}(z)))_\mathcal{O} - (\mathbf{v}^*, \mathbf{v}(z))_\mathcal{O}. \end{aligned} \quad (5.58)$$

Now, using the first resolvent relation in (5.10) and invoking the respective solution maps for (5.48)–(5.52) and (5.53)–(5.57), we may express the (prospective) solution component (p, \mathbf{v}) of (5.9) as

$$\mathbf{v} = \mathbf{v}(\xi w_1 - w_1^*; p^*; \mathbf{v}^*) = \mathbf{v}(\xi w_1 - w_1^*) + \bar{\mathbf{v}} \quad (5.59)$$

$$p = p(\xi w_1 - w_1^*; p^*; \mathbf{v}^*) = p(\xi w_1 - w_1^*) + \bar{p} \quad (5.60)$$

(cf. (5.46)–(5.47)). With (5.58) and (5.59)–(5.60) in mind: Accordingly, if we define an operator $B \in \mathcal{L}(H_0^2(\Omega), H^{-2}(\Omega))$ as

$$\begin{aligned} (B(w), z) &\equiv \xi^2 \langle w, z \rangle_\Omega + \langle \Delta w, \Delta z \rangle_\Omega \\ &\quad + \xi^2 (\mathbf{v}(w), \mathbf{v}(z))_\mathcal{O} + \xi (\mathbf{U} \cdot \nabla \mathbf{v}(w), \mathbf{v}(z))_\mathcal{O} + \frac{\xi}{2} (\operatorname{div}(\mathbf{U})\mathbf{v}(w), \mathbf{v}(z))_\mathcal{O} \\ &\quad + (\xi \sigma(\mathbf{v}(w)), \epsilon(\mathbf{v}(z)))_\mathcal{O} + \eta \xi (\mathbf{v}(w), \mathbf{v}(z))_\mathcal{O} - \xi (p(w), \operatorname{div}(\mathbf{v}(z)))_\mathcal{O}, \end{aligned} \quad (5.61)$$

– where $(p(w), \mathbf{v}(w))$ solves (5.48)–(5.52) with $H_0^2(\Omega)$ boundary data w – then finding solution $w_1 \in H_0^2(\Omega)$ of the structural PDE component (5.43)–(5.44) is tantamount to finding solution $w_1 \in H_0^2(\Omega)$ of the variational equation

$$\langle B(w_1), z \rangle_{(H^{-2} \times H_0^2)(\Omega)} = \mathcal{F}(z), \text{ for all } z \in H_0^2(\Omega); \quad (5.62)$$

where the functional $\mathcal{F} \in H^{-2}(\Omega)$ is given by

$$\begin{aligned} \mathcal{F}(z) \equiv & (\mathbf{v}^*, \mathbf{v}(z))_{\mathcal{O}} + \langle w_2^* + \xi w_1^*, z \rangle_{\Omega} \\ & + \xi(\mathbf{v}(w_1^*), \mathbf{v}(z))_{\mathcal{O}} - \xi(\bar{\mathbf{v}}, \mathbf{v}(z))_{\mathcal{O}} \\ & + (\mathbf{U} \cdot \nabla \mathbf{v}(w_1^*), \mathbf{v}(z))_{\mathcal{O}} - (\mathbf{U} \cdot \nabla \bar{\mathbf{v}}, \mathbf{v}(z))_{\mathcal{O}} \\ & + \frac{1}{2}(\text{div}(\mathbf{U})\mathbf{v}(w_1^*), \mathbf{v}(z))_{\mathcal{O}} - \frac{1}{2}(\text{div}(\mathbf{U})\bar{\mathbf{v}}, \mathbf{v}(z))_{\mathcal{O}} \\ & + \eta(\mathbf{v}(w_1^*), \mathbf{v}(z))_{\mathcal{O}} - \eta(\bar{\mathbf{v}}, \mathbf{v}(z))_{\mathcal{O}} \\ & + (\sigma(\mathbf{v}(w_1^*)), \epsilon(\mathbf{v}(z)))_{\mathcal{O}} - (\sigma(\bar{\mathbf{v}}), \epsilon(\mathbf{v}(z)))_{\mathcal{O}} \\ & - (p(w_1^*), \text{div}(\mathbf{v}(z)))_{\mathcal{O}} + (\bar{p}, \text{div}(\mathbf{v}(z)))_{\mathcal{O}}. \end{aligned}$$

Recall that by Lemma 5.1(iv), one has the following estimate for the pressure term in (5.61), for $\xi = \xi(\mathbf{U})$ large enough:

$$\|p(w)\|_{L^2(\mathcal{O})} \leq \frac{C}{\xi} \|w\|_{H_0^2(\Omega)}, \quad \text{for all } w \in H_0^2(\Omega).$$

By means of this estimate and Korn's inequality, we will have, in a manner analogous to that in the proof of Lemma 5.1, that the operator B is $H_0^2(\Omega)$ -elliptic, for $\xi = \xi(\mathbf{U})$ large enough. Thus we can use the Lax-Milgram Theorem to solve the variational equation (5.62), or what is the same, recover the solution component w_1 of the resolvent equations (5.43)–(5.44). In turn, we will have

$$\begin{aligned} w_2 &= \xi w_1 - w_1^*, \\ \mathbf{v} &= \mathbf{v}(w_2) + \bar{\mathbf{v}}, \\ p &= p(w_2) + \bar{p}. \end{aligned}$$

where again $[p(w_2), \mathbf{v}(w_2)]$ is the solution to (5.48)–(5.52), and $[\bar{p}, \bar{\mathbf{v}}]$ solves the system (5.53)–(5.57).

This finally establishes the range condition in (5.7) for $\xi > 0$ sufficiently large. A subsequent application of Lumer-Philips Theorem yields a contraction semigroup for the $\hat{\mathcal{A}}: D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$. As a consequence, the application of Theorem 1.1 [41, Chapter 3.1], p.76, gives the desired result for the (unperturbed) compressible flow-structure generator \mathcal{A} .

6 Stationary Problem

Since the stationary problem associated with a dissipative dynamical system is of interest when studying long-time behavior of solutions [22], we discuss the linear stationary problem associated with (2.4)–(2.6). We briefly discuss the inclusion of nonlinearity in the plate for the stationary problem in

Section 7. (Such a discussion is in line with [18, p.658].) Formally, we introduce the following problem:

$$\begin{cases} \mathbf{U} \cdot \nabla p + \operatorname{div} \mathbf{u} = 0 & \text{in } \mathcal{O} \times (0, \infty) \\ \mathbf{U} \cdot \nabla \mathbf{u} - \operatorname{div} \sigma(\mathbf{u}) + \eta \mathbf{u} + \nabla p = 0 & \text{in } \mathcal{O} \times (0, \infty) \\ (\sigma(\mathbf{u})\mathbf{n} - p\mathbf{n}) \cdot \boldsymbol{\tau} = 0 & \text{on } \partial\mathcal{O} \times (0, \infty) \\ \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \partial\mathcal{O} \times (0, \infty) \end{cases} \quad (6.1)$$

$$\begin{cases} \Delta^2 w + [2\nu \partial_{x_3}(\mathbf{u})_3 + \lambda \operatorname{div}(\mathbf{u}) - p]_{\Omega} = 0 & \text{on } \Omega \times (0, \infty) \\ w = \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty) \end{cases} \quad (6.2)$$

(We note here, as in Remark 3.1, the boundary condition $(\sigma(\mathbf{u})\mathbf{n} - p\mathbf{n}) \cdot \boldsymbol{\tau} = 0$ —for $\boldsymbol{\tau} \in TH^{1/2}(\partial\mathcal{O})$ —is to be interpreted in the sense of distributions.)

Note that in terms of the fluid-structure generator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$, solving the PDE system (6.1)–(6.2) is equivalent to identifying an element in $\operatorname{Null}(\mathcal{A})$. Alternatively, as in [18], the problem (6.1)–(6.2) can be interpreted variationally. This is to say, we define a *weak solution* to (6.1)–(6.2) to be a triple $(p, \mathbf{u}, w) \in L^2(\mathcal{O}) \times \mathbf{V}_0 \times H_0^2(\Omega)$, which must satisfy,

$$(\operatorname{div} \mathbf{u}, q)_{\mathcal{O}} = \int_{\mathcal{O}} (\operatorname{div} \mathbf{U})(pq) dx + \int_{\mathcal{O}} (\mathbf{U} \cdot \nabla q) p dx \quad (6.3)$$

$$a_{\mathcal{O}}(\mathbf{u}, \boldsymbol{\psi}) - (p, \operatorname{div} \boldsymbol{\psi})_{\mathcal{O}} = - \int_{\mathcal{O}} (\mathbf{U} \cdot \nabla \mathbf{u}) \boldsymbol{\psi} dx - (\Delta w, \Delta \beta)_{\Omega}, \quad (6.4)$$

where the bilinear form $a_{\mathcal{O}}(\cdot, \cdot)$ is as given in (3.7) for all $q \in W_{\mathbf{U}}$ and all $\boldsymbol{\psi} \in \mathbf{V}_{\Omega}$. We take:

$$W_{\mathbf{U}} = \{q \in L^2(\mathcal{O}) : \mathbf{U} \cdot \nabla q \in L^2(\mathcal{O})\}. \quad (6.5)$$

$$\mathbf{V}_{\Omega} \equiv \left\{ \mathbf{v} = \mathbf{v}(\beta) \in \mathbf{H}^1(\mathcal{O}) : [\mathbf{v} \cdot \mathbf{n}]_{\partial\mathcal{O}} = \begin{cases} 0 & \text{on } S \\ \beta \in H_0^2(\Omega) & \text{on } \Omega. \end{cases} \right\} \quad (6.6)$$

Note that by Theorem 5 of [12], and extension by zero of $\beta \in H_0^2(\Omega)$, the space \mathbf{V}_{Ω} is well-defined on the Lipschitz geometry of \mathcal{O} .

We note that (6.3)–(6.4) is a *natural* definition of a weak (variational) solution, in line with *weak* solutions [18] to the dynamic equations (2.4)–(2.6). Indeed, we demonstrate that such weak solutions—**should they exist**—are classical solutions of the PDE system (6.1)–(6.2).

Lemma 6.1. *Suppose that $(p, \mathbf{u}, w) \in L^2(\mathcal{O}) \times \mathbf{V}_0 \times H_0^2(\Omega)$ and satisfies (6.3)–(6.4). Then (p, \mathbf{u}, w) satisfies (6.1)–(6.2) almost everywhere, and in fact $[p, \mathbf{u}, w, 0] \in \operatorname{Null}(\mathcal{A})$.*

Proof. Here, we essentially mimic the final part of the proof of Lemma 5.1.

Firstly, in (6.3) we consider $q \in \mathcal{D}(\mathcal{O})$. An invocation of Green's Theorem then yields

$$\begin{aligned} (\operatorname{div}(\mathbf{u}), q)_{\mathcal{O}} &= \int_{\mathcal{O}} \operatorname{div}(\mathbf{U})(pq) dx + 0 - \left(\int_{\mathcal{O}} \operatorname{div}(\mathbf{U})(qp) dx + \int_{\mathcal{O}} (\mathbf{U} \cdot \nabla p) q dx \right) \\ &= - \int_{\mathcal{O}} (\mathbf{U} \cdot \nabla p) q dx \quad \text{for every } q \in \mathcal{D}(\mathcal{O}). \end{aligned}$$

The density of $\mathcal{D}(\mathcal{O})$ in $L^2(\mathcal{O})$ then yields that

$$\mathbf{U} \cdot \nabla p + \operatorname{div}(\mathbf{u}) = 0 \quad \text{in the } L^2(\mathcal{O})\text{-sense} \quad (6.7)$$

(and so, with $\mathbf{u} \in \mathbf{V}_0$, in particular, $p \in W_{\mathbf{U}}$ of (6.5)).

Secondly, if in (6.4) we take test function $\psi \in [\mathcal{D}(\mathcal{O})]^3$, then we infer, upon integration by parts, that

$$-\operatorname{div} \sigma(\mathbf{u}) + \eta \mathbf{u} + \nabla p + \mathbf{U} \cdot \nabla \mathbf{u} = 0 \quad \text{in } \mathbf{L}^2(\mathcal{O})\text{-sense.} \quad (6.8)$$

Subsequently, from (6.8), the fact that $\{p, \mathbf{u}\} \in L^2(\mathcal{O}) \times \mathbf{H}^1(\mathcal{O})$, and by an integration by parts, we can (as before) assign a meaning to the boundary trace term $[\sigma(\mathbf{u})\mathbf{n} - p\mathbf{n}]_{\partial\mathcal{O}}$, viz.,

$$\sigma(\mathbf{u})\mathbf{n} - p\mathbf{n} \in \mathbf{H}^{-\frac{1}{2}}(\partial\mathcal{O}). \quad (6.9)$$

Applying this boundary trace to the relation (6.4), we then have for a test function $\psi \in \mathbf{V}_0 \subset \mathbf{V}_\Omega$, upon an integration by parts (and considering (6.8)) we make the inference

$$\langle \sigma(\mathbf{u})\mathbf{n} - p\mathbf{n}, \psi \rangle_{\partial\mathcal{O}} = 0 \quad \text{for every } \psi \in \mathbf{V}_0, \quad (6.10)$$

Thus, the following tangential boundary condition is satisfied:

$$[\sigma(\mathbf{u})\mathbf{n} - p\mathbf{n}] \cdot \boldsymbol{\tau} = 0 \quad \text{for all } \boldsymbol{\tau} \in TH^{1/2}(\partial\mathcal{O}).^6 \quad (6.11)$$

Thirdly, with respect to (6.4), we have upon integration by parts with variational relation, and an invocation of (6.8),

$$\langle \sigma(\mathbf{u})\mathbf{n} - p\mathbf{n}, \psi \rangle_{\partial\mathcal{O}} = -(\Delta w, \Delta \beta)_\Omega \quad \text{for every } \psi \in \mathbf{V}_\Omega. \quad (6.12)$$

In particular, if for given $\beta \in H_0^2(\Omega)$, we set

$$\psi = \begin{cases} 0 & \text{in } S, \\ \beta \mathbf{n} & \text{in } \Omega \end{cases}, \quad (6.13)$$

then this $\psi \in \mathbf{V}_\Omega$ (see e.g., Theorem 3.33 of [38]). Applying this test function in (6.12) to (6.12) – and using $\mathbf{n}|_\Omega = [0, 0, 1]$ —we have

$$([2\nu\partial_{x_3}(\mathbf{u})_3 + \lambda\operatorname{div}(\mathbf{u}) - p]_\Omega, \beta)_\Omega = -(\Delta w, \Delta \beta)_\Omega \quad \text{for every } \beta \in H_0^2(\Omega).$$

In particular, this holds for $\beta \in \mathcal{D}(\Omega)$, whence we obtain

$$\Delta^2 w + [2\nu\partial_{x_3}(\mathbf{u})_3 + \lambda\operatorname{div}(\mathbf{u}) - p] = 0 \quad \text{in } L^2(\Omega)\text{-sense.} \quad (6.14)$$

Upon collecting (6.7), (6.8), (6.11) and (6.14), we then have that variational solution $[p, \mathbf{u}, w]$ of (6.3)–(6.4) solves the coupled PDE system (6.1)–(6.2) a.e. (Note that since solution component $\mathbf{u} \in \mathbf{V}_0$, then its normal component on $\partial\mathcal{O}$ is zero.) \square

We note that for any ψ and β as above, we can utilize the weak identities to see that a stationary solution of (6.3)–(6.4) must satisfy, for all $q \in W_{\mathbf{U}}$ and $\psi \in \mathbf{V}_\Omega$,

$$\begin{aligned} a_{\mathcal{O}}(\mathbf{u}, \psi) - (p, \operatorname{div} \psi)_{\mathcal{O}} + (\Delta w, \Delta \beta)_\Omega + (\operatorname{div} \mathbf{u}, q) \\ = - \int_{\mathcal{O}} (\mathbf{U} \cdot \nabla \mathbf{u}) \psi dx + \int_{\mathcal{O}} (\operatorname{div} \mathbf{U})(pq) dx + \int_{\mathcal{O}} (\mathbf{U} \cdot \nabla q) p dx. \end{aligned} \quad (6.15)$$

⁶See again Remark 3.1.

Since $p \in W_{\mathbf{U}}$ (as we saw in the proof of Lemma 6.1), we may choose $q = p$ and $\psi = \mathbf{u} \in \mathbf{V}_0$ in (6.15), invoke Green's theorem and the divergence theorem as above, to see that

$$a_{\mathcal{O}}(\mathbf{u}, \mathbf{u}) = \frac{1}{2} \int_{\mathcal{O}} (\operatorname{div} \mathbf{U}) [|p|^2 + |\mathbf{u}|^2] dx. \quad (6.16)$$

As discussed above, it is clear that the long-time behavior (stability) properties of solutions to the dynamics (6.1)–(6.2) depend on the structure of the flow field \mathbf{U} . Thus, from (6.16) it is not an unwelcome assumption (see [7]) to consider a divergence-free flow field \mathbf{U} . This yields the following theorem.

Lemma 6.2. *If the ambient vector field \mathbf{U} is divergence free, a weak solution to (6.3)–(6.4) (or equivalently, a solution to (6.1)–(6.2)) will be a triple of the form $(c, \mathbf{0}, w_c)$, with $c = \text{const}$, and where $w_c \in H_0^2(\Omega)$ solves the boundary value problem*

$$\Delta^2 w = c \quad \text{in } \Omega, \quad w = \nabla w = 0 \quad \text{on } \partial\Omega. \quad (6.17)$$

Proof. If $\operatorname{div}(\mathbf{U}) = 0$, then from (6.16) and Korn's Inequality we have that $\mathbf{u} = 0$. Subsequently, the fluid equation in (6.1) gives $\nabla p = 0$, and so $p = c$. In turn, the structural equation in (6.2) with such $\{\mathbf{u}, p\}$ becomes (6.17). \square

7 Plate Nonlinearity

The treatment of semilinear, cubic-type nonlinearities in fluid-plate problems has become popular (see the surveys [29, 25] and references therein). In this section we demonstrate well-posedness of mild solutions to the dynamic problem, as well as discuss the stationary problem, *in the presence of the scalar von Karman nonlinearity* [22]. We begin with some basic facts about the von Karman nonlinearity, introduced in (2.12)–(2.13). The first of which revolves around the *local Lipschitz* property of f from $H_0^2(\Omega) \rightarrow L^2(\Omega)$. By way of availing ourselves of said Lipschitz continuity for von Karman plates, we further assume that bounded $\Omega \subset \mathbb{R}^2$ is sufficiently smooth.

This property relies on the so called sharp regularity of the Airy stress function: Corollary 1.4.4 in [22]. To begin, one has the estimate,

$$\| (\Delta_D^2)^{-1} [u, w] \|_{W^{2,\infty}(\Omega)} \leq C \|u\|_{2,\Omega} \|w\|_{2,\Omega}, \quad (7.1)$$

where Δ_D^2 denotes the biharmonic operator with clamped boundary conditions. With $v(w) = v(w, w)$ in (2.13), the estimate (7.1) above yields

$$\|v(w)\|_{W^{2,\infty}(\Omega)} \leq C \|w\|_{2,\Omega}^2,$$

which, in turn, implies that the Airy stress function $v(w)$ satisfies the inequality

$$\|[u_1, v(u_1)] - [u_2, v(u_2)]\|_{L^2(\Omega)} \leq C \left(\|u_1\|_{H_0^2(\Omega)}^2 + \|u_2\|_{H_0^2(\Omega)}^2 \right) \|u_1 - u_2\|_{H_0^2(\Omega)} \quad (7.2)$$

(see Corollary 1.4.5 in [22]). Thus, the nonlinearity $f(w) = [w, v(w) + F_0]$ is locally Lipschitz from $H_0^2(\Omega)$ into $L^2(\Omega)$.

The second critical property of the nonlinearity involves the existence of a potential energy functional Π associated with f . In the case of the von Karman nonlinearity, it has the form

$$\Pi(w) = \frac{1}{4} \int_{\Omega} \left(|\Delta v(w)|^2 - 2w[w, F_0] \right) dx,$$

and possesses the properties that Π is a C^1 -functional on $H_0^2(\Omega)$ such that f is a Fréchet derivative of Π : $-f(w) = \Pi'(w)$. From this it follows that for a smooth function w :

$$\frac{d}{dt} \Pi(w) = (\Pi'(w), w_t) = -(f(w), w_t)_{\Omega}.$$

Moreover $\Pi(\cdot)$ is locally bounded on $H_0^2(\Omega)$, and there exist $\eta < 1/2$ and $C \geq 0$ such that

$$\eta \|\Delta w\|_{\Omega}^2 + \Pi(w) + C \geq 0, \quad \forall w \in H_0^2(\Omega). \quad (7.3)$$

The latter fact follows from the bound [22, Chapter 1.4]

$$\|w\|_{\theta}^2 \leq \epsilon [\|\Delta w\|^2 + \|\Delta v(w)\|^2] + C_{\epsilon}, \quad \theta \in [0, 2).$$

Remark 7.1. We note that the Berger and Kirchhoff nonlinearities, for instance discussed in [23, 18], satisfy the above properties; they: (i) are locally Lipschitz $H_0^2(\Omega) \rightarrow L^2(\Omega)$, (ii) have a C^1 antiderivative Π satisfying the above properties.

7.1 Nonlinear Dynamic Problem

We now address the system (2.4)–2.6, taken with active plate nonlinearity:

$$w_{tt} + \Delta^2 w + [2\nu \partial_{x_3}(\mathbf{u})_3 + \lambda \operatorname{div}(\mathbf{u}) - p]_{\Omega} = [w, v(w) + F_0] \quad \text{on } \Omega \times (0, \infty).$$

We will show the well-posedness of *mild solutions* (in the sense of [41]) in the presence of the von Karman nonlinearity. To this end, we define a nonlinear operator $\mathcal{F} : \mathcal{H} \rightarrow \mathcal{H}$, given by

$$\mathcal{F}([p, \mathbf{u}, w_1, w_2]) = [0, \mathbf{0}, 0, f(w_1)].$$

This mapping is locally Lipschitz (by the properties of f above), and thus will be considered as a perturbation to the linear fluid-structure Cauchy problem which is modelled by generator $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{H}$. In particular we have the abstract problem in variable $\mathbf{y} \in \mathcal{H}$,

$$\mathbf{y}' = \mathcal{A}\mathbf{y} + \mathcal{F}(\mathbf{y}) \quad (7.4)$$

$$\mathbf{y}(0) = \mathbf{y}_0 \in \mathcal{H}. \quad (7.5)$$

Theorem 7.1. *The nonlinear Cauchy problem in (7.4)–(7.5) is well-posed in the sense of mild solutions. This is to say: there is a unique local-in-time mild solution $\mathbf{y}(t)$ on $t \in [0, t_{\max})$ (which is also a weak solution). Moreover, for $\mathbf{y}_0 \in D(\mathcal{A})$, the corresponding solution is strong.*

In either case, when $t_{\max}(\mathbf{y}_0) < \infty$, we have that $\|\mathbf{y}(t)\|_{\mathcal{H}} \rightarrow \infty$ as $t \nearrow t_{\max}(\mathbf{y}_0)$.

Proof. With the fluid-structure semigroup $e^{\mathcal{A}t}$ in hand from Theorem 3.1, this is a direct application of Theorem 1.4 [41, p.185] and localized version of Theorem 1.6 [41, p.189], pertaining to locally Lipschitz perturbations of semigroup solutions. \square

In order to guarantee global solutions, i.e., valid solutions of (7.4)–(7.5) on $[0, T]$ for any $T > 0$, we must utilize the “good” structure of Π . The energy identity, in the presence of nonlinearity (i.e., when (2.5) has the term $f(w)$), is obtained in a standard way using the properties of Π (see [23, 18]). Consider $\mathbf{y}_0 = (p_0, \mathbf{u}_0, w_0, w_1) \in \mathcal{H}$ and $\mathbf{U} \in \mathbf{V}_0$. Any mild solution corresponding to (7.4)–(7.5) satisfies:

$$\begin{aligned} \mathcal{E}(p(t), \mathbf{u}(t), w(t), w_t(t)) + \int_0^t a_{\mathcal{O}}(\mathbf{u}(\tau), \mathbf{u}(\tau)) d\tau + \Pi(w(t)) &= \mathcal{E}(p_0, \mathbf{u}_0, w_0, w_1) + \Pi(w(0)) \\ &+ \frac{1}{2} \int_0^t \int_{\mathcal{O}} \operatorname{div}(\mathbf{U}) [|p(\tau)|^2 + |\mathbf{u}(\tau)|^2] dx d\tau. \end{aligned} \quad (7.6)$$

From this a priori relation, Gronwall’s inequality, and the bound on the Π in (7.3), we have the final (nonlinear) well-posedness theorem.

Theorem 7.2. *For any $T > 0$, the Cauchy problem (7.4)–(7.5) is well-posed on \mathcal{H} for all $[0, T]$. This is to say that the PDE problem in (2.4)–(2.6), taking into account the nonlinear plate equation (2.12), is well-posed in the sense of mild solutions.*

Moreover, in the case of $\operatorname{div} \mathbf{U} \equiv 0$, we have the global-in-time estimate for solutions:

$$\sup_{t \in [0, \infty)} \mathcal{E}(p(t), \mathbf{u}(t), w(t), w_t(t)) \leq \mathbf{C}(p_0, \mathbf{u}_0, w_0, w_1, F_0). \quad (7.7)$$

Proof. The proof follows a standard tack, and is along the lines of [18] (in the case of these dynamics, taken with $\mathbf{U} = 0$), or [23, 42] (for the case of compressible, inviscid gas dynamics). See also [22, Chapter 2.3] for an abstract discussion of nonlinear second order evolutions with locally Lipschitz perturbations. \square

7.2 Nonlinear Stationary Problem

We now briefly mention the nonlinear stationary problem in the case when $\operatorname{div} \mathbf{U} \equiv 0$. As noted above (and, as is evident from (7.6) and (7.7)), this is the primary case of interest for *long time behavior* analysis of the dynamics (2.4)–(2.5).

We note that the analysis above in Section 6 obtains identically in the presence of plate nonlinearity. Thus, with \mathbf{U} divergence free as above, we have the equivalence of weak solutions to the system

$$\begin{cases} \mathbf{U} \cdot \nabla p + \operatorname{div} \mathbf{u} = 0 & \text{in } \mathcal{O} \times (0, \infty) \\ \mathbf{U} \cdot \nabla \mathbf{u} - \operatorname{div} \sigma(\mathbf{u}) + \eta \mathbf{u} + \nabla p = 0 & \text{in } \mathcal{O} \times (0, \infty) \\ (\sigma(\mathbf{u}) \mathbf{n} - p \mathbf{n}) \cdot \boldsymbol{\tau} = 0 & \text{on } \partial \mathcal{O} \times (0, \infty) \\ \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } S \times (0, \infty) \\ \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \Omega \times (0, \infty) \end{cases} \quad (7.8)$$

$$\begin{cases} \Delta^2 w + [2\nu \partial_{x_3}(\mathbf{u})_3 + \lambda \operatorname{div}(\mathbf{u}) - p]_{\Omega} = [w, v(w) + F_0] & \text{on } \Omega \times (0, \infty) \\ w = \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, \infty) \end{cases} \quad (7.9)$$

and the following biharmonic problem: Find $w \in H_0^2(\Omega)$ such that

$$(\Delta w, \Delta v)_{\Omega} - (f(w), v) = (c, v)_{\Omega}, \quad \forall v \in H_0^2(\Omega). \quad (7.10)$$

(Note, this reduction is equivalent to that in [18, Section 3.3].) With property (7.3) of Π , it is well known that (for a given c) the solutions to (7.10), denoted by \mathcal{N}_c , form a nonempty, compact set in $H_0^2(\Omega)$ [18, 28, 22]⁷. This leaves us with the final theorem:

Theorem 7.3. *Assume $\operatorname{div} \mathbf{U} \equiv 0$. Weak solutions to (7.8)–(7.9) are fully characterized by points of the form:*

$$(c, \mathbf{0}, w_c), \quad w_c \in \mathcal{N}_c \subset H_0^2(\Omega).$$

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9 Appendix

For the reader's convenience, we provide an explicit proof for the well-posedness of the (uncoupled) pressure equation (5.21).

Theorem 9.1. *(See [30] and [37]) Consider the linear equation*

$$k\lambda + \mathbf{v} \cdot \nabla \lambda + \frac{1}{2}\lambda \operatorname{div}(\mathbf{v}) = G \quad \text{in } \mathcal{O}, \quad (9.1)$$

(as before, $\mathcal{O} \subset \mathbb{R}^3$ is a bounded, convex domain). Also, the parameter $k > 0$ and forcing term $G \in L^2(\mathcal{O})$. Moreover, the fixed vector field \mathbf{v} in (9.1) is in $\mathbf{H}^3(\mathcal{O})$ and further satisfies:

$$\begin{aligned} (i) \quad & \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \partial\mathcal{O}; \\ (ii) \quad & 2\|\nabla \mathbf{v}\|_{L^\infty(\mathcal{O})} + \frac{C_S}{2} \operatorname{meas}(\mathcal{O})^{\frac{1}{6}} \|\Delta \operatorname{div}(\mathbf{v})\|_{\mathcal{O}} \leq k, \end{aligned} \quad (9.2)$$

where $C_S > 0$ is a constant which gives rise to the Sobolev embedding inequality,

$$\|f\|_{L^6(\mathcal{O})} \leq \sqrt{C_S} \|\nabla f\|_{\mathcal{O}} \quad \text{for all } f \in H_0^1(\mathcal{O}). \quad (9.3)$$

Then for given $G \in L^2(\mathcal{O})$, there exists a unique $\lambda \in L^2(\mathcal{O})$ which is a weak solution of (9.1). Moreover, the solution satisfies the bound

$$\|\lambda\|_{\mathcal{O}} \leq \frac{1}{k} \|G\|_{\mathcal{O}}. \quad (9.4)$$

By the $L^2(\mathcal{O})$ -function λ being a weak solution of (9.1), we mean that it satisfies the following variational relation:

$$k \int_{\mathcal{O}} \lambda \varphi d\mathcal{O} - \int_{\mathcal{O}} \lambda (\mathbf{v} \cdot \nabla \varphi) d\mathcal{O} - \frac{1}{2} \int_{\mathcal{O}} \lambda \operatorname{div}(\mathbf{v}) \varphi d\mathcal{O} = \int_{\mathcal{O}} G \varphi d\mathcal{O} \quad \text{for every } \varphi \in H^1(\mathcal{O}). \quad (9.5)$$

⁷The structure of \mathcal{N}_c is dependent upon the in-plane forcing F_0

Proof of Theorem 9.1: In large part, the present proof is taken from that of Appendix I of [30] (on p.541)—see also [37]—which however was undertaken on the assumption that geometry \mathcal{O} is smooth. Accordingly, adjustments are made here for general convex domain \mathcal{O} , as well as for the perturbation $\frac{\lambda}{2} \operatorname{div}(\mathbf{v})$.

Given $\epsilon > 0$, we denote λ_ϵ to be the solution of the following regularized boundary value problem:

$$\begin{cases} -\epsilon \Delta \lambda_\epsilon + k \lambda_\epsilon + \mathbf{v} \cdot \nabla \lambda_\epsilon + \frac{1}{2} \lambda_\epsilon \operatorname{div}(\mathbf{v}) = G & \text{in } \mathcal{O}, \\ \lambda_\epsilon|_{\partial \mathcal{O}} = 0. \end{cases} \quad (9.6)$$

Assume initially that data $G \in H_0^1(\mathcal{O})$. We note that the Lax-Milgram Theorem insures the existence of the $H_0^1(\mathcal{O})$ -function λ_ϵ which solves (9.6): Indeed, multiplying the left hand side of the equation in (9.6) by solution variable λ_ϵ , integrating, and then integrating by parts, we have

$$\begin{aligned} & \epsilon \|\nabla \lambda_\epsilon\|_{\mathcal{O}}^2 + \kappa \|\lambda_\epsilon\|_{\mathcal{O}}^2 - \frac{1}{2} \int_{\mathcal{O}} \operatorname{div}(\mathbf{v}) |\lambda_\epsilon|^2 d\mathcal{O} + \frac{1}{2} \int_{\mathcal{O}} \operatorname{div}(\mathbf{v}) |\lambda_\epsilon|^2 d\mathcal{O} \\ &= \epsilon \|\nabla \lambda_\epsilon\|_{\mathcal{O}}^2 + \kappa \|\lambda_\epsilon\|_{\mathcal{O}}^2. \end{aligned} \quad (9.7)$$

Thus we infer $H_0^1(\mathcal{O})$ -ellipticity for the bilinear form associated with the PDE (9.6).

Subsequently, since the bounded domain \mathcal{O} is convex, we can invoke Theorem 3.2.1.2, p.147, of [33] to have that solution $\lambda_\epsilon \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$. This extra regularity allows for a multiplication of both sides of (9.6) by term $\Delta \lambda_\epsilon$, and a subsequent integration and integration by parts, so as to yield the relation

$$-\epsilon \|\Delta \lambda_\epsilon\|_{\mathcal{O}}^2 - \kappa \|\nabla \lambda_\epsilon\|_{\mathcal{O}}^2 + \int_{\mathcal{O}} \Delta \lambda_\epsilon (\mathbf{v} \cdot \nabla \lambda_\epsilon) d\mathcal{O} + \frac{1}{2} \int_{\mathcal{O}} \operatorname{div}(\mathbf{v}) \lambda_\epsilon \Delta \lambda_\epsilon = \int_{\mathcal{O}} G \Delta \lambda_\epsilon d\mathcal{O}. \quad (9.8)$$

To handle the third term on left hand side: Using classic vector field identities for the wave equation—see e.g., [39], [14], or [43] p.459—we have

$$\begin{aligned} \int_{\mathcal{O}} \Delta \lambda_\epsilon (\mathbf{v} \cdot \nabla \lambda_\epsilon) d\mathcal{O} &= \int_{\partial \mathcal{O}} \frac{\partial \lambda_\epsilon}{\partial \mathbf{n}} \mathbf{v} \cdot \nabla \lambda_\epsilon d\mathcal{O} - \int_{\mathcal{O}} (\nabla \mathbf{v} \nabla \lambda_\epsilon) \cdot \nabla \lambda_\epsilon d\mathcal{O} \\ &\quad + \frac{1}{2} \int_{\mathcal{O}} |\nabla \lambda_\epsilon|^2 \operatorname{div}(\mathbf{v}) d\mathcal{O}. \end{aligned} \quad (9.9)$$

(In stating this relation, we are using assumption (i) of (9.2).) With respect to the first term on right hand side of (9.9): using Proposition 4, p.702 of [12], and the fact that $\lambda_\epsilon|_{\partial \mathcal{O}} = 0$, we have on $\partial \mathcal{O}$

$$\begin{aligned} \nabla \lambda_\epsilon|_{\partial \mathcal{O}} &= \nabla_{\partial \mathcal{O}} (\lambda_\epsilon|_{\partial \mathcal{O}}) + \mathbf{n} \frac{\partial \lambda_\epsilon}{\partial \mathbf{n}} \\ &= \mathbf{n} \frac{\partial \lambda_\epsilon}{\partial \mathbf{n}}. \end{aligned}$$

(Above, $\nabla_{\partial \mathcal{O}} (\lambda_\epsilon|_{\partial \mathcal{O}}) \in \mathbf{L}^2(\partial \mathcal{O})$ denotes the tangential gradient of $\lambda_\epsilon|_{\partial \mathcal{O}}$; see [40] or p.701 of [12].) Applying this relation to (9.9), and considering $\mathbf{v} \cdot \mathbf{n}|_{\partial \mathcal{O}} = 0$, we then have

$$\int_{\mathcal{O}} \Delta \lambda_\epsilon (\mathbf{v} \cdot \nabla \lambda_\epsilon) d\mathcal{O} = - \int_{\mathcal{O}} (\nabla \mathbf{v} \nabla \lambda_\epsilon) \cdot \nabla \lambda_\epsilon d\mathcal{O} + \frac{1}{2} \int_{\mathcal{O}} |\nabla \lambda_\epsilon|^2 \operatorname{div}(\mathbf{v}) d\mathcal{O}. \quad (9.10)$$

For the fourth term on the left hand side of (9.8): Using Green's Theorem, we have

$$\begin{aligned} (\lambda_\epsilon \operatorname{div}(\mathbf{v}), \Delta \lambda_\epsilon)_{\mathcal{O}} &= -(\nabla [\lambda_\epsilon \operatorname{div}(\mathbf{v})], \nabla \lambda_\epsilon)_{\mathcal{O}} \\ &= - \int_{\mathcal{O}} |\nabla \lambda_\epsilon|^2 \operatorname{div}(\mathbf{v}) d\mathcal{O} - (\lambda_\epsilon \nabla [\operatorname{div}(\mathbf{v})], \nabla \lambda_\epsilon)_{\mathcal{O}}. \end{aligned} \quad (9.11)$$

A further integration by parts then yields

$$(\lambda_\epsilon \operatorname{div}(\mathbf{v}), \Delta \lambda_\epsilon)_\mathcal{O} = - \int_\mathcal{O} |\nabla \lambda_\epsilon|^2 \operatorname{div}(\mathbf{v}) d\mathcal{O} - \frac{1}{2} \int_\mathcal{O} \lambda_\epsilon^2 \Delta \operatorname{div}(\mathbf{v}) d\mathcal{O}. \quad (9.12)$$

Applying the relations (9.10) and (9.12) to (9.8), we then have

$$\epsilon \|\Delta \lambda_\epsilon\|_\mathcal{O}^2 + \kappa \|\nabla \lambda_\epsilon\|_\mathcal{O}^2 + \int_\mathcal{O} (\nabla \mathbf{v} \nabla \lambda_\epsilon) \cdot \nabla \lambda_\epsilon d\mathcal{O} + \frac{1}{4} \int_\mathcal{O} \lambda_\epsilon^2 \Delta \operatorname{div}(\mathbf{v}) d\mathcal{O} = - \int_\mathcal{O} G \Delta \lambda_\epsilon d\mathcal{O}. \quad (9.13)$$

Now, concerning the fourth term on left hand side:

$$\left| \int_\mathcal{O} \lambda_\epsilon^2 \Delta \operatorname{div}(\mathbf{v}) d\mathcal{O} \right| \leq \|\lambda_\epsilon^2\|_\mathcal{O} \|\Delta \operatorname{div}(\mathbf{v})\|_\mathcal{O};$$

and subsequently applying Hölder's inequality with conjugates $p = 3/2$ and $p^* = 3$, we have then

$$\begin{aligned} \left| \int_\mathcal{O} \lambda_\epsilon^2 \Delta \operatorname{div}(\mathbf{v}) d\mathcal{O} \right| &\leq \operatorname{meas}(\mathcal{O})^{\frac{1}{6}} \|\lambda_\epsilon\|_{L^6(\mathcal{O})}^2 \|\Delta \operatorname{div}(\mathbf{v})\|_\mathcal{O} \\ &\leq C_S \operatorname{meas}(\mathcal{O})^{\frac{1}{6}} \|\nabla \lambda_\epsilon\|_\mathcal{O}^2 \|\Delta \operatorname{div}(\mathbf{v})\|_\mathcal{O}, \end{aligned} \quad (9.14)$$

where positive constant C_S is that in (9.3).

Estimating the relation (9.13), by means of (9.14), we then obtain

$$\epsilon \|\Delta \lambda_\epsilon\|_\mathcal{O}^2 + \left(\kappa - \left[\frac{C_S}{4} \operatorname{meas}(\mathcal{O})^{\frac{1}{6}} \|\Delta \operatorname{div}(\mathbf{v})\|_\mathcal{O} + \|\nabla \mathbf{v}\|_{L^\infty(\mathcal{O})} \right] \right) \|\nabla \lambda_\epsilon\|_\mathcal{O}^2 \leq \|\nabla G\|_\mathcal{O} \|\nabla \lambda_\epsilon\|_\mathcal{O};$$

and so after using assumption (9.2)(ii), we arrive at

$$\epsilon \|\Delta \lambda_\epsilon\|_\mathcal{O}^2 + \frac{k}{2} \|\nabla \lambda_\epsilon\|_\mathcal{O}^2 \leq \|\nabla G\|_\mathcal{O} \|\nabla \lambda_\epsilon\|_\mathcal{O}. \quad (9.15)$$

From this estimate, we obtain

$$\|\nabla \lambda_\epsilon\|_\mathcal{O} \leq \frac{2}{k} \|\nabla G\|_\mathcal{O}. \quad (9.16)$$

In turn, applying this uniform bound to (9.15), we have also

$$\epsilon \|\Delta \lambda_\epsilon\|_\mathcal{O} \leq \sqrt{\frac{2\epsilon}{k}} \|\nabla G\|_\mathcal{O}.$$

Consequently, there exists a subsequence $\{\lambda_\epsilon\}$ and function $\lambda \in H_0^1(\mathcal{O})$ such that

$$\begin{aligned} \text{(i)} \quad &\lim_{\epsilon \rightarrow 0} \lambda_\epsilon = \lambda \text{ weakly in } H_0^1(\mathcal{O}) \text{ and strongly in } L^2(\mathcal{O}); \\ \text{(ii)} \quad &\lim_{\epsilon \rightarrow 0} \epsilon \Delta \lambda_\epsilon = 0 \end{aligned} \quad (9.17)$$

(see e.g., Theorem 3.27(ii), p.87, of [38]).

With the convergences above in hand, we multiply the (9.6) by test function $\varphi \in H^1(\mathcal{O})$, and integrate. (Recall that each λ_ϵ is a strong solution of (9.6).) This gives the relation

$$\epsilon \int_\mathcal{O} \Delta \lambda_\epsilon \varphi d\mathcal{O} + k \int_\mathcal{O} \lambda_\epsilon \varphi d\mathcal{O} - \frac{1}{2} \int_\mathcal{O} \lambda_\epsilon \operatorname{div}(\mathbf{v}) \varphi d\mathcal{O} - \int_\mathcal{O} \lambda_\epsilon \mathbf{v} \cdot \nabla \varphi d\mathcal{O} = \int_\mathcal{O} G \varphi d\mathcal{O} \text{ for every } \varphi \in H^1(\mathcal{O}). \quad (9.18)$$

Passing to the limit on left hand side, and invoking (9.17) we have that strong L^2 -limit λ satisfies (9.5).

Moreover, from the relations (9.6) and (9.7) for the regularized problem, and the strong limit posted in (9.17), we have the estimate

$$k \|\lambda\|_{\mathcal{O}} \leq \|G\|_{\mathcal{O}}. \quad (9.19)$$

In sum: we have justified the existence of a operator \mathcal{L}_0 , say, which satisfies, for given $G \in H_0^1(\mathcal{O})$, $\mathcal{L}_0(G) = \lambda$, where $\lambda \in L^2(\mathcal{O})$ solves (9.5), and which yields the estimate

$$\|\mathcal{L}_0 G\|_{\mathcal{O}} \leq \frac{1}{k} \|G\|_{\mathcal{O}}, \quad (9.20)$$

after using (9.19). An extension by continuity now yields the solvability of equation (9.1) for given L^2 -data G , and this solvability is unique because of the inherent dissipativity in this equation. Lastly, the estimate (9.4) is just (9.20). \square

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