DOI:
https://doi.org/10.1007/s00245-021-09798-0

Theory of Solutions for An Inextensible Cantilever

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April 6, 2021

Abstract

Recent equations of motion for the large deflections of a cantilevered elastic beam are analyzed. In the traditional theory of beam (and plate) large deflections, nonlinear restoring forces are due to the effect of stretching on bending; for an inextensible cantilever, the enforcement of arc-length preservation leads to quasilinear stiffness effects and inertial effects that are both nonlinear and nonlocal. For this model, smooth solutions are constructed via a spectral Galerkin approach. Additional compactness is needed to pass to the limit, and this is obtained through a complex procession of higher energy estimates. Uniqueness is obtained through a non-trivial decomposition of the nonlinearity. The confounding effects of nonlinear inertia are overcome via the addition of structural (Kelvin-Voigt) damping to the equations of motion. Local well-posedness of smooth solutions is shown first in the absence of nonlinear inertial effects, and then shown with these inertial effects present, taking into account structural damping. With damping in force, global-in-time, strong well-posedness result is obtained by achieving exponential decay for small data.

Key terms: inextensible cantilever, elasticity, quasilinear, structural damping, well-posedness

MSC 2010: 74B20, 35L77, 35B65, 74H20

1 Introduction

1.1 Motivation and Overview

The large deflections of elastic beams and plates have broad applicability in engineering and other physical sciences, and they have been intensely studied from the modeling, analytical, and computational points of view (see, e.g., [6, 7, 13, 21]). Specifically, with respect to fluid-structure interaction models, the large deflections of panel, airfoil, and flap structures are of particular interest [5, 9] (and references therein). In these circumstances, the presence of a fluid flow can act as a destabilizing mechanism, giving rise to self-excitation instabilities (i.e., aeroelastic flutter [5, 28]) that manifest as *limit cycle oscillations* (LCOs). In such applications, relevant large deflection models require nonlinear restoring forces that take into account higher order effects, typically appearing via a potential energy above the "quadratic" level. The choice of nonlinearity dictates the qualitative features of the post-onset dynamics—which is to say, the dynamics in the nonlinear regime of interest. Traditional large deflection theory for *panels* (i.e., fully restricted boundary conditions) is that of von Karman [6], producing semilinear, cubic-type nonlinearities based on a quadratic strain-displacement law [7, 24].

The configuration of a *cantilever in axial flow*, whereby an elastic beam (or thin plate) has a flow of gas running *along its principal axis*, has been historically overlooked. Until about 15 years ago, interest in this configuration was minimal [19], while interest in airfoil and panel flutter has been *immense* for more than 75 years [5, 9]. A cantilever in axial flow is particularly prone to aeroelastic instability, with the bifurcation leading to sustained LCOs. This fact is useful in the development of vibration-based energy harvesting devices [12, 14]. In such applications, dynamic instability is encouraged to extract energy from LCOs of the elastic cantilever, after the onset of flutter. The main idea for large displacement harvesters is to capture mechanical energy via piezoelectric laminates or patches (for which oscillating strains induce current [14]). The feasibility of such a system has been recently demonstrated with affixed piezo (SMART) materials [12, 14, 16, 33].

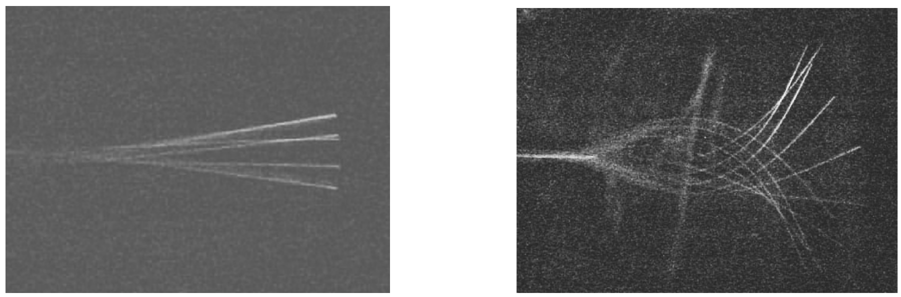


Figure 1: Temporal snapshots of post onset, small amplitude LCO (left) and large amplitude LCO (right) for a cantilever in axial flow. Captured from wind-tunnel simulations [35, 36]. In these experiments, the airflow runs from left to right.

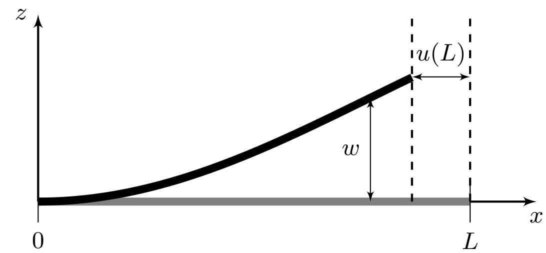
To effectively and efficiently harvest energy in this manner, one must be able capture and predict the post-onset behaviors of a cantilever, and thus, *one must have a viable model for the large deflections of cantilever*. Tradition nonlinear elastic models are based on local stretching effects, which are dominant when the entire structural boundary is restricted. However, for a cantilever, nonlinear effects are decidedly not due to stretching [10, 12, 31, 36]. An appropriate nonlinear cantilever model, then, should account for in-plane displacements and variable stiffness and inertia. Thus, dominant nonlinear effects should come from the cantilever's *inextensibility*, rather than extensible effects (stretching). Inextensible cantilever models are rather recent [8, 10, 36], and have not been addressed—even in vacuo—in the rigorous mathematical literature. Thus, in this paper, we discuss the model derivation for an inextensible cantilever, and we produce a rigorous theory of solutions for the corresponding PDE model. The treatment at hand is a rigorous follow up to the recent [8], where an inextensible cantilever is discussed and analyzed numerically. In that paper, the results proven here were announced.

1.2 Large Deflections of Cantilevers

Cantilever flutter is associated with large deflections on the order of the beam's length [35, 36]. For such deflections, structural nonlinearity arises in modeling as a byproduct of the inclusion of higher order terms in strain and energetic expressions. From a mathematical point of view, the presence of non-conservative flow-effects gives rise to a beam bifurcation (this is flutter [5, 8, 17, 18]), which would yield exponential growth in time for a linear model, according to destabilized eigenvalues. To ensure that flow-destabilized trajectories remain bounded, one must consider a nonlinear restoring force, active for large displacements, slopes, or curvatures.

The typical way to achieve this in the theory of elasticity is through the inclusion of cubic-type forces that arise from the effect of local stretching on bending [18, 24] . Since extensibility is not physically dominant for cantilevers, engineers have posited that the prevailing nonlinear forces result from inextensibility [10, 32, 36]. Although enforcing inextensibility—as a nonlinear constraint—can be quite challenging, the recent modeling work [10, 36] utilizes a simplified approach that accounts for both nonlinear stiffness and nonlinear inertia effects. The result is a beam theory that is both quasilinear and nonlocal in space, as well as implicit in the transverse acceleration, which is to say the dynamics are not a traditional second order evolution. The model was recently considered in the mathematical paper [8], where solutions were defined and qualitatively investigated from a numerical point of view under the influence of non-conservative flow effects. **In the paper at hand we consider this inextensible elastic cantilever model and develop a rigorous well-posedness theory for smooth solutions**.

For the remainder of this treatment, let $(u, w) \in \mathbb{R}^2$ denote the Lagrangian displacement of a beam whose centerline equilibrium position is $x \in [0, L]$. This is to say that $u(x, t)$ is the axial (longitudinal) displacement from equilibrium and $w(x, t)$ is the transverse beam deflection.



Then, the **inextensible equations of motion** of interest—derived later in Section 2—are:

$$\begin{cases} w_{tt} + D\partial_x^4 w + k_2 \partial_x^4 w_t + \mathbf{A}[w, u_{tt}] = p(x,t) & \text{in } (0,L)\times(0,T); \\ w(0) = w_x(0) = 0; \quad w_{xx}(L) = w_{xxx}(L) = 0 & \text{in } (0,T); \\ w(t=0) = w_0, \quad w_t(t=0) = w_1, \end{cases} \tag{1}$$

with
$$\mathbf{A}[w, u_{tt}] = -D\partial_x\left[w_{xx}^2 w_x\right] + D\partial_{xx}\left[w_{xx} w_x^2\right] + \partial_x\left[w_x \int_x^L u_{tt} d\xi\right] \tag{2}$$

$$u_{tt}(x) = -\int_0^x [w_{xt}^2 + w_x w_{xtt}] d\xi. \tag{3}$$

We denote by D the standard (mass-normalized) beam stiffness coefficient [10], L is the beam's length, and $k_2 \geq 0$ corresponds to structural damping of Kelvin-Voigt type (discussed further in Section 2). The RHS $p(x, t)$ constitutes a given transverse pressure differential across the deflected beam.

We now mention the only (to the best of our knowledge) large deflection beam models in the PDE literature that accommodate a cantilever configuration. First is the extensible system found in [24]; in addition to standard elasticity assumptions, it invokes a quadratic strain-displacement law consistent with von Karman theory [37]. As a system, it is nonlinearly coupled in u and w via the beam's *extensionality*: $[u_x + \frac{1}{2}w_x^2]$.

$$\begin{cases} u_{tt} - D_1\partial_x\left[u_x + \frac{1}{2}(w_x)^2\right] = 0; \\ (1-\alpha\partial_x^2)w_{tt} + D_2\partial_x^4 w + (k_0 - k_1\partial_x^2)w_t - D_1\partial_x\left[w_x(u_x + \dfrac{1}{2}w_x^2)\right] = p(x,t); \\ u(0) = 0,\ w(0) = w_x(0) = 0; \quad \left[u_x(L) + \frac{1}{2}w_x^2(L)\right] = 0,\ w_{xx}(L) = 0,\ \alpha\partial_x w_t(L) = D_2 w_{xxx}(L); \\ u(t=0) = u_0,\ u_t(t=0) = u_1; \quad w(t=0) = w_0,\ w_t(t=0) = w_1. \end{cases} \tag{4}$$

The above *Lagnese-Leugering* system is the beam analog of the so called *full von Karman plate equations* [22]. Above, $D_1, D_2 > 0$ are two different mass-normalized stiffness parameters and $\alpha \geq 0$ represents (linearized) rotational inertia in the filaments of the plate[1]. The coeffcients $k_i \geq 0$ correspond to damping of various strengths. The paper [24] considers a variant of this model that allows for

[1] D_1, D_2, and α are not necessarily independent in the presentation of these equations with physical coefficients.

boundary feedbacks and takes $k_0 = k_1 = 0$; the results include nonlinear semigroup well-posedness, as well as a stabilization result.

One further consideration can be made as a simplification of the above system when we take negligible in-plane accelerations, $u_{tt} \approx 0$. Elementary simplifications then produce a scalar extensible cantilever, as was studied in [18]. In that reference, well-posedness and long-time behavior of the scalar system are analyzed in the presence of a non-conservative $p(x,t)$ representing an inviscid potential flow. A principal consideration in that analysis is whether $\alpha > 0$ or $\alpha = 0$. In the case where $\alpha > 0$, stabilization-type estimates require damping strength to be tailored to the inertia, i.e., $\alpha > 0 \implies k_1 > 0$.

Each of these extensible beam models above is reasonable under certain modeling hypotheses, in particular contexts. However, **as is clear from the engineering literature discussed above, large deflections of a flow-driven cantilever should be appropriately modeled with inextensibility**. Lastly, we point to some work that addresses the 2 or 3-D deflections of inextensible rods in the seminal reference [1] (and many references therein), as well as [2]. The principal inextensible models therein are linear of wave type (or their linearizations are, i.e., second order in space, perhaps with strong damping), and hence fundamentally distinct from the nonlinear models considered here which account for nonlinear inertia and stiffness effects.

1.3 The Analysis At Hand

In this section we outline the remainder of the paper and state our results informally.

We conclude the introduction in Section 1.4, where we discuss the novel mathematical contributions of this treatment and technical challenges of the analysis. The remaining preliminary sections are 2 and 3. Section 2 presents a derivation of the equations of motion, as well as a brief discussion of structural damping central to this analysis. Section 3 presents the functional setup for the analysis and technical definitions of solutions used in subsequent well-posedness and stability proofs.

Each of the remaining sections corresponds to a main result. In those latter sections, we will: (i) state a main result technically, using the terminology and concepts established in Section 3; (ii) outline the proof briefly; and (iii) execute the proof in detail. Below, we give a short, nontechnical description of each of these main sections.

Section 4 provides a *local*[2] *well-posedness for strong solutions* for (1) *in the absence of nonlinear inertia with no imposed damping*. The resulting system is a conservative, quasilinear beam system. The result—Theorem 4.1—is built upon higher order energy estimates used to obtain additional compactness needed in executing a Galerkin procedure with cantilever eigenfunctions..

Section 5 provides a *local well-posedness result—in the presence of nonlinear inertia—for strong solutions* in Theorem 5.1; in this case, *some damping is required* ($k_2 > 0$) to obtain estimates in the construction of solutions. The damping addresses the nonlocal and implicit nature of the inertial terms.

Section 6 provides our final main result: *global existence of strong solutions for small data* (in the presence of both inertia and damping). This result is typical for quasilinear hyperbolic dynamics, whereby the presence of damping and small data allow for stabilization estimates that ensure exponential decay, yielding an arbitrary time of existence.

Lastly, Section 7 gives a brief discussion of open problems related to the model at hand, and Section 8 gives the authors' declarations and acknowledgements.

1.4 Novel Contributions and Technical Challenges

The model we focus on here only appeared for the first time in the context of elastic cantilevers in the recent papers [10, 36]. These (and other earlier works focusing on inextensible pipes conveying fluid

[2] local, in the sense that the time of existence depends on the size of the initial data in the associated solution topology.

such as [29, 32]) are largely engineering-oriented, making use of finite dimensional analyses via modal truncation or the Rayleigh-Ritz method at the energetic level. And, although the present authors' recent work [8] discusses solutions and states well-posedness theorems, it is numerically-focused, without proofs. Thus, **the existing body of work on inextensible elasticity does not address**:

- a construction of PDE solutions (at the infinite dimensional level)
- (Hadamard) well-posedness
- the effects of damping in relation to nonlinear inertial terms
- the time of existence for solutions or quantitative restrictions on data.

To the knowledge of the authors, *this is the first treatment to rigorously address the theory of solutions for inextensible elasticity.*

Although the central problem here is a 1-D beam, the following issues render the analysis quite challenging. Some of these issues are common for quasilinear dynamics, but many are not (e.g., those associated with nonlinear inertia), and we also point to the non-trivial *interaction* between (high order) free boundary conditions, nonlinear stiffness, and nonlocal inertial terms.

The **technical challenges faced in the analysis** are:

- Despite a good, conservative structure for the baseline equations of motion, quasilinear and semilinear terms do not straightforwardly admit (semigroup or fixed point) perturbation methods.
- The term $\partial_x[w_{xx}^2 w_x]$ precludes weak limit point identification at the baseline energy level.
- Nonlinear terms and free boundary conditions (i) do not readily permit differentiation of the equations to obtain higher energy estimates, and (ii) convolute the standard technique of *going back through the equations* to trade time and space regularity.
- Nonlinear inertial terms (i) present themselves at a level above finite energy, (ii) are also nonlocal, and (iii) are implicit terms in w_{tt}, and hence do not constitute a traditional evolution. The truncated version of the dynamics is in fact *quasilinear in time* (62).

In addressing the issues above, we note the following specific **novelties of this analysis**:

- The sequence of multipliers used to close estimates in obtaining compactness are non-standard, including the use of stabilization-type multipliers.
- A novel decomposition of nonlinear differences exploits polynomial symmetry for a non-obvious uniqueness proof, relying critically on smooth trajectory estimates obtained earlier.
- The inclusion of damping to permit appropriate estimates for well-posedness of the full model is a peculiarity, one that, at present, we cannot avoid. On the other hand, including damping in the full model (1)–(2) successfully obtains global solutions for small data.

2 PDE Model Derivation

Recall that $w(x,t)$ is the transverse deflection and $u(x,t)$ is the in-axis displacement from equilibrium of a beam at $t \in [0,T]$ and a spatial point $x \in [0,L]$. Let $\varepsilon(x,t)$ describe the *axial strain* along the centerline of the beam. In this section we derive the in-vacuo equations of motion via Hamilton's principle. The inextensibility condition is simplified to an *effective inextensibility constraint*, which is enforced via a Lagrange multiplier. Our derivation tracks the one first appearing in [10], and we point to the earlier references [29, 32] for inextensibility treated in the context of pipes conveying fluid.

2.1 Inextensibility

According to classical work (e.g., [32, 34]) we have the Lagrangian strain relation [32]

$$\left[1+\varepsilon\right]^2 = (1+u_x)^2 + w_x^2.$$

When the beam is *inextensible*, we take $\varepsilon(x,t) = 0$, which immediately yields the condition

$$1 = (1+u_x)^2 + w_x^2. \tag{5}$$

From [7, 10, 32], large deflections dictate that higher order nonlinear terms should be retained, namely, *up to cubic order*. (For variational purposes, then, energetic expressions will be accurate up to quartic order.) By expanding the inextensibility condition (5), we see that if $w_x \sim \epsilon$, we will have $u_x \sim \epsilon^2$:

$$2u_x + u_x^2 + w_x^2 = 0.$$

As in [10], we drop $u_x^2 \sim \epsilon^4$, owing to its relative order being above cubic. Approximating, then

$$0 = 2u_x + w_x^2 \quad \Longrightarrow \quad u_x = -\frac{1}{2}w_x^2.$$

This yields what we henceforth refer to as the *effective inextensibility constraint*, providing a direct relationship between u and w:

$$u(x,t) = -\frac{1}{2}\int_0^x [w_x(\xi,t)]^2 d\xi. \tag{6}$$

2.2 Nonlinear Elasticity

Define the elastic potential energy (E_P) via beam curvature κ and constant stiffness D (flexural rigidity) [32] in the standard way

$$E_P \equiv \frac{D}{2}\int_0^L \kappa^2 dx.$$

Owing to inextensibility, we may take the beam's displaced state, $\{(x+u(x), w(x)) \ : \ x \in [0,L]\}$, as a parametrized curve. The standard expression for curvature in this scenario is:

$$\kappa = \frac{(1+u_x)w_{xx} - u_{xx}w_x}{[(w_x)^2 + (1+u_x)^2]^{3/2}}.$$

From inextensibility (5) (without approximation), we see that the denominator is one. From (5), we can also write $u_x = \sqrt{1-w_x^2} - 1$ which leads to $u_{xx} = -w_x w_{xx}(1-w_x^2)^{-1/2}$. Substituting in κ, we obtain:

$$\kappa = (1+u_x)w_{xx} - w_x u_{xx} = (1-w_x^2)^{1/2}w_{xx} + w_x(w_x w_{xx}(1-w_x^2)^{-1/2}) = \frac{w_{xx}}{(1-w_x^2)^{1/2}}.$$

To be consistent with the approximation that yields (6), we must retain terms at the level of w_x^2 in approximating E_P [10, 32]. Via a Taylor expansion, we take $\kappa \approx w_{xx}\sqrt{1+w_x^2}$.

Remark 2.1. This point distinguishes the derivation from linear elasticity in w, where $\kappa \approx w_{xx}$.

Finally, the effective potential energy for the problem at hand becomes

$$E_P = \frac{D}{2}\int_0^L w_{xx}^2\left(1+w_x^2\right)dx. \tag{7}$$

The kinetic energy (E_K) for the dynamics taken in the standard way for a mass-normalized beam:

$$E_K = \frac{1}{2}\int_0^L \left(u_t^2 + w_t^2\right)dx. \tag{8}$$

2.3 Hamilton's Principle

To derive the equations of motion and the associated boundary conditions, we utilize Hamilton's Principle [10, 24]. We consider displacements u and w (and hence virtual displacements δu and δw) which are smooth and respect the essential boundary conditions at $x = 0$, namely:

$$w,\ w_x,\ \delta w,\ \delta w_x :\ 0 \ \text{ at } \ x = 0; \qquad\qquad u,\ \delta u :\ 0 \ \text{ at } \ x = 0.$$

The effective inextensibility constraint, $f \equiv u_x + (1/2)w_x^2 = 0$, will be appended to the system via a *Lagrange multiplier* λ. Thus, we express the Lagrangian in the usual way:

$$\mathcal{L} = E_K - E_P + \int_0^L \lambda f dx. \tag{9}$$

Taking the variation of (9) and performing the necessary integration by parts with respect to both time and space, Hamilton's principle provides the Euler-Lagrange equations of motion and the associated boundary conditions. Virtual changes are considered for both displacements, u and w.[3]

To minimize the Lagrangian, we set $\delta \int_{t_1}^{t_2} \mathcal{L} dt \equiv 0$ and utilize the arbitrariness of the virtual changes δu and δw. For interior terms, we gather virtual changes and set the totals equal. The relevant calculation pertains to the E_P:

$$\delta E_P = D \int_0^L \big[(1 + w_x^2)w_{xx}\big]\delta w_{xx} + \big[(w_x w_{xx}^2)\big]\delta w_x dx. \tag{10}$$

Integrating by parts until only δw appears, and utilizing the arbitrariness of the virtual changes, we obtain the unforced equations of motion:

$$\text{from } \ \delta u : \quad u_{tt} + \lambda_x = 0 \tag{11}$$

$$\text{from } \ \delta w : \quad w_{tt} - D\partial_x \left(w_{xx}^2 w_x\right) + D\partial_{xx}\left(w_{xx}\left[1 + w_x^2\right]\right) + \partial_x\left(\lambda w_x\right) = 0. \tag{12}$$

For the (natural) boundary conditions at $x = L$, the relevant calculations pertain to w (the u and λ conditions can then be inferred). In the integration by parts proceeding from (10), we obtain by the arbitrariness of δw, δw_x and δu at $x = L$:

$$\lambda(L) = 0; \quad (1 + w_x^2(L))w_{xx}(L) = 0; \quad (1 + w_x^2(L))w_{xxx}(L) + w_x(L)w_{xx}^2(L) = 0. \tag{13}$$

From (13), we infer that $w_{xx}(L) = w_{xxx}(L) = 0$—the standard free boundary conditions.

Remark 2.2. This fact is both critical and somewhat surprising, as the nonlinear effects (and their previously discussed simplifications) do not alter the standard *linear* boundary conditions for a cantilever. Note that in extensible elasticity, this is not always the case [6, 18].

Now, using the equation (11) we can formally write

$$\lambda(x) = -\int_0^x u_{tt}(\xi)d\xi + \lambda(0).$$

We then utilize the fact that $\lambda(L) = 0$ to conclude $\lambda(0) = \displaystyle\int_0^L u_{tt}(\xi)d\xi$. From this we deduce:

$$\lambda(x) = \int_x^L u_{tt}(\xi)d\xi.$$

Substituting the above expression in (12) we finally obtain the equations of motion (1)–(2), and the corresponding boundary conditions for w, as well as for u and λ at $x = L$.

[3] Note that virtual change in λ simply produces the effective inextensibility constraint.

2.4 Damping

Discussion of damping in beams goes far back in both the engineering literature [3, 32] as well as the mathematical literature [4, 30]. In the treatment at hand, some additional velocity regularization is needed to address the nonlinear inertial terms; namely w_t must be "better" than $C([0,T];L^2(0,L))$. We obtain this by imposing Kelvin-Voigt type structural damping. Note, this type of damping is in fact invoked in the engineering-oriented references [29, 32] for improving numerical simulations. The recent [31] addresses local damping and stiffness in a cantilever from a modeling and experimental point of view.

Let us here refer to the damped, linear Euler-Bernoulli beam equation

$$w_{tt} + D\partial_x^4 w + [k_0 - k_1\partial_x^2 + k_2\partial_x^4]w_t = p.$$

Weak (frictional) damping has the form $k_0 w_t$, providing no velocity regularization. In the elasticity context, Kelvin-Voigt damping $k_2\partial_x^4 w_t$ is strain-rate type, and mirrors the principal (linear) operator, providing a strong dissipative effect. In fact, this damping transforms the underlying dynamics to be of parabolic type [4,26]. Square root-like damping, $-k_1\partial_x^2 w_t$ [16], interpolates between the previous two damping types. (See [8, 18] for more discuss of damping in the context of nonlinear cantilevers.)

Remark 2.3. Square root-type damping corresponds to modal damping models [9], as one finds frequently in the engineering literature [3, 27, 28]. However, the boundary conditions for a given problem affect the physical interpretation of square-root type damping; in [30] it is noted that square-root type damping has a questionable physical interpretation for a cantilevered configuration. See also [17] for more recent discussion. In the analysis here, we utilize the (strong) Kelvin-Voigt damping.

Remark 2.4. It is of course of interest to discuss damping in the context of the stiffness-only model $\iota = 0$. On the other hand, in this treatment the damping is primarily included to mitigate the effects of nonlinear inertia. We discuss this further in Section 7.

3 Functional Setup and Key Notions

3.1 Equations of Motion

With the derivation above, we recall the equations of motion, allowing for Kelvin-Voigt damping $k_2 \geq 0$, and including *flags* for the nonlinear terms:

$$\begin{cases} w_{tt} + D\partial_x^4 w + k_2\partial_x^4 w_t + \mathbf{A}_{\iota,\sigma}(w, u_{tt}) = p(x,t) & \text{in } (0,L)\times(0,T) \\ w(t=0) = w_0(x), w_t(t=0) = w_1(x) \\ w(x=0) = w_x(x=0) = 0;\ w_{xx}(x=L) = w_{xxx}(x=L) = 0, \end{cases} \tag{14}$$

$$\mathbf{A}_{\iota,\sigma}(w, u_{tt}) = -\,\sigma D\partial_x\left[w_{xx}^2 w_x\right] + \sigma D\partial_{xx}\left[w_{xx}w_x^2\right] + \iota\partial_x\left[w_x\int_x^L u_{tt}(\xi)d\xi\right] \tag{15}$$

$$u(x) = -\frac{1}{2}\int_0^x [w_x(\xi)]^2\, d\xi. \tag{16}$$

To simplify terminology, we use the following language from here on:

$$\textbf{[NL Stiffness]} = -\,D\partial_x\left[w_{xx}^2 w_x\right] + D\partial_{xx}\left[w_x^2 w_{xx}\right]$$

$$\textbf{[NL Inertia]} = \partial_x\left[w_x\int_x^L u_{tt}(\xi)d\xi\right],$$

the latter of which is nonlocal, when written in w through (16). The flags, $\iota, \sigma = 0$ or 1, in (15), easily isolate particular nonlinear effects. This is to say, when $\iota = 0$, we say that **[NL Inertia]** is turned off.

Remark 3.1. For convenience, we note two expansions. First

$$[\textbf{NL Stiffness}] = D[w_{xxx}^3 + 4w_x w_{xx} w_{xxx} + w_x^2 \partial_x^4 w],$$

which highlights the quasilinear nature of the PDE (with high order semilinearity). Secondly,

$$[\textbf{NL Inertia}] = -w_x u_{tt} + w_{xx} \int_x^L u_{tt} d\xi, \quad \text{with} \qquad u_{tt} = -\int_0^x [w_{xt}^2 + w_x w_{xtt}] d\xi, \tag{17}$$

which highlights that, when closed in w, (i) there is high temporal regularity required to interpret the strong form of the PDE, and (ii) the equation is implicit in the acceleration w_{tt}.

3.2 Notation and Conventions

For a given spatial domain D, its associated $L^2(D)$ will be denoted as $||\cdot||_D$ (or simply $||\cdot||$ when the context is clear). Inner products in a Hilbert space are written $(\cdot,\cdot)_H$ (or simply $(\cdot,\cdot)$ when $H = L^2(D)$ and the context is clear). We will also denote pertinent duality pairings as $\langle\cdot,\cdot\rangle_{X\times X'}$, for a given Banach space X, as well as the general notation for a norm, $||\cdot||_X$. The open ball of radius R in X will be denoted $B_R(X)$. The space $H^s(D)$ will indicate the standard Sobolev space of order s, defined on domain D, and $H_0^s(D)$ will be the closure of $C_0^\infty(D)$ in the $H^s(D)$-norm $\|\cdot\|_{H^s(D)}$, also written as $\|\cdot\|_s$. For $\Gamma \subset \partial D$, boundary restrictions $u\big|_\Gamma$ are taken in the sense of the trace theorem for $u \in H^{1/2^+}(D)$.

The constant C we take to mean a generic constant that may change from line to line. In estimates where dependencies are critical, we will write $C(q_i)$, where q_i are relevant quantities. Additionally, in our involved estimates below, for situations where $||q_1||_X \le C||q_2||_Y$ for some quantities q_1, q_2 in spaces X and Y, with C having no critical dependencies, *we will simply write* $||q_1||_X \lesssim ||q_2||_Y$.

Finally, we will frequently make use of standard Sobolev embeddings (in particular, that of $H^{1/2^+}(0,L) \hookrightarrow L^\infty(0,L)$) as well as the Sobolev interpolation inequalities [15].

3.3 Energies

With reference to Section 2, we employ the following energies:

$$E(t) \equiv E_K(t) + E_P(t) \equiv \frac{1}{2}\left[||w_t||^2 + \iota||u_t||^2\right] + \frac{D}{2}\left[||w_{xx}||^2 + \sigma||w_x w_{xx}||^2\right]. \tag{18}$$

The energies now include the nonlinear flags. This can be written in w explicitly using $u_t = -\int_0^x w_x w_{xt} d\xi$.

In the unforced situation, with $p(x,t) \equiv 0$, the formal energy identity is obtained by the velocity multiplier w_t on (14) taken with the relation (16), yielding

$$E(t) + k_2 \int_s^t ||w_{xxt}||^2_{L^2(0,L)} d\tau = E(s), \quad 0 \le s \le t.$$

Higher order energies corresponding to smooth solutions will be defined in later sections.

3.4 Spaces and Operators

The principal state space for cantilevered beam displacement takes into account the clamped conditions:

$$H_*^2 = \{v \in H^2(0,L) : v(0) = 0, \quad v_x(0) = 0\}.$$

This space is equipped with an $H^2(0,L)$ equivalent inner product:

$$(v,w)_{H_*^2} = D(v_{xx}, w_{xx}). \tag{19}$$

Denoting R as the Riesz isomorphism $H^2_* \to [H^2_*]'$, we see it is given by:

$$R(v)(w) \equiv (v,w)_{H^2_*}. \tag{20}$$

This framework is conveniently induced by the generator of the linear cantilever dynamics:

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2(0,L) \to L^2(0,L), \quad \mathcal{A}f \equiv D\partial_x^4 f,$$
$$\mathcal{D}(\mathcal{A}) = \{w \in H^4(0,L) : w(0) = w_x(0) = 0;\ w_{xx}(L) = w_{xxx}(L) = 0\}. \tag{21}$$

From this we have in a standard fashion [26]:

$$\mathcal{D}(\mathcal{A}^{1/2}) = H^2_*, \quad \mathcal{D}(\mathcal{A}^{-1/2}) = [H^2_*]' \text{ and } \mathcal{A}^{1/2} = R \quad \text{in (20)}.$$

Then $(u,\cdot)_{H^2_*}$ is the extension of $(\mathcal{A}u,\cdot)$ from $\mathcal{D}(\mathcal{A})$ to H^2_* which gives (19).

Using the above spaces we can define the appropriate state space(s) for our dynamics. The finite energy space will be denoted as:

$$\mathscr{H} \equiv \mathcal{D}(\mathcal{A}^{1/2}) \times L^2(0,L) = H^2_* \times L^2(0,L),$$

with the inner product $y = (y_1, y_2),\ \tilde{y} = (\tilde{y}_1, \tilde{y}_2) \in \mathscr{H}$

$$(y,\tilde{y})_{\mathscr{H}} = (y_1,\tilde{y}_1)_{H^2_*} + (y_2,\tilde{y}_2)_{L^2(0,L)}. \tag{22}$$

In our discussions, we will also require stronger state spaces (corresponding to *strong* solutions):

$$\mathscr{H}_s \equiv \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{1/2}), \quad \text{for } \iota = k_2 = 0, \tag{23}$$

$$\mathscr{H}_s^I \equiv \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}), \quad \text{for } \iota = 1, k_2 > 0. \tag{24}$$

The norm in $\mathscr{H}_s$ is taken (equivalent[4] to the natural operator-induced norm) to be:

$$||y||^2_{\mathscr{H}_s} = ||\partial_x^4 y_1||^2 + ||\partial_x^2 y_2||^2, \quad \text{for } \iota = k_2 = 0,$$
$$||y||^2_{\mathscr{H}_s^I} = ||\partial_x^4 y_1||^2 + ||\partial_x^4 y_2||^2, \quad \text{for } \iota = 1, k_2 > 0.$$

3.5 Mode Functions

We will utilize the so called *in vacuo modes* (eigenfunctions) associated to the operator $\mathcal{A}$. Specifically, we work with the Euler-Bernoulli cantilever eigenfunctions as our approximants in H^2_*; namely, the eigenvalues and eigenfunctions $\{\lambda_n, s_n(x)\}_{n=1}^\infty$ of $\mathcal{A}$ on $L^2(0,L)$. These modes and associated eigenvalues are computed in an elementary way. The $C^\infty([0,L])$ mode shapes take the form

$$s_n(x) \equiv c_n[\cos(\kappa_n x) - \cosh(\kappa_n x)] + C_n[\sin(\kappa_n x) - \sinh(\kappa_n x)],\ \kappa_n^4 = \lambda_n, \tag{25}$$

where the κ_n are obtained (numerically) by solving the associated characteristic equation

$$\cos(\kappa_n L)\cosh(\kappa_n L) = -1.$$

The C_n are obtained by invoking the boundary conditions:

$$C_n = \frac{-c_n\big(\cos(\kappa_n L) + \cosh(\kappa_n L)\big)}{\sin(\kappa_n L) + \sinh(\kappa_n L)},$$

and the c_n values are chosen to normalize the functions in $L^2(0,L)$.

Via the spectral theorem, these functions are *complete* and *orthonormal* in $L^2(0,L)$, as well as complete and orthogonal in H^2_* (with respect to $(\cdot,\cdot)_{H^2_*}$). These eigenvalues have the property that $0 < \lambda_1 < \lambda_2 < ... \to \infty$.

[4] The topological equivalences on $\mathcal{D}(\mathcal{A})$ follow from repeated applications of Poincaré.

3.6 Definition of Solutions

We provide the natural setting for the weak formulation of the problem; this will yield the appropriate starting point for our Galerkin procedure to construct solutions. Ultimately, we will construct weak solutions that possess additional regularity; these, in turn, will be strong solutions.

We begin with the *weak form* of (14) which we define for functions that are smooth in time:

$$(w_{tt},\phi)+D(w_{xx},\phi_{xx})+k_2(w_{xxt},\phi_{xx})+\sigma D\big(w_{xx}w_x,w_x\phi_{xx}\big)+\sigma D(w_{xx}w_x,w_{xx}\phi_x)-\iota\left(w_x\int_x^L u_{tt},\phi_x\right)$$
$$=(p,\phi),\ \forall\ \phi\in H^2_*. \qquad (26)$$

When $\sigma>0$, the [**NL Stiffness**] is in force; similarly, when $\iota>0$, [**NL Inertia**] is in force. When $k_2>0$, Kelvin-Voigt damping is imposed.

We now give precise definitions of solutions making reference to the weak form (26) above:

Definition 1. *We say a* weak *solution to* (14)*, with $k_2=\iota=0$ and $\sigma=1$ is a function w, with*

$$w\in L^2\left(0,T;H^2_*\right);\ w_t\in L^2\left(0,T;L^2(0,L)\right);\ w_{tt}\in L^2\left(0,T;[H^2_*]'\right)$$

that satisfies (26)*, replacing $L^2(0,L)$ inner products with $(H^2_*,[H^2_*]')$ duality pairings where necessary.*

Moreover, for any $\chi\in H^2_*$, $\psi\in L^2(0,L)$, we require

$$(w,\chi)_{H^2_*}\big|_{t\to0^+}=(w_0,\chi)_{H^2_*},\quad (w_t,\psi)\big|_{t\to0^+}=(w_1,\psi). \qquad (27)$$

Definition 2. *A* weak *solution to* (14) *with $k_2>0$ and $\iota=\sigma=1$ is a function w, with*

$$w\in L^2\left(0,T;H^2_*\right);\ w_t\in L^2\left(0,T;H^2_*\right);\ w_{tt}\in L^2\left(0,T;[H^2_*]'\right),$$

such that (26) *holds, replacing $L^2(0,L)$ inner products with $(H^2_*,[H^2_*]')$ duality pairings where necessary.*

Moreover, for any $\chi\in H^2_*$, $\psi\in L^2(0,L)$, we require

$$(w,\chi)_{H^2_*}\big|_{t\to0^+}=(w_0,\chi)_{H^2_*},\quad (w_t,\psi)\big|_{t\to0^+}=(w_1,\psi). \qquad (28)$$

Remark 3.2. For $k_2>0$ and $\iota>0$, the definition of weak solution is self-consistent; this is to say, for such a function w, all terms in (26) are well-defined. We note that for $k_2=0$, there are complications with the a priori regularity of $w_t\in L^2(0,T;L^2(0,L))$ and the interpretation of the [**NL Inertia**] terms.

Now, we define strong solutions as weak solutions with additional regularity.

Definition 3. *A* strong *solution to* (14) *with $k_2=\iota=0$ and $\sigma=1$ is a weak solution (as in Definition 1) with the additional regularity*

$$w\in L^2\left(0,T;\mathcal{D}(\mathcal{A})\right);\ w_t\in L^2(0,T;H^2_*);\ w_{tt}\in L^2\left(0,T;L^2(0,L)\right).$$

Definition 4. *A* strong *solution to* (14) *with $k_2>0$, $\iota=\sigma=1$ is a weak solution (as in Definition 2) with the additional regularity*

$$w\in L^2\left(0,T;\mathcal{D}(\mathcal{A})\right);\ w_t\in L^2\left(0,T;\mathcal{D}(\mathcal{A})\right);\ w_{tt}\in L^2\left(0,T;H^2_*\right).$$

As we will show below in Corollaries 4.2 and 5.2, strong solutions will satisfy the pointwise form of the PDE in (14) as well as the higher order boundary conditions at $x=L$.

4 The Case of Only Stiffness Effects: $\sigma = 1$, $\iota = k_2 = 0$

4.1 Precise Statement of the Theorem

Theorem 4.1. *Take $\sigma = 1$ with $\iota = k_2 = 0$, and consider $p \in H^2_{loc}(0,\infty; L^2(0,L))$. For smooth data $(w_0, w_1) \in \mathscr{H}_s = \mathcal{D}(\mathcal{A}) \times H^2_*$, strong solutions exist up to some time $T^*(w_0, w_1, p)$. For all $t \in [0, T^*)$, the solution w is unique and obeys the energy identity*

$$E(t) = E(0) + \int_0^t (p, w_t)_{L^2(0,L)} d\tau.$$

Restricting to $B_R(\mathscr{H}_s)$, for any $T < T^*(R,p)$ solutions depend continuously on the data in the sense of $C([0,T]; \mathscr{H})$ with an estimate on the difference of two trajectories, $z = w^1 - w^2$:

$$\sup_{t \in [0,T]} \big|\big|(z(t), z_t(t))\big|\big|_{\mathscr{H}} \leq C(R,T) \big|\big|\big(z(0), z_t(0)\big)\big|\big|_{\mathscr{H}}, \quad \forall\, t \in [0,T].$$

Remark 4.1. The time of existence T^* depends on the data in the sense of

$$T^* = T^* \left(||(w_0, w_1)||_{\mathscr{H}_s}, ||p||_{H^2(0,T;L^2(0,L))} \right),$$

namely, the size of the data in the appropriate space, rather than the individual data itself.

4.2 Proof Outline

We will commence with a Galerkin procedure, using the mode functions $\{s_j\}_{j=1}^\infty$ described above. This will yield approximate solutions, with the baseline energy identity providing associated weak limit points. Identifying the nonlinear weak limits is non-trivial, hence, two higher-order multipliers will be used to provide more regular a priori bounds; one is an energy estimate corresponding to the time-differentiated version of the equation, and the other is a stability type estimate resulting from the multiplier $\partial_x^4 w$. Additional compactness is obtained through these estimates with appropriately smooth initial data. With a weak solution in hand corresponding to smooth data, we will show that this strong solution satisfies the PDE pointwise, along with all four cantilever boundary conditions. Lastly, we will tackle the uniqueness and continuous dependence in this case through a particular decomposition of the polynomial structure of the nonlinear stiffness.

4.3 Proof of Theorem 4.1

4.3.1 Existence

Consider the positive eigenfunctions of $\mathcal{A}$ described in Section 3.5, with $\lambda_n \to \infty$; these constitute an *orthonormal* basis for $L^2(0,L)$ and *orthogonal* basis for any $\mathcal{D}(\mathcal{A}^s)$, $s \in \mathbb{R}$. Now, for each $n = 1, 2, \ldots$, we denote

$$S_n \equiv \text{span}\{s_1, s_2, \ldots, s_n\}. \tag{29}$$

Step 1 - Approximants: For fixed smooth data, $w_0 \in \mathcal{D}(\mathcal{A})$ and $w_1 \in H^2_*$, we can construct two approximating sequences $\{w_0^n\}_{n=1}^\infty$ and $\{w_1^n\}_{n=1}^\infty$ such that

$$w_0^n := \sum_{j=1}^n (w_0, s_j)\, s_j \ \in S_n \quad \text{and} \quad w_1^n := \sum_{j=1}^n (w_1, s_j)\, s_j \ \in S_n. \tag{30}$$

By construction: $$w_0^n \to w_0 \ \text{ in } \mathcal{D}(\mathcal{A}), \qquad w_1^n \to w_1 \ \text{ in } H^2_*. \tag{31}$$

and we can proceed to define smooth finite-dimensional approximations,

$$w^n(x,t) \coloneqq \sum_{j=1}^{n} q_j(t) s_j(x),$$

where each $q_j(t)$ is a smooth function of time.

From the weak form, (26), we construct a corresponding matrix system by taking $\phi = s_j$. We define the following spatial four tensor for ease of writing:

$$\mathcal{S}_{ijkl} = \ (\phi_{i,xx}\phi_{j,xx}, \phi_{k,x}\phi_{l,x}). \tag{32}$$

Interpreting $q_i s_i$ as a sum, we have the separated form of the equations:

$$\left[q_i''(s_i,s_j)\right] + Dq_i\left[\kappa_i^4(s_i,s_j)\right] + Dq_i^3\left[\mathcal{S}_{iiij}+\mathcal{S}_{jiii}\right] = (p,s_j), \tag{33}$$

where primes represent ∂_t. Initialization is given by

$$q_j(0) = (w_0,s_j)\,, \quad q_j'(0) = (w_1,s_j)\,, \quad j = 1,2,\dots\,.$$

We may then invoke standard ODE existence and uniqueness for this finite dimensional system. Noting the hypotheses on p_t, p_{tt}, we obtain a solution $\{q_j\}_{j=1}^n \in C^3(0,t^*)$, for some small $t^*(n)$.

Step 2 - Energy Level 0: The estimate below for (33) on the approximant w^n follows immediately using w_t as the multiplier in the equations (14)–(16), taken with $\iota = k_2 = 0$:

$$E_0^n(t) = E_0^n(0) + \int_0^t (p, w_t^n)\, d\tau \quad \text{for all} \quad t > 0,$$

where

$$E_0^n(t) = \frac{1}{2}\left[||w_t^n||^2 + D||w_{xx}^n||^2 + D||w_x^n w_{xx}^n||^2\right]. \tag{34}$$

Now, via Young's inequality and Grönwall applied to (34), and noting that by (31) that $\{E_0^n(0)\}_{n=1}^\infty$ is uniformly bounded in terms of the initial data $||(w_0,w_1)||^2_{\mathscr{H}}$, we obtain:

$$E_0^n(t) \le f_0\left(||p||_{L^2(0,t;L^2(0,L))}, ||w_0||_{\mathcal{D}(\mathcal{A})}, ||w_1||_{H^2_*}\right) e^{t/2} \quad \text{for all} \quad t > 0. \tag{35}$$

The function f_0 is increasing in its arguments. The estimate in (35) ensures that the time of existence for the approximants, t^*, is independent of n.

Step 3 - Boundedness of $w_{tt}^n(0)$: We will consider $E_1(t)$ as the natural "energy" corresponding to the *time*-differentiated version of the stiffness-only equation ($\iota = k_2 = 0$). For this calculation it is pivotal to establish boundedness of the sequence $\{w_{tt}^n(0)\}_{n=1}^\infty$ in $L^2(0,L)$. To that end, it is true that the following holds for all $\phi \in S_n$, $n = 1,2,\ldots$:

$$\left(w_{tt}^n + D\partial_x^4 w^n - D\partial_x\left[(w_{xx}^n)^2 w_x^n\right] + D\partial_{xx}\left[w_{xx}^n(w_x^n)^2\right] - p, \phi\right) = 0. \tag{36}$$

We consider $\phi = s_j(x),\ j = 1,2,\ldots,n$. Then, multiplying (36) by $q_j''(t)$, summing over the $j's$, and rearranging the terms we obtain:

$$||w_{tt}^n||^2 = (p,w_{tt}^n) - D(\partial_x^4 w^n, w_{tt}^n) + D(\partial_x([w_{xx}^n]^2 w_x^n), w_{tt}^n) - D(\partial_{xx}([w_x^n]^2 w_{xx}^n), w_{tt}^n). \tag{37}$$

Owing to the C^3 temporal regularity of w^n, we can take $t = 0$ in the above expression. Therefore, using (i) the expanded version of **[NL Stiffness]** shown in *Remark* 3.1, (ii) the Sobolev embedding into $H^1 \hookrightarrow L^\infty$, (iii) and Poincaré for various derivatives, we have:

$$||w_{tt}^n(0)|| \lesssim ||p(0)|| + ||w_{xxx}^n(0)|| \; ||\partial_x^4 w^n(0)||^2 + ||w_{xx}^n(0)|| \; ||w_{xxx}^n(0)||^2 + \left(1 + ||w_{xx}^n(0)||^2\right) ||\partial_x^4 w^n(0)||.$$

The expression on the right-hand side is bounded. Indeed, by (31), $||\partial_x^4 w^n(0)|| \lesssim ||w_0||_{\mathcal{D}(\mathcal{A})}$. Moreover, by hypothesis, since $p, p_t \in L^2(0,T; L^2(0,L))$, $||p(0)||$ is interpreted as a temporal trace [15], with $||p(0)|| \lesssim ||p||_{H^1(0,T;L^2(0,L))}$. Hence we conclude that

$$||w_{tt}^n(0)|| \leq f\left(||p||_{H^1(0,T;L^2(0,L))}, ||w_0||_{\mathcal{D}(\mathcal{A})}\right). \tag{38}$$

Step 4 - Energy Level 1: Our goal now is to form the $E_1(t)$ energy which will correspond to time differentiation of the stiffness dynamics. We note that time differentiation does not affect the boundary conditions for $w^n(x,t) = \sum_{i=1}^n q_i(t) s_i(x)$. Hence, after proceeding with appropriate integration by parts, isolating conserved quantities, and gathering similar terms, we obtain the a priori identity:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[||w_{tt}||^2 + D||w_{xxt}||^2 + D||w_{xx}w_{xt}||^2 + D||w_{xxt}w_x||^2\right] \\ &\qquad = -\frac{d}{dt}\Big[4D(w_x w_{xx}, w_{xt} w_{xxt})\Big] + (p_t, w_{tt}) + 3D(w_{xx}w_{xxt}, w_{xt}^2) + 3D(w_x w_{xt}, w_{xxt}^2). \end{aligned} \tag{39}$$

We have omitted the superscript n here and in the estimation below for ease of presentation. The identity above is integrated in time on $(0,t)$, with an eye to utilize a version of Grönwall's inequality.

Remark 4.2. As an a priori estimate, the equality above holds for approximate solutions, which are appropriately smooth; this can be seen by operating directly on the ODE system (33), differentiating in time, multiplying by q_j'' and integrating in time.

Accordingly, we define the energy $E_1^n(t)$ precisely, corresponding to smoother norms for a solution:

$$E_1^n(t) = \frac{1}{2}\left[||w_{tt}^n||^2 + D||w_{xxt}^n||^2 + D||w_{xt}^n w_{xx}^n||^2 + D||w_x^n w_{xxt}^n||^2\right]. \tag{40}$$

Now, we must bound/absorb the unsigned quantities in the energy identity (39) above. We first note some important intermediate inequalities. (We have freely used: Young's inequality, Poincaré, Sobolev interpolation, and the continuous embedding $H^{1/2^+}(0,L) \hookrightarrow L^\infty(0,L)$.)

1. $3D\left|(w_{xx}w_{xxt}, w_{xt}^2)\right| \leq 3D||w_{xt}||_{L^\infty}^2 ||w_{xx}|| \; ||w_{xxt}|| \lesssim ||w_{xx}||^4 + ||w_{xxt}||^4$
2. $3D\left|(w_x w_{xt}, w_{xxt}^2)\right| \leq 3D||w_x||_{L^\infty}||w_{xt}||_{L^\infty}||w_{xxt}||^2 \lesssim ||w_{xx}||^4 + ||w_{xxt}||^4$
3. $4D\left|(w_x w_{xx}, w_{xt}w_{xxt})\right| \leq \varepsilon_1 ||w_x w_{xxt}||^2 + C_{\varepsilon_1}||w_{xx}w_{xt}||^2 \leq \varepsilon_1||w_x w_{xxt}||^2 + C_{\varepsilon_1}||w_{xt}||_{L^\infty}^2 ||w_{xx}||^2$.

To continue our estimation of 3 above, we interpolate the term $||w_{xt}||_{L^\infty}^2$ as follows:

$$||w_{xt}||_{L^\infty}^2 \lesssim ||w_t||_{3/2+\epsilon}^2 \lesssim ||w_t||^{1/2-\epsilon} \; ||w_{xxt}||^{3/2+\epsilon}.$$

Substituting the above in 3 and then utilizing Young's inequality in the (p,q) setting we obtain:

$$\begin{aligned} 4D\left|(w_x w_{xx}, w_{xt}w_{xxt})\right| &\leq \varepsilon_1||w_x w_{xxt}||^2 + C_{\varepsilon_1}||w_t||^{1/2-\epsilon} \; ||w_{xxt}||^{3/2+\epsilon} \; ||w_{xx}||^2 \\ &\leq \varepsilon_1||w_x w_{xxt}||^2 + C_{\varepsilon_1}C_{\varepsilon_p}||w_t||^{(1/2-\epsilon)q} \;\; ||w_{xx}||^{2q} + C_{\varepsilon_1}\varepsilon_p||w_{xxt}||^{(3/2+\epsilon)p}. \end{aligned} \tag{41}$$

We choose $p > 1$ such that $(3/2+\epsilon)p = 2$. Hence, by fixing $\epsilon = 1/4$, we obtain $p = 8/7$ and $q = 8$. Inequality in (41) becomes:

$$4D\,|(w_x w_{xx}, w_{xt} w_{xxt})| \le \varepsilon_1 ||w_x w_{xxt}||^2 + C_{\varepsilon_1} C_{\varepsilon_p} ||w_t||^4 + C_{\varepsilon_1} C_{\varepsilon_p} ||w_{xx}||^{32} + C_{\varepsilon_1} \varepsilon_p ||w_{xxt}||^2.$$

Choosing ε_1 and ε_p sufficiently small, we can absorb terms by $E_1^n(t)$ on the LHS of (39). Thus, using (31) in passing to the limit on the RHS, and invoking the result from (38), we arrive at the estimate:

$$E_1^n(t) \le f_1\left(p_t, ||w_0||_{\mathcal{D}(\mathcal{A})}, ||w_1||_{H_*^2}\right) + f_2\left(p, ||w_0||_{\mathcal{D}(\mathcal{A})}, ||w_1||_{H_*^2}\right) t + C\int_0^t \left[E_1^n(\tau)\right]^2 d\tau. \qquad (42)$$

Note that $C > 0$ above *does not depend* on w_0, w_1 or p. The functions f_1 and f_2 are smooth, real-valued functions, increasing in their arguments. In particular, the function f_2 is obtained after we apply (35) to the norms $||w_{xx}||^4$, $||w_t||^4$ and $||w_{xx}||^{32}$ that appear on the RHS of the estimates (1)–(4). Dependence on p we take to mean dependence on the norm $||p||_{L^2(0,t;L^2(0,L))}$ (mutatis mutandis for p_t), as in the previous step.

Hence, using the *nonlinear* version of Grönwall's inequality [11], we obtain a local-in-time estimate:

$$E_1^n(t) \le \frac{f_1 + f_2 t}{1 - C\left[f_1 t + f_2 t^2\right]} \equiv M_1(t), \quad 0 \le t < T^* \;\text{ where }\; T^* = \sup_{t>0}\left\{C\left[f_1 t + f_2 t^2\right] < 1\right\}. \qquad (43)$$

Remark 4.3. Following the assumptions of Theorem 4.1, requiring $p \in H^2_{loc}(0,\infty;L^2)$ is done here since the version of Grönwall we utilize for (42) requires f_1 and f_2 to be continuous functions in time.

Then, for any fixed $T < T^*$, we have that (43) constitutes a uniform-in-n a priori bound on $E_1^n(t) < M_1^*(T)$, $\;t \in [0,T]$, where

$$M_1^*(T) = \max_{t\in[0,T]} M_1(t); \qquad (44)$$

this quantity depends only on fixed norms of the data and T.

Remark 4.4. It is also important to note that, for a fixed t, $M_1(t)$ is an *increasing* function in $||(w_0,w_1)||_{\mathscr{H}_s}$ that vanishes when $p = w_0 = w_1 = 0$; this is used for continuous dependence.

From (43), we conclude that the Galerkin approximations satisfy a local-in-time bound by the data on any interval $[0,T]$ with $T < T^*$[5]).

Whenever the initial data $(w_0,w_1) \in \mathscr{H}$, as well as p, are fixed, then T^* is fixed; *hence, for the existence portion of the proof of Theorem 4.1, we take* $T < T^*$ *fixed and consider* $t \in [0,T]$.

Step 5 - Additional Spatial Regularity: Unlike the standard approach, we cannot obtain the needed additional boundedness of $\partial_x^4 w$ by going back through the equation (with additional regularity of w_{tt} established). To obtain further regularity of solutions, spatial differentiation is used.

Remark 4.5. Owing to the high order boundary conditions, one must take care in this process. We note energy identities associated with one spatial differentiation result in problematic trace terms that cannot be controlled by the conservative energetic terms. Moreover, as spatial differentiation produces mixed time-space terms, we do not proceed to obtain an energy estimate in this scenario; rather, we utilize an equipartition multiplier and integrate in space-time, which will provide control of the term

$$||\partial_x^4 w||^2_{L^2(0,t;L^2(0,L))} - ||w_{xxt}||^2_{L^2(0,t;L^2(0,L))}$$

the latter term is controlled by the estimate in the previous step.

[5]Conversely, given $T > 0$, there is a ball of data small in the sense of $\sum E_i(0)$ for which solutions exist up to T.

To obtain the a priori bound, we multiply the equation by $\partial_x^4 w$ and estimate.

$$\left(w_{tt}, \partial_x^4 w\right) + D\left(\partial_x^4 w, \partial_x^4 w\right) - D\left(\partial_x[w_{xx}^2 w_x], \partial_x^4 w\right) + D\left(\partial_x^2[w_{xx} w_x^2], \partial_x^4 w\right) = \left(p, \partial_x^4 w\right). \tag{45}$$

Note that as in Step 3, this can be justified by multiplying the weak ODE form (33) by $\lambda_j q_j$ (see [23]). We integrate the above in time on $(0,t)$. For the first term of (45) we integrate by parts:

$$\int_0^t \left(w_{tt}, \partial_x^4 w\right) = \int_0^t (w_{xxtt}, w_{xx}) = (w_{xxt}, w_{xx})\big|_0^t - \int_0^t ||w_{xxt}||^2.$$

For the remaining of the terms in (45) we identify positive quantities and gather terms.

$$\begin{aligned} &D\int_0^t \left[||\partial_x^4 w||^2 + ||w_x \partial_x^4 w||^2\right] d\tau - \int_0^t ||w_{xxt}||^2 d\tau \\ &\quad = \int_0^t \left[\left(p, \partial_x^4 w\right) - D\left(w_{xx}^3, \partial_x^4 w\right) - 4D\left(w_x w_{xx} w_{xxx}, \partial_x^4 w\right)\right] d\tau - (w_{xxt}, w_{xx})\Big|_0^t. \end{aligned} \tag{46}$$

We now bound the expressions that appear on the RHS above.

1. $\left|(w_{xxt}, w_{xx})\big|_0^t\right| \le ||w_{xxt}(t)||\ ||w_{xx}(t)|| + ||w_{xxt}(0)||\ ||w_{xx}(0)||$
2. $\left|\left(p, \partial_x^4 w\right)\right| \le\ C_\varepsilon ||p||^2 + \varepsilon ||\partial_x^4 w||^2$
3. $$\begin{aligned} D\left|\left(w_{xx}^3, \partial_x^4 w\right)\right| &\le \delta ||\partial_x^4 w||^2 + C_\delta ||w_{xx}||_{L^6}^6 \\ &\le \delta ||\partial_x^4 w||^2 + C_\delta ||w_{xx}||_{L^\infty}^4 ||w_{xx}||^2 \le\ \delta ||\partial_x^4 w||^2 + C_\delta ||w_{xx}||_{1/2+\epsilon}^4 ||w_{xx}||^2 \\ &\le \delta ||\partial_x^4 w||^2 + C_\delta \left[||w||_4^{1+2\epsilon} ||w_{xx}||^{3-2\epsilon}\right] ||w_{xx}||^2 \quad (\text{take } \epsilon = 1/4) \\ &\le \delta ||\partial_x^4 w||^2 + C_\delta \delta_p ||w||_4^{(3/2)p} + C_{\delta_1,\delta_p} ||w||_2^{(9/2)q} \quad (\text{take } p = 4/3) \\ &\le\ (\delta + C_\delta \delta_p)\, ||\partial_x^4 w||^2 + C_{\delta,\delta_p} ||w_{xx}||^{18} \end{aligned}$$
4. $$\begin{aligned} 4D\left|(w_x w_{xx} w_{xxx}, \partial_x^4 w)\right| &\le \eta ||\partial_x^4 w||^2 + C_\eta ||w_x||_{L^\infty}^2 ||w_{xx}||_{L^\infty}^2 ||w_{xxx}||^2 \\ &\le \eta ||\partial_x^4 w||^2 + C_\eta ||w_{xx}||^2 \left[||w_{xx}||^{3/2-\epsilon} ||w||_4^{1/2+\epsilon}\right] \left[||w_{xx}||\ ||w||_4\right] \quad (\text{take } \epsilon = 1/4) \\ &\le \eta ||\partial_x^4 w||^2 + C_\eta \eta_p ||w||_4^{(7/4)p} + C_{\eta,\eta_p} ||w_{xx}||^{(17/4)q} \quad (\text{take } p = 8/7) \\ &\le\ (\eta + C_\eta \eta_p)\, ||\partial_x^4 w||^2 + C_{\eta,\eta_p} ||w_{xx}||^{34}. \end{aligned}$$

We choose $\varepsilon, \delta, \delta_p, \eta, \eta_p$ so that, upon integration, $\int_0^t ||\partial_x^4 w||^2 d\tau$ is absorbed by the LHS of (46). Hence, by denoting

$$V(t) = D\left|\left|\partial_x^4 w\right|\right|^2 + D\left|\left|w_x \partial_x^4 w\right|\right|^2,$$

and $V^n(t)$ the above functional evaluated on w^n, we estimate (46) as:

$$\int_0^t V^n(\tau) d\tau \le f_3\left(t, p, E_0^n(0), E_1^n(0), E_0^n(t), E_1^n(t)\right) \quad \text{for all } t \in [0,T], \tag{47}$$

where we have invoked the estimates from the previous level (35) and (43), and $T < T^*$. Again, f_3 is increasing in its arguments, and dependence on p is taken as in the previous sections.

Remark 4.6. Note that (47) *is not a true energy estimate* in the sense of pointwise-in-time control of an "energy". The estimate above highlights the need to first close the higher time estimate for solutions in order to use the equipartition approach.

Based on the boundedness of of $E_0^n(0)$ and $E_1^n(0)$, along with the combination of (35),(43), (47), we deduce that

$$\int_0^t V^n(\tau)d\tau \leq f_3\left(t, p, p_t, ||w_0||_{\mathcal{D}(\mathcal{A})}, ||w_1||_{H_*^2}, M_1^*(T)\right) \quad \text{for all } t \in [0,T]. \tag{48}$$

Combining (43) and (48), we arrive at the final energy estimate for boundedness of

$$||w^n||_{L^2(0,T;\mathcal{D}(\mathcal{A}))} + ||w_t^n||_{L^\infty(0,T;\mathcal{D}(\mathcal{A}^{1/2}))} + ||w_{tt}^n||_{L^\infty(0,T;L^2(0,L))} \leq C(\text{data}, T), \tag{49}$$

where $T < T^*$ as in (43) and the dependence on "data" is as in the RHS of (48). This bound holds for the associated subsequential weak limit points and provides additional compactness below.

Remark 4.7. Denoting w as the function corresponding to the weak/weak-* limit above, we see that $w \in L^2(0,T;H^4(0,L))$ and $w_t \in L^2(0,T;H_*^2)$; hence we obtain in the standard way [15] the auxiliary bound for $w \in C([0,T];H^3(0,L))$.

Step 6 - Limit Passage and Weak Solution: With higher a priori bounds in hand for smooth data $w_0 \in \mathcal{D}(\mathcal{A}),\ w_1 \in H_*^2$, we proceed to pass with the limit and construct a weak solution satisfying (26) with $k_2 = \iota = 0$ and $\sigma = 1$ on any $[0,T]$ for $T < T^*(w_0, w_1, p)$.

From (49), Banach-Aloaglu yields existence of a subsequence $\{w^{n_k}\}_{k=1}^\infty$ and associated weak limit point

$$w \in L^2\left(0,T;H^4(0,L)\right) \cap H^1\left(0,T;H_*^2\right) \cap H^2\left(0,T;L^2(0,L)\right), \quad \text{such that} \tag{50}$$

$$w^{n_k} \rightharpoonup w \in L^2\left(0,T;\mathcal{D}(\mathcal{A})\right); \quad w_t^{n_k} \rightharpoonup w_t \in L^2\left(0,T;H_*^2\right); \quad w_{tt}^{n_k} \rightharpoonup w_{tt} \in L^2\left(0,T;L^2(0,T)\right), \tag{51}$$

with compactness of the Sobolev embeddings and Aubin-Lions ensuring strong convergence for w_{n_k} in $L^2(0,T;H_*^2)$.

Now, based on Definition 1, in order to identify w as a *weak solution*, it must satisfy the weak formulation (26) with $k_2 = \iota = 0$ and $\sigma = 1$. Identification for linear terms in (26) immediately follows from the above weak convergence, whereas the two [**NL Stiffness**] terms require more attention. For $\phi \in H_*^2$, adding and subtracting mixed terms, we obtain (omitting temporal integration):

$$\begin{aligned}
\left([w_x^{n_k}]^2 w_{xx}^{n_k} - w_x^2 w_{xx}, \phi_{xx}\right) &\leq \left([w_x^{n_k}]^2(w_{xx}^{n_k} - w_{xx}), \phi_{xx}\right) + \left(w_{xx}([w_x^{n_k}]^2 - w_x^2), \phi_{xx}\right) \\
&\leq ||w_x^{n_k}||_{L^\infty}^2 (w_{xx}^{n_k} - w_{xx}, \phi_{xx}) + ||\phi_{xx}||\ ||w_{xx}||\ ||[w_x^{n_k}]^2 - w_x^2||_{L^\infty} \\
&\leq ||w_{xx}^{n_k}||^2 (w_{xx}^{n_k} - w_{xx}, \phi_{xx}) + ||\phi_{xx}||\ ||w_{xx}||\ ||w_x^{n_k} + w_x||_{L^\infty} ||w_x^{n_k} - w_x||_{L^\infty} \\
&\leq ||w_{xx}||^2 (w_{xx}^{n_k} - w_{xx}, \phi_{xx}) + 2\ ||\phi_{xx}||\ ||w_{xx}||^3 ||w_x^{n_k} - w_x||_{1/2+\epsilon} \\
&\quad \to 0 \ \text{ as } \ k \to \infty.
\end{aligned}$$

The above calculation requires no additional regularity of solutions, and follows from bounds at the baseline energy level E_0, i.e., $w, w^n \in L^\infty(0,T;H_*^2) \cap W^{1,\infty}(0,T;L^2(0,L))$. Below, we isolate the problematic nonlinear difference, and critically use additional regularity gained in the preceding steps.

$$\begin{aligned}
\left([w_{xx}^{n_k}]^2 w_x^{n_k} - w_{xx}^2 w_x, \phi_x\right) &\leq \left([w_{xx}^{n_k}]^2(w_x^{n_k} - w_x), \phi_x\right) + \left(w_x([w_{xx}^{n_k}]^2 - w_{xx}^2), \phi_x\right) \\
&\leq ||\phi_x||_{L^\infty} ||w_{xx}^{n_k}||^2 ||w_x^{n_k} - w_x||_{L^\infty} + ||\phi_x||_{L^\infty} ||w_x||_{L^\infty} ||[w_{xx}^{n_k}]^2 - w_{xx}^2|| \\
&\leq ||\phi_{xx}||\ ||w_{xx}^{n_k}||^2 ||w_{xx}^{n_k} - w_{xx}|| + ||\phi_{xx}||\ ||w_{xx}||\ ||w_{xx}^{n_k} + w_{xx}||_{L^\infty} ||w_{xx}^{n_k} - w_{xx}|| \\
&\leq ||\phi_{xx}||\ ||w_{xx}||^2 ||w_{xx}^{n_k} - w_{xx}|| + 2\ ||\phi_{xx}||\ ||w_{xx}||\ ||w_{xxx}||\ ||w_{xx}^{n_k} - w_{xx}|| \\
&\quad \to 0 \ \text{ as } \ k \to \infty.
\end{aligned}$$

We emphasize the need for strong convergence for $w_{xx}^{n_k}$ in $L^2(0,T;L^2(0,T))$ obtained through compactness of the higher estimates.

Remark 4.8. Algebraic manipulations of the difference $[w_{xx}^{n_k}]^2 w_x^{n_k} - w_{xx}^2 w_x$ reveal a clear compactness gap for limit passage at the level of only $||w_{xx}^{n_k}||$ boundedness. An alternative approach for the identification of limit points for $\{[w_{xx}^{n_k}]^2 w_x^{n_k}\}_{k=1}^{\infty}$ (which uniformly bounded in L^1), would be to utilize the Dunford-Pettis weak compactness criterion in L^1. However, associated multiplier estimates bring about non-trivial commutators corresponding to the quasilinear nature of [**NL Stiffness**].

We conclude that the limit point w, as above, satisfies the weak formulation (26) with $k_2 = \iota = 0$ and $\sigma = 1$, and is thusly a *weak solution.*

Step 7 - Strong Solution and Free Boundary Condition: With a weak solution $w(x,t)$ in hand corresponding to smooth initial data, we have immediately that the solution is strong, by Definition 3 and the regularity afforded by (51). This concludes the proof of Theorem 4.1. □

Naturally, we would like to show that the strong solution constructed above satisfies the PDE pointwisedly, as well as the higher order boundary conditions.

Corollary 4.2. *Strong solutions $w(x,t)$ as described in Definition 3, satisfy equation* (14) *with $\sigma = 1$ and $\iota = k_2 = 0$ almost everywhere in space and in time. Additionally, they satisfy the free boundary conditions: $w_{xx}(L,t) = w_{xxx}(L,t) = 0$ for all $0 \leq t \leq T$.*

Proof of Corollary 4.2. The weak limit w constructed above satisfies:

$$(w_{tt},\phi) + D(w_{xx},\phi_{xx}) + D(w_{xx}^2 w_x,\phi_x) + D(w_x^2 w_{xx},\phi_{xx}) = (p,\phi), \quad \forall \phi \in H_*^2, \;\; a.e.\ t. \tag{52}$$

Having in hand the regularity given in (50), we undo integration by parts in (52) evaluated on test functions to obtain the strong form of the PDE. That is:

$$\big(w_{tt} + D\partial_x^4 w - D\partial_x\big[w_{xx}^2 w_x\big] + D\partial_{xx}\big[w_{xx}w_x^2\big] - p,\phi\big) = 0, \quad \forall \phi \in C_0^{\infty}(0,L).$$

Via density, we have:

$$w_{tt} + D\partial_x^4 w - D\partial_x\big[w_{xx}^2 w_x\big] + D\partial_{xx}\big[w_{xx}w_x^2\big] = p \quad a.e.\ x, \;\; a.e.\ t, \tag{53}$$

and thus the PDE in (14) is satisfied $a.e.\ x$ pointwisedly for $a.e.\ t$.

Since $w \in H_*^2$ by construction, we must verify the free boundary conditions. Undoing the integration by parts procedure and invoking (53) results the following boundary terms:

$$\phi_x(L)\left(w_{xx}(L) + w_x^2(L)w_{xx}(L)\right) - \phi(L)\left(w_{xxx}(L) + w_x(L)w_{xx}^2(L) + w_x^2(L)w_{xxx}(L)\right) = 0, \tag{54}$$

for all $\phi \in H_*^2$, holding $a.e.$ in t. But, as in Remark 4.7, $w \in C([0,T];H^3(0,L))$, and so we can write:

$$\phi(L)(1+w_x^2(L))w_{xxx}(L) = \phi_x(L)\left(w_{xx}(L) + w_x^2(L)w_{xx}(L)\right) - \phi(L)w_x(L)w_{xx}^2(L),$$

where the RHS is continuous function of time. Now, consider the subclass of $\phi \in H_0^1 \cap H_*^2 \subseteq H_*^2$. Then,

$$w_{xx}(L)\left(1+w_x^2(L)\right)\phi_x(L) = 0 \;\text{ for all such } \phi.$$

By the surjectivity of the trace theorem, there exists one function so that $\phi_x(L) \neq 0$, and thus

$$w_{xx}(L)\left(1+w_x^2(L)\right) = 0 \implies w_{xx}(L) = 0.$$

Now, consider $\phi \in H^2_*$. Again, by the surjectivity of the trace theorem, there exists at least one ϕ so that $\phi(L) \neq 0$. Using this ϕ and the fact that $w_{xx}(L) = 0$, (54) yields:

$$w_{xxx}(L)\left(1 + w_x^2(L)\right) = 0 \implies w_{xxx}(L) = 0.$$

Thus, we have verified that the free boundary terms $w_{xx}(L) = w_{xxx}(L) = 0$ are satisfied. □

Remark 4.9. It is particularly important that strong solutions remain in $\mathscr{H}_s \equiv \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{1/2})$ for data (w_0, w_1) emanating therefrom—namely, exhibiting regularity and satisfying all four boundary conditions. This, for instance, allows us to use Poincaré repeatedly on the solution, so, for a strong solution w, we have the norm equivalences: $||\partial_x^4 w|| \sim ||w||_{H^4(0,L)} \sim ||w||_{\mathcal{D}(\mathcal{A})}$.

4.3.2 Uniqueness and Continuous Dependence

Now, consider two strong solutions, w and v whose difference $z = w - v$ satisfies:

$$z_{tt} + D\partial_x^4 z - D\partial_x\left(w_{xx}^2 w_x - v_{xx}^2 v_x\right) + D\partial_x^2\left(w_{xx}w_x^2 - v_{xx}v_x^2\right) = 0, \tag{55}$$

as well as the strong form of the boundary conditions at $x = 0$ and $x = L$ and associated initial conditions $z_0 = w_0 - v_0$ and $z_1 = w_1 - v_1$. We consider the dynamics above on $t \in [0, T]$, where $T < T^* = \min\{T^*(w_0, w_1),\ T^*(v_0, v_1)\}$. We multiply (55) by z_t and integrate over $x \in (0, L)$.

For linear terms we have standard conserved quantities, $||z_t||^2;\ \ D||z_{xx}||^2$. We now take a closer look at the nonlinear differences. Note that the regularity of strong solutions in Definition 3 is sufficient—specifically $w_t \in L^2(0, T; H^2_*)$—to permit the calculations below.

1. $\left(\partial_x^2\left[w_{xx}w_x^2\right], z_t\right) - \left(\partial_x^2\left[v_{xx}v_x^2\right], z_t\right) = \left(w_{xx}w_x^2 - w_x^2 v_{xx}, z_{xxt}\right) + \left(w_x^2 v_{xx} - v_{xx}v_x^2, z_{xxt}\right),$

   Examining each of the resulting terms above yields:

   (i) $\left(w_{xx}w_x^2 - w_x^2 v_{xx}, z_{xxt}\right) = \left(w_x^2, z_{xx}z_{xxt}\right) = \dfrac{1}{2}\dfrac{d}{dt}||w_x z_{xx}||^2 - \left(w_x w_{xt}, z_{xx}^2\right)$

   (ii) $\left(w_x^2 v_{xx} - v_{xx}v_x^2, z_{xxt}\right) = \left(v_{xx}\left[w_x^2 - v_x^2\right], z_{xxt}\right) = \left(v_{xx}\left[w_x + v_x\right], z_x z_{xxt}\right).$

2. $-\left(\partial_x\left[w_{xx}^2 w_x\right], z_t\right) + \left(\partial_x\left[v_{xx}^2 v_x\right], z_t\right) = \left(w_{xx}^2 w_x - w_{xx}^2 v_x, z_{xt}\right) + \left(w_{xx}^2 v_x - v_{xx}^2 v_x, z_{xt}\right)$

   Like before, we examine each term separately:

   (i) $\left(w_{xx}^2 w_x - w_{xx}^2 v_x, z_{xt}\right) = \left(w_{xx}^2, z_x z_{xt}\right) = \dfrac{1}{2}\dfrac{d}{dt}||w_{xx}z_x||^2 - \left(w_{xx}w_{xxt}, z_x^2\right)$

   (ii) $\left(w_{xx}^2 v_x - v_{xx}^2 v_x, z_{xt}\right) = \left(v_x\left[w_{xx}^2 - v_{xx}^2\right], z_{xt}\right) = \left(v_x\left[w_{xx} + v_{xx}\right], z_{xx}z_{xt}\right).$

Combining the linear terms along with 1 and 2 we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[||z_t||^2 + D||z_{xx}||^2 + D||w_x z_{xx}||^2 + D||w_{xx}z_x||^2\right] \\ &\quad = D\left(w_x w_{xt}, z_{xx}^2\right) - D\left(v_{xx}\left[w_x + v_x\right], z_x z_{xxt}\right) + D\left(w_{xx}w_{xxt}, z_x^2\right) - D\left(v_x\left[w_{xx} + v_{xx}\right], z_{xx}z_{xt}\right).\end{aligned} \tag{56}$$

The expression above cannot be directly estimated, but we exploit symmetry in the polynomial nature of the nonlinearity by swapping the roles of w and v in the previous calculation (equivalent to subtracting v from w), adding the two identities, yielding the (now) symmetric identity:

$$
\frac{d}{dt}\Big[||z_t||^2 + D||z_{xx}||^2\Big] + \frac{D}{2}\frac{d}{dt}\Big[||w_x z_{xx}||^2 + ||v_x z_{xx}||^2 + ||w_{xx} z_x||^2 + ||v_{xx} z_x||^2\Big] \tag{57}
$$
$$
= D\Big(w_x w_{xt} + v_x v_{xt}, z_{xx}^2\Big) + D\Big(w_{xx} w_{xxt} + v_{xx} v_{xxt}, z_x^2\Big) - D\Big(\,[w_{xx} + v_{xx}]\,[w_x + v_x]\,, z_x z_{xxt} + z_{xx} z_{xt}\Big).
$$

Now, the third term in the above line sees z-regularity higher than that appearing in the "energetic" (i.e., positive, conservative) portion of the identity. The key step is to rewrite this term, moving time derivatives onto individual trajectories treated as coefficients—so as to exploit bounds in higher norms for individual trajectories, as well as the particular quadratic factorization appearing here.

$$
\begin{aligned}
D\Big(\,[w_{xx} + v_{xx}]\,[w_x + v_x]\,, z_x z_{xxt} + z_{xx} z_{xt}\Big) = {} & D\frac{d}{dt}\Big(\,(w_{xx} + v_{xx})\,(w_x + v_x)\,, z_x z_{xx}\Big) \\
& - D\Big(\partial_t\,[(w_{xx} + v_{xx})(w_x + v_x)], z_x z_{xx}\Big).
\end{aligned}
$$

Denote:

$$
E(t) = ||z_t||^2 + D||z_{xx}||^2 + \frac{D}{2}\Big[||w_x z_{xx}||^2 + ||v_x z_{xx}||^2 + ||w_{xx} z_x||^2 + ||v_{xx} z_x||^2\Big].
$$

Then, (57) becomes upon temporal integration:

$$
\begin{aligned}
E(t) = {} & E(0) - D\left((w_{xx} + v_{xx})\,(w_x + v_x)\,, z_x z_{xx}\right)\Big|_0^t \\
& + \int_0^t \Big[D\left(w_x w_{xt} + v_x v_{xt}, z_{xx}^2\right) + D\left(w_{xx} w_{xxt} + v_{xx} v_{xxt}, z_x^2\right) \\
& + D\Big([w_{xxt} + v_{xxt}][w_x + v_x], z_x z_{xx}\Big) + D\Big([w_{xx} + v_{xx}][w_{xt} + v_{xt}], z_x z_{xx}\Big)\Big] d\tau.
\end{aligned}
$$

The RHS terms are estimated in the following way, using the Sobolev embeddings and Poincaré, with an eye to use Grönwall:

1. $D\left|\big(w_x w_{xt} + v_x v_{xt}, z_{xx}^2\big)\right| \le ||w_x w_{xt} + v_x v_{xt}||_{L^\infty} ||z_{xx}||^2$
$$\lesssim \Big(||w_{xx}||\,||w_{xxt}|| + ||v_{xx}||\,||v_{xxt}||\Big)||z_{xx}||^2$$
2. $D\left|\big(w_{xx} w_{xxt} + v_{xx} v_{xxt}, z_x^2\big)\right| \le ||z_x||_{L^\infty}^2 \Big(||w_{xx}||\,||w_{xxt}|| + ||v_{xx}||\,||v_{xxt}||\Big)$
$$\lesssim \Big(||w_{xx}||\,||w_{xxt}|| + ||v_{xx}||\,||v_{xxt}||\Big)||z_{xx}||^2$$
3. $D\left|([w_{xxt} + v_{xxt}][w_x + v_x], z_x z_{xx})\right| \le ||w_x + v_x||_{L^\infty}\ ||z_x||_{L^\infty}\ ||w_{xxt} + v_{xxt}||\ ||z_{xx}||$
$$\lesssim \ (||w_{xx}|| + ||v_{xx}||)\,(||w_{xxt}|| + ||v_{xxt}||)\,||z_{xx}||^2$$
4. $D\left|([w_{xx} + v_{xx}][w_{xt} + v_{xt}], z_x z_{xx})\right| \le ||w_{xx} + v_{xx}||\ ||z_x||_{L^\infty}\ ||w_{xt} + v_{xt}||_{L^\infty}\ ||z_{xx}||$
$$\lesssim \ (||w_{xx}|| + ||v_{xx}||)\,(||w_{xxt}|| + ||v_{xxt}||)\,||z_{xx}||^2$$

5. $D\,|([w_{xx}+v_{xx}][w_x+v_x]z_x, z_{xx})| \leq C_{\varepsilon_1}(w,v)||z_x||^2 + \varepsilon_1||z_{xx}||^2$

$$\begin{aligned} &\lesssim C_{\varepsilon_1}(w,v)||z||\,||z_{xx}|| + \varepsilon_1||z_{xx}||^2 \\ &\leq C_{\varepsilon_1,\varepsilon_2}(w,v)||z||^2 + (\varepsilon_1+\varepsilon_2)||z_{xx}||^2 \\ &\lesssim C_{\varepsilon_1,\varepsilon_2}(w,v)\left[\int_0^t ||z_t||^2 d\tau + ||z(0)||^2\right] + (\varepsilon_1+\varepsilon_2)||z_{xx}||^2, \end{aligned}$$

where above we have used interpolation and H_*^2 norm equivalence in the second inequality, and the fundamental theorem of calculus in the fourth line. The dependence of C above is in the sense that $C(w,v) \equiv C\left(||w_{xxx}+v_{xxx}||\,||w_{xx}+v_{xx}||\right) \leq C\left(\sup_{0\leq t\leq T}\left[||w(t)||_3^2 + ||v(t)||_3^2\right]\right)$.

Thus, choosing $\varepsilon_1, \varepsilon_2$ sufficiently small, and putting 1–5 together, we obtain:

$$E(t) \leq c(1+C(w,v))E(0) + C(w,v)\int_0^t E(\tau)d\tau + \int_0^t K(w,v)E(\tau)d\tau. \tag{58}$$

We again note the dependence of $K(w,v)$ in the sense of:

$$K(w,v) \equiv K\big(||w_{xx}+v_{xx}||\,||w_{xxt}+v_{xxt}||\big) \leq K\big(||w||_2^2, ||w_t||_2^2, ||v||_2^2, ||v_t||_2^2\big).$$

The constant c in (58) does not depend on the initial data, nor the trajectories w, v.

Finally, we note the $C([0,T])$ boundedness (for $T < T^*(\text{data}_w, \text{data}_v)$) of the quantities $C(w,v),\ K(w,v)$ from the regularity of strong solutions, along with Remark 4.7 on the individual trajectories, $(w,w_t),\ (v,v_t)$. Taking $\sup_{[0,T]}$, we obtain:

$$E(t) \leq \mathcal{C}_1 E(0) + \mathcal{C}_2\int_0^t E(\tau)d\tau,$$

where $t\in[0,T]$ and we have the dependencies $\mathcal{C}_i\Big(||(w_0,w_1)||_{\mathscr{H}_s}, ||(v_0,v_1)||_{\mathscr{H}_s}, ||p||_{H^1(0,T;L^2(0,L))}\Big)$.

The standard Grönwall lemma yields:

$$E(t) \leq \mathcal{C}_1 E(0)e^{\mathcal{C}_2 t}, \quad t\in[0,T]. \tag{59}$$

Uniqueness of strong solutions follows immediately, since if $(w_0,w_1) = (v_0,v_1)$, the times of existence are identified and $E(0)=0$ gives $z=0$ in the sense of $L^2(0,T;L^2(0,L))$ for all valid T.

Continuous dependence also follows from (59), but is somewhat more subtle. Upon inspection, the constants above $\mathcal{C}_i$ are continuous, real-valued, positive functions of their arguments. Namely, the $\mathcal{C}_i(\cdots),\ i=1,2$ are bounded when restricting to $\overline{B_R(\mathscr{H}_s)}$—see Remark 4.4. Hence, for $(w_n, w_{n,t}), (w,w_t) \in \overline{B_R(\mathscr{H}_s)}$ we see that $z_n = w - w_n$ has the property that

$$(z_n(0), z_{n,t}(0)) \to (0,0) \in \mathscr{H} \implies (z_n, z_{n,t}) \to (0,0) \in C([0,T];\mathscr{H}).$$

5 The Case with Nonlinear Inertia: $\sigma = \iota = 1,\ k_2 > 0$

5.1 Precise Statement of the Theorem

Theorem 5.1. *Take $\sigma = \iota = 1$ and $k_2 > 0$, and consider $p \in H^3_{loc}(0,\infty;L^2(0,L))$. For initial data $(w_0,w_1)\in\mathcal{D}(\mathcal{A}^2)^2$, strong solutions exist up to some time $T^*(w_0,w_1,p)$ and are unique on their existence interval. For all $t\in[0,T^*)$, a solution obeys the energy identity*

$$E(t) + k_2\int_0^t ||w_{xxt}||^2_{L^2(0,L)} = E(0) + \int_0^t (p, w_t)_{L^2(0,L)}d\tau,$$

where $E(t)$ is as in (18) with $\sigma = \iota = 1$.

Restricting to $B_R(\mathcal{D}(\mathcal{A}^2)^2)$, for any $T < T^*(R,p)$ solutions depend continuously on the data in the sense of $C([0,T];\mathscr{H})$ with an estimate on the difference of two trajectories, $z = w^1 - w^2$:

$$\sup_{t\in[0,T]} \big|\big|(z(t), z_t(t))\big|\big|_{\mathscr{H}} \le C(R,T)\big|\big|(z(0), z_t(0))\big|\big|_{\mathscr{H}}, \quad \forall\, t \in [0,T].$$

Remark 5.1. The dependence $T^* = T^*\big(||(w_0, w_1)||_{\mathcal{D}(\mathcal{A}^2)\times\mathcal{D}(\mathcal{A}^2)}, ||p||_{H^3(0,T;L^2(0,L))}\big)$.

5.2 Proof Outline

For this proof we utilize a modified strategy from the previous section, as the presence of inertia (and damping) change the sequence of multipliers. Indeed, with the addition of damping (as per the discussion above), we can obtain a sequence of true energy estimates at various levels, and again exploit the techniques in the proof of the *Theorem* 4.1 after closing estimates. Due to the structure of **[NL Inertia]**, even in the presence of velocity-regularizing Kelvin-Voigt damping, further additional regularity (hence higher estimates) will be needed in the construction of solutions and their uniqueness.

5.3 Proof of Theorem 5.1

5.3.1 Existence

The setup here is the same as in Section 4.3.1. Since the inertial term and damping ($\iota = 1$ and $k_2 > 0$) are additional terms to the stiffness equation, we will proceed through the relevant calculations corresponding only to **[NL Inertia]** (**[NL Stiffness]** calculations are unchanged). *Kelvin-Voigt damping* appears in the final estimates with no discussion, owing to its linearity.

Step 1 - Approximants: Again, consider smooth data, $w_0 \in \mathcal{D}(\mathcal{A}^2)$ and $w_1 \in \mathcal{D}(\mathcal{A}^2)$, and take Fourier partial sums as $\{w_0^n\}_{n=1}^\infty$ and $\{w_1^n\}_{n=1}^\infty$. Then, as before, we have:

$$w_0^n \;\to w_0 \;\text{ in } \mathcal{D}(\mathcal{A}^2); \quad w_1^n \to w_1 \;\text{ in } \mathcal{D}(\mathcal{A}^2) \tag{60}$$

and

$$w^n(x,t) := \sum_{j=1}^n q_j(t) s_j(x),$$

for $q_j(t)$ smooth functions of time. Throughout this section we freely use $u^n = -(1/2)\int_0^x [w_x^n]^2 d\xi$.

From the weak form, (26) (this time taken with $\iota = 1$ and $k_2 > 0$), we construct the corresponding matrix system using the tensors $\mathcal{S}_{ijkl}$ from (32) and

$$\mathcal{I}_{ijkl} = \left(\int_0^x \phi_{i,x}\phi_{j,x}, \int_0^x \phi_{k,x}\phi_{l,x}\right). \tag{61}$$

Remark 5.2. The following calculation for the inertial tensor connects $\mathcal{I}_{ijkl}$ back to the weak form (26):

$$\mathcal{I}_{ijkl} = -\int_0^L \left[\left(\partial_x \int_x^L \int_0^\xi \phi_{i,x}\phi_{j,x} d\xi_2 d\xi\right)\int_0^x \phi_{k,x}\phi_{l,x} d\xi\right] dx = \int_0^L \left[\left(\int_x^L \int_0^\xi \phi_{i,x}\phi_{j,x} d\xi_2 d\xi\right)\phi_{k,x}\phi_{l,x}\right] dx.$$

Analogously to (33), we then have the separated form of the ODE system:

$$q_i''(s_i,s_j) + \left[q_i''(q_i)^2 + (q_i')^2 q_i\right]\mathcal{I}_{iiij} + k_2 q_i'\left[\delta_i^4(s_i,s_j)\right] + Dq_i\left[k_i^4(s_i,s_j)\right] + Dq_i^3\left[\mathcal{S}_{iiij} + \mathcal{S}_{jiii}\right] = (p, s_j). \tag{62}$$

Although this ODE system is not an evolution (it is quasilinear in time), it is polynomially nonlinear in the q_i's. Thus, via the implicit function theorem, we have *local* solvability for q_i'' in terms of the other quantities and lower order terms in q. Therefore, local-in-time, there are $C^4(0,t^*(n))$ solutions, again noting the regularity assumption on p.

Step 2 - Energy Level 0: For this step we examine the inertial term that corresponds to **Level 0** which was described in **Step 2** of Section 4.3.1 for the stiffness-only equation ($\iota = k_2 = 0$).

$$\left(\partial_x \left[w_x \int_x^L u_{tt}\right], w_t\right) = -\left(\int_x^L u_{tt}, w_x w_{xt}\right) = -\left(\int_x^L u_{tt}, \partial_x \int_0^x w_x w_{xt}\right) = (u_{tt}, u_t) = \frac{1}{2}\frac{d}{dt}||u_t||^2.$$

Denote $\mathcal{E}_0^n(t) = E_0^n(t) + I_0^n(t) \geq 0$, where $I_0^n(t) = \dfrac{1}{2}||u_t^n||^2$ and $E_0^n(t)$ is as in (34). Estimating conservatively, we have:

$$\mathcal{E}_0^n(t) + k_2 \int_0^t ||w_{xxt}^n||^2 d\tau \leq \mathcal{E}_0^n(0) + \frac{1}{2}\int_0^t ||p||^2 + \frac{1}{2}\int_0^t \mathcal{E}_0^n(\tau) d\tau \quad \text{for all} \quad t > 0. \tag{63}$$

From (60) and $u_t = -\int_0^x w_x w_{xt}$, so $||u_t|| \lesssim ||w||_{\mathcal{D}(\mathcal{A}^{1/2})}||w_t||_{\mathcal{D}(\mathcal{A}^{1/2})}$. It is immediate that $\{\mathcal{E}_0^n(0)\}_{n=1}^\infty$ is uniform-in-n controlled by $||(w_0,w_1)||_{\mathcal{D}(\mathcal{A}^{1/2})}$. Hence, the standard Grönwall inequality yields:

$$\mathcal{E}_0^n(t) \leq g_0\left(p, ||w_0||_{\mathcal{D}(\mathcal{A})}, ||w_1||_{H_*^2}\right) e^{t/2} \quad \text{for all} \quad t > 0. \tag{64}$$

The function g_0 is analogous as that described in (35).

Step 3 - Uniform Boundedness of Initial Inertia: To utilize the additional a priori bound described in the next step, we need the quantity $||w_{tt}^n(0)||^2 + ||u_{tt}^n(0)||^2$ to be uniformly bounded by appropriate norms on w_0 and w_1. Our proof of uniform $L^2(0,L)$ boundedness $\{w_{tt}^n(0)\}_{n=1}^\infty$ in **Step 3** in the proof of *Theorem* 4.1 *cannot be invoked for this calculation*, since additional terms now appear in the equation for $\iota = 1,\ k_2 > 0$.

From the equation, approximate solutions satisfy the relation

$$\begin{aligned} ||w_{tt}^n||^2 + D(\partial_x^4 w^n, w_{tt}^n) + k_2\left(\partial_x^4 w_t^n, w_{tt}^n\right) - D(\partial_x([w_{xx}^n]^2 w_x^n), w_{tt}^n) + D(\partial_{xx}([w_x^n]^2 w_{xx}^n), w_{tt}^n) \\ + \left(\partial_x\left[w_x^n \int_x^L u_{tt}^n\right], w_{tt}^n\right) = (p, w_{tt}^n). \end{aligned} \tag{65}$$

Examining the inertial term:

$$\left(\partial_x\left[w_x^n \int_x^L u_{tt}^n\right], w_{tt}^n\right) = -\left(u_{tt}^n, \int_0^x w_x^n w_{xtt}^n\right) = ||u_{tt}^n||^2 + \left(u_{tt}^n, \int_0^x [w_{xt}^n]^2\right),$$

where we used the expansion of u_{tt} in terms of w as in (17). Combining everything, we have the identity:

$$\begin{aligned} ||w_{tt}^n||^2 + ||u_{tt}^n||^2 = (p, w_{tt}^n) - \left(u_{tt}^n, \int_0^x [w_{xt}^n]^2\right) - k_2\left(\partial_x^4 w_t^n, w_{tt}^n\right) \\ - D(\partial_x^4 w^n, w_{tt}^n) + D(\partial_x([w_{xx}^n]^2 w_x^n), w_{tt}^n) - D(\partial_{xx}([w_x^n]^2 w_{xx}^n), w_{tt}^n). \end{aligned} \tag{66}$$

Since approximate solutions (and p) are continuous in time, we take the time-trace at $t=0$ in (66) and use Young's inequality to obtain the estimate:

$$\begin{aligned} ||w_{tt}^n(0)||^2 + ||u_{tt}^n(0)||^2 \leq{}& \delta||u_{tt}^n(0)||^2 + c_\delta||w_{xxt}^n(0)||^4 + \varepsilon||w_{tt}^n(0)||^2 \\ &+ c_\varepsilon\Big[||p(0)||^2 + ||\partial_x^4 w_t^n(0)||^2 + ||\partial_x^4 w^n(0)||^4||w_{xxx}^n(0)||^2 \\ &+ ||w_{xx}^n(0)||^2||w_{xxx}^n(0)||^4 + \left(1 + ||w_{xx}^n(0)||^4\right)||\partial_x^4 w^n(0)||^2\Big]. \end{aligned}$$

Choosing sufficiently small δ and ε, and using (60), we can finally conclude that

$$||w^n_{tt}(0)||^2 + ||u^n_{tt}(0)||^2 \leq C\left(||w_0||_{\mathcal{D}(\mathcal{A})}, ||w_1||_{\mathcal{D}(\mathcal{A})}, p(0)\right). \tag{67}$$

This fact will be used below in the next energy level.

Step 4 - Energy Level 1: In this step we proceed with examining the inertial term from the *Energy Level 1* estimate described in **Step 3** of Section 4.3.1. Our aim is to control the conserved quantity $||u_{tt}||^2$, corresponding to a (formal) time differentiation of the equations. Differentiating the inertial term in time and multiplying by w_{tt} we form:

$$\begin{aligned}\left(\partial_{xt}\left[w_x \int_x^L u_{tt}\right], w_{tt}\right) &= -\left(w_{xt}\int_x^L u_{tt}, w_{xtt}\right) - \left(w_x \int_x^L u_{ttt}, w_{xtt}\right)\\ &\equiv \qquad \mathcal{I}_1 \qquad + \qquad \mathcal{I}_2.\end{aligned}$$

For $\mathcal{I}_2$, we have $\mathcal{I}_2 = -\left(\int_x^L u_{ttt}, w_x w_{xtt}\right) = -\left(u_{ttt}, \int_0^x w_x w_{xtt}\right)$. Recalling $u_{tt}(x) = -\int_0^x \left[w_{xt}^2 + w_x w_{xtt}\right] d\xi$, we obtain:

$$\mathcal{I}_2 = \ (u_{ttt}, u_{tt}) + \left(u_{ttt}, \int_0^x w_{xt}^2\right) = \frac{1}{2}\frac{d}{dt}||u_{tt}||^2 + \frac{d}{dt}\left(u_{tt}, \int_0^x w_{xt}^2\right) - 2\left(u_{tt}, \int_0^x w_{xt}w_{xtt}\right).$$

The second term above will be estimated so that it can be absorbed by pointwise-in-time conserved quantities. The third term is identical to $\mathcal{I}_1$, and

$$\mathcal{I}_1 = \left(\int_0^x \left[w_{xt}^2 + w_x w_{xtt}\right], \int_0^x w_{xt}w_{xtt}\right) = \frac{1}{4}\frac{d}{dt}\left|\left|\int_0^x w_{xt}^2\right|\right|^2 + \left(\int_0^x w_x w_{xtt}, \int_0^x w_{xt}w_{xtt}\right).$$

Combining these calculations, we obtain:

$$\frac{d}{dt}\left[\frac{1}{2}||u_{tt}||^2 + \frac{3}{4}\left|\left|\int_0^x w_{xt}^2\right|\right|^2 + \left(u_{tt}, \int_0^x w_{xt}^2\right)\right] = -3\left(\int_0^x w_x w_{xtt}, \int_0^x w_{xt}w_{xtt}\right). \tag{68}$$

Utilizing once more the approximate inextensibility relation, we can rewrite:

$$\frac{d}{dt}\left[\frac{1}{2}||u_{tt}||^2 + \frac{3}{4}\left|\left|\int_0^x w_{xt}^2\right|\right|^2 + \left(u_{tt}, \int_0^x w_{xt}^2\right)\right] = 3\left(u_{tt}, \int_0^x w_{xt}w_{xtt}\right) + 3\left(\int_0^x w_{xt}^2, \int_0^x w_{xt}w_{xtt}\right).$$

Poincaré and the Sobolev embedding into L^∞ yields:

$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{2}||u_{tt}||^2 + \frac{3}{4}\left|\left|\int_0^x w_{xt}^2\right|\right|^2 + \left(u_{tt}, \int_0^x w_{xt}^2\right)\right] &\leq C_{\varepsilon_1}\left[||u_{tt}||^4 + \left|\left|\int_0^x w_{xt}^2\right|\right|^4\right] + C_{\varepsilon_2}||w_{xxt}||^4\\ &\quad + (\varepsilon_1 + \varepsilon_2)\,||w_{xxtt}||^2.\end{aligned} \tag{69}$$

For the unsigned, conservative term on the LHS we utilize Young's inequality with precise coefficients:

$$\left|\left(u_{tt}, \int_0^x w_{xt}^2\right)\right| \leq \ \frac{3}{8}||u_{tt}||^2 + \frac{2}{3}\left|\left|\int_0^x w_{xt}^2\right|\right|^2,$$

which is sufficient for absorption on the LHS of (69).

Now, let us introduce more notation for the estimate resulting from the above formal calculations:

$$I_1^n(t) = \frac{1}{2}||u_{tt}^n||^2 + \frac{3}{4}\left|\left|\int_0^x \left[w_{xt}^2\right]^n\right|\right|^2 \quad \text{and} \quad \mathcal{E}_1^n(t) = E_1^n(t) + I_1^n(t), \tag{70}$$

with $E_1^n(t)$ given in the stiffness analysis by (40). Compiling everything together and absorbing damping terms on the RHS, we then have that the approximate solutions w^n satisfy:

$$\mathcal{E}_1^n(t) + k_2\int_0^t ||w_{xxtt}^n||^2 \leq g_1\left(p_t, ||w_0||_{\mathcal{D}(\mathcal{A})}, ||w_1||_{\mathcal{D}(\mathcal{A})}\right) + g_2\left(p, ||w_0||_{\mathcal{D}(\mathcal{A})}, ||w_1||_{H_*^2}\right)t \tag{71}$$
$$+ C\int_0^t \left[\mathcal{E}_1^n(\tau)\right]^2 d\tau.$$

The dependencies for g_1 and g_2 follow after the application of (64) and (67). Note that $C > 0$ here *does not depend* on w_0, w_1 or p. Dependence on p is taken in the sense of (42).

Step 5 - Energy Level 2: In contrast to what was done in the stiffness-only estimate for **Step 5** of Section 4.3, we proceed to obtain an actual *energy estimate* for higher spatial regularity. Indeed, the inclusion of the strong damping $k_2 > 0$ allows us to consider improved regularity of the solution by employing the multiplier $\partial_x^4 w_t$, not permissible when $k_2 = 0$. Thus the calculations for the from Section 4.3 are modified below.

We proceed by multiplying the equation by $\partial_x^4 w_t$ and spatially integrating, with appropriate integration by parts. Here it is important to take note of the boundary conditions associated to eigenfunctions in Section 3.5 and hence to approximants w^n and all of their time derivatives as well.

Isolating conserved quantities and gathering terms yields:

$$\frac{1}{2}\frac{d}{dt}\Big[||w_{xxt}||^2 + D||\partial_x^4 w||^2 + D||w_x\partial_x^4 w||^2\Big] + k_2||\partial_x^4 w_t||^2$$
$$= \left(p, \partial_x^4 w_t\right) - 4D\left(w_x w_{xx} w_{xxx}, \partial_x^4 w_t\right) - D\left(w_{xx}^3, \partial_x^4 w_t\right) - \left(\partial_x\left[w_x\int_x^L u_{tt}\right], \partial_x^4 w_t\right).$$

We first estimate quantities associated with stiffness using (as before) interpolation, the Sobolev embeddings, and Young's inequality:

1. $4D\left|\left(w_x w_{xx} w_{xxx}, \partial_x^4 w_t\right)\right| \leq C_{\delta_1}||w_{xx}||^8 + C_{\delta_1}||\partial_x^4 w||^4 + \delta_1||\partial_x^4 w_t||^2$
2. $D\left|\left(w_{xx}^3, \partial_x^4 w_t\right)\right| \leq C_{\delta_2}||w_{xx}||_{L^\infty}^4||w_{xx}||^2 + \delta_2||\partial_x^4 w_t||^2 \leq C_{\delta_2}\left(||w_{xx}||_{L^\infty}^{16/3} + ||w_{xx}||^8\right) + \delta_2||\partial_x^4 w_t||^2.$

where we have used Young's Inequality $p = 4/3$ and $q = 4$. Subsequently, we interpolate $||w_{xx}||_{L^\infty}^{16/3}$ as:

$$||w_{xx}||_{L^\infty}^{16/3} \leq ||w_{xx}||_{1/2+\epsilon}^{16/3} \leq ||w_{xx}||^{10/3}||w_{xx}||_2^2 \lesssim ||w_{xx}||^{20/3} + ||\partial_x^4 w||^4, \tag{72}$$

where we chose $\epsilon = 1/4$ and used Young's inequality again with $p = 2$ and $q = 2$.

According to the above, we introduce the notation:

$$E_2^n(t) = ||w_{xxt}^n||^2 + D||\partial_x^4 w^n||^2 + D||w_x^n\partial_x^4 w^n||^2. \tag{73}$$

We now estimate the *inertial* contribution above, aiming to control the term $||u_{xxt}||^2$:

$$\left(\partial_x\left[w_x\int_x^L u_{tt}\right], \partial_x^4 w_t\right) = \left(w_{xx}\int_x^L u_{tt}, \partial_x^4 w_t\right) - \left(w_x u_{tt}, \partial_x^4 w_t\right)$$
$$\equiv \qquad \mathcal{J}_1 \qquad + \qquad \mathcal{J}_2.$$

We can directly bound $\mathcal{J}_1$ as follows:

$$|\mathcal{J}_1| \leq C_{\delta_3}||w_{xx}||^2_{L^\infty}||u_{tt}||^2 + \delta_3||\partial_x^4 w_t||^2 \leq C_{\delta_3}||\partial_x^4 w||^4 + C_{\delta_3}||u_{tt}||^4 + \delta_3||\partial_x^4 w_t||^2.$$

For $\mathcal{J}_2$, we note that $u_{xxt} = -w_x w_{xxt} - w_{xx}w_{xt}$, and use this expression to integrate by parts twice:

$$\mathcal{J}_2 = -\left(u_{tt}, w_{xxx}w_{xxt}\right) - 2\left(u_{xxt}, w_{xx}w_{xxt}\right) + \frac{1}{2}\frac{d}{dt}||u_{xxt}||^2 + \left(u_{xxtt}, w_{xx}w_{xt}\right). \tag{74}$$

We estimate the remaining unsigned terms:

1. $-2\left(u_{xtt}, w_{xx}w_{xxt}\right) = 2\left(u_{tt}, w_{xxx}w_{xxt}\right) + 2\left(u_{tt}, w_{xx}w_{xxxt}\right),$

   we can combine the first term on the RHS with the first in (74), then control each term:

   (i) $|(u_{tt}, w_{xxx}w_{xxt})| \lesssim ||u_{tt}||^4 + ||\partial_x^4 w||^4 + ||w_{xxt}||^4$
   (where we used Young's inequality with 1 for the u_{tt} term)

   (ii) $|(u_{tt}, w_{xx}w_{xxxt})| = |(w_{xx}u_{tt}, w_{xxxt})| \leq C_{\delta_4}||\partial_x^4 w||^4 + C_{\delta_4}||u_{tt}||^4 + \delta_4||\partial_x^4 w_t||^2.$

2. $\left(u_{xxtt}, w_{xx}w_{xt}\right) = \left(\partial_t[u_{xxt}], w_{xx}w_{xt}\right) = \dfrac{d}{dt}\left(u_{xxt}, w_{xx}w_{xt}\right) - \left(u_{xxt}, w_{xxt}w_{xt}\right) - \left(u_{xxt}, w_{xx}w_{xtt}\right),$
   where each term is bounded as follows:

   (i) $|(u_{xxt}, w_{xxt}w_{xt})| \lesssim ||u_{xxt}||^4 + ||w_{xt}||^2_{L^\infty}||w_{xxt}||^2 \lesssim ||u_{xxt}||^4 + ||w_{xxt}||^4$

   (ii) $|(u_{xxt}, w_{xx}w_{xtt})| = |(w_{xx}u_{txx}, w_{xtt})| \leq C_{\varepsilon_3}||\partial_x^4 w||^4 + C_{\varepsilon_3}||u_{xxt}||^4 + \varepsilon_3||w_{xxtt}||^2$

   (iii) $|(u_{xxt}, w_{xx}w_{xt})| \leq \varepsilon||u_{xxt}||^2 + C_\varepsilon||w_{xt}||^2_{L^\infty}||w_{xx}||^2 \leq \varepsilon||u_{xxt}||^2 + C_\varepsilon||w_{xt}||^{9/4}_{L^\infty} + C_\varepsilon||w_{xx}||^{18},$

   where we used Young's inequality with $p = 9/8$ and $q = 9$. Then we use interpolation for $||w_{xt}||^{9/4}_{L^\infty}$:

   $$||w_{xt}||^{9/4}_{L^\infty} \leq ||w_t||^{9/4}_{3/2+\epsilon} \leq ||w_t||^{9/32}||w_{xxt}||^{63/32} \leq C_{\varepsilon_p}||w_t||^{18} + \varepsilon_p||w_{xxt}||^2, \tag{75}$$

   where we chose $\epsilon = 1/4$ and Young's inequality with $p = 64$ and $q = 64/63$.

   By choosing ε and ε_p sufficiently small, the above terms can be absorbed. Additionally, we note that from the previous energy bounds, (64), $||w_t^n||$ and $||w_{xx}^n||$ are bounded in any power in which they appear.

Denoting:

$$I_2^n(t) = \frac{1}{2}||u_{xxt}^n||^2 \quad \text{and} \quad \mathcal{E}_2^n(t) = E_2^n(t) + I_2^n(t),$$

where $E_2^n(t)$ is given by (73), we can obtain a clean estimate. It is true from (60) that, as before, $\{\mathcal{E}_2^n(0)\}_{n=1}^\infty$ is uniformly bounded in terms of $||(w_0, w_1)||_{\mathcal{D}(\mathcal{A})^2}$. Thus, combining (71) with a compilation of the calculations described in this step and absorbing damping terms on the RHS, we have the estimate

$$\begin{aligned}\mathcal{E}_1^n(t) + &\mathcal{E}_2^n(t) + k_2\int_0^t \left[||w_{xxtt}^n||^2 + ||\partial_x^4 w_t^n||^2\right] d\tau \\ &\lesssim g_3\left(p_t, ||w_0||_{\mathcal{D}(\mathcal{A})}, ||w_1||_{\mathcal{D}(\mathcal{A})}\right) + g_4\left(p, ||w_0||_{\mathcal{D}(\mathcal{A})}, ||w_1||_{H^2_*}\right)t + \int_0^t \left[\mathcal{E}_1^n(\tau) + \mathcal{E}_2^n(\tau)\right]^2 d\tau.\end{aligned} \tag{76}$$

We point out once again that the $C > 0$ associated to '$\lesssim$' above *does not depend* on w_0, w_1 or p and that the denoted dependence on p (and its derivative) is taken in the sense of (42).

Hence, disregarding the damping integral and invoking *nonlinear* Grönwall [11], we obtain:

$$\mathcal{E}_1^n(t)+\mathcal{E}_2^n(t)\le \frac{g_3+g_4t}{1-C\left[g_3t+g_4t^2\right]}\equiv M_2(t)\quad 0\le t<T_1^* \quad \text{where} \quad T_1^*=\sup_t\left\{C\left[g_3t+g_4t^2\right]<1\right\}. \tag{77}$$

From (77), we deduce that the Galerkin approximations w^n satisfy a uniform-in-n a priori bound on $[0,T]$ for any $T<T_1^*$:

$$0\le \mathcal{E}_1^n(t)+\mathcal{E}_2^n(t)\le M_2^*(T)\equiv \max_{t\in[0,T]} M_2(t).$$

This, along with (64), provides uniform-in-n boundedness in the associated norms of $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$ for a finite time depending on the initial data.

Step 6 - Boundedness of Initial Jerk: It is apparent from the expression of the **[NL Inertia]** in (17) that the existence of strong solutions requires higher regularity of w_{tt}. We obtain this via yet another energy level, corresponding to two temporal differentiations of the equations. To begin, we again need uniform estimates of $t=0$ quantities appearing in the energy estimates. We remark that the resulting regularity of solutions obtained here is requisite also in the latter proof of uniqueness. Lastly, we note that in order to obtain this estimate (as well as that in the previous sections for $\mathcal{E}_1^n$ and $\mathcal{E}_2^n$) with $\iota=1$, the presence of the damping term $k_2>0$ is critical.

For the upcoming energy inequality for $\mathcal{E}_3^n(t)$, we must justify boundedness in n of $\{||w_{ttt}^n(0)||\}_{n=1}^\infty$, $\{||u_{ttt}^n(0)||\}_{n=1}^\infty$. To that end, the weak equations of motion (26) hold on approximants w^n and can be differentiated in time for any fixed test function ϕ. Then, by choosing $\phi=s_j(x)$, multiplying (26) by $q_j'''(t)$ and summing over $j=1,2,\dots,n$, we obtain:

$$\begin{aligned}||w_{ttt}^n||^2+D\left(\partial_x^4w_t^n,w_{ttt}^n\right)+k_2\left(\partial_x^4w_{tt}^n,w_{ttt}^n\right)-D\left(\partial_{xt}\left[(w_{xx}^n)^2w_x^n\right],w_{ttt}^n\right)+D\left(\partial_{xxt}\left[w_{xx}^n(w_x^n)^2\right],w_{ttt}^n\right)\\ +\left(\partial_{xt}\left[w_x^n\int_x^L u_{tt}^n\right],w_{ttt}^n\right)-(p_t,w_{ttt}^n)=0.\end{aligned} \tag{78}$$

Differentiating directly, we have $u_{ttt}=-\int_0^x\left[3w_{xt}w_{xtt}+w_xw_{xttt}\right]d\xi$, which yields:

$$\begin{aligned}\left(\partial_{xt}\left[w_x^n\int_x^L u_{tt}^n\right],w_{ttt}^n\right)=-\left(\partial_t\left[w_x^n\int_x^L u_{tt}^n\right],w_{xttt}^n\right)&=-\left(w_{xt}^n\int_x^L u_{tt}^n,w_{xttt}^n\right)-\left(w_x^n\int_x^L u_{ttt}^n,w_{xttt}^n\right)\\ &\equiv \qquad \mathcal{K}_1 \qquad + \qquad \mathcal{K}_2.\end{aligned}$$

For $\mathcal{K}_1$ we proceed by undoing the integration by parts which yields:

$$\mathcal{K}_1=\left(w_{xxt}^n\int_x^L u_{tt}^n,w_{ttt}^n\right)-\left(w_{xt}^nu_{tt}^n,w_{ttt}^n\right).$$

These two terms can now be transferred to the right hand side and be estimated. For $\mathcal{K}_2$ we recall the expression for u_{ttt} above, and by adding and subtracting appropriate terms we have:

$$\mathcal{K}_2=-\left(u_{ttt}^n,\int_0^x w_x^nw_{xttt}^n\right)=||u_{ttt}^n||^2+3\left(u_{ttt}^n,\int_0^x w_{xt}^nw_{xtt}^n\right).$$

Grouping everything together, and absorbing $||w_{ttt}^n(0)||^2$ and $||u_{ttt}^n(0)||^2$ from the RHS, we obtain:

$$||w_{ttt}^n(0)||^2+||u_{ttt}^n(0)||^2\le h_1\left(p_t(0),\partial_x^kw^n(0),\partial_x^lw_t^n(0),\partial_x^4w_{tt}^n(0)\right),\quad k,l=1,2,3,4, \tag{79}$$

with h_1 is polynomial in its slots. As we can see from the above expression, it is now crucial to establish the boundedness of the sequence $\left\{\partial_x^4w_{tt}^n(0)\right\}_{n=1}^\infty$ in $L^2(0,L)$ in terms of the data, $(w_0,w_1)\in\mathcal{D}(\mathcal{A}^2)$.

To achieve this bound, we revisit the weak form (26) and test with $\phi = \partial_x^8 s_j(x)$, then multiplying by $q_j''(t)$ and summing over $j = 1, 2, \ldots, n$, yielding (after some integration by parts):

$$||\partial_x^4 w_{tt}^n||^2 + \left(\partial_x^5 \left[w_x^n \int_x^L u_{tt}^n\right], \partial_x^4 w_{tt}^n\right) = \left(p, \partial_x^4 w_{tt}^n\right) - D\left(\partial_x^8 w^n, \partial_x^4 w_{tt}^n\right) - k_2\left(\partial_x^8 w_t^n, \partial_x^4 w_{tt}^n\right)$$
$$+ D\left(\partial_x^5\left[(w_{xx}^n)^2 w_x^n\right], \partial_x^4 w_{tt}^n\right) - D\left(\partial_x^6\left[w_{xx}^n (w_x^n)^2\right], \partial_x^4 w_{tt}^n\right).$$

Brute force yields:

$$\partial_x^5 \left[w_x^n \int_x^L u_{tt}^n\right] = \partial_x^6 w^n \int_x^L u_{tt}^n - 5[\partial_x^5 w^n u_{tt}^n + w_{xx}^n u_{xxxtt}^n] - 10[\partial_x^4 w^n u_{xtt}^n + w_{xxx}^n u_{xxtt}^n] - w_x^n \partial_x^4 u_{tt}^n.$$
$$\partial_x^4 u_{tt} = -\left[6 w_{xxt} w_{xxxt} + 2 w_{xt} \partial_x^4 w_t + \partial_x^4 w w_{xtt} + 3 w_{xxx} w_{xxtt} + 3 w_{xx} w_{xxxtt} + w_x \partial_x^4 w_{tt}\right].$$
$$-\left(w_x^n \partial_x^4 u_{tt}^n, \partial_x^4 w_{tt}^n\right) = \left(\partial_x^4 u_{tt}^n, -w_x^n \partial_x^4 w_{tt}^n\right)$$
$$= ||\partial_x^4 u_{tt}^n||^2 + 6\left(\partial_x^4 u_{tt}^n, w_{xxt}^n w_{xxxt}^n\right) + 2\left(\partial_x^4 u_{tt}^n, w_{xt}^n \partial_x^4 w_t^n\right) + \left(\partial_x^4 u_{tt}^n, \partial_x^4 w^n w_{xtt}^n\right)$$
$$+ 3\left(\partial_x^4 u_{tt}^n, w_{xxx}^n w_{xxtt}^n\right) + 3\left(\partial_x^4 u_{tt}^n, w_{xx}^n w_{xxxtt}^n\right).$$

Combining the terms above, we can extract $||\partial_x^4 w_{tt}^n||^2$ and $||\partial_x^4 u_{tt}^n||^2$ on the LHS. We group the RHS terms into different categories based on the actions that are necessary to control them. ***Type 1*** is first:

$$T_1 \equiv \left(p, \partial_x^4 w_{tt}^n\right) - D\left(\partial_x^8 w^n, \partial_x^4 w_{tt}^n\right) - k_2\left(\partial_x^8 w_t^n, \partial_x^4 w_{tt}^n\right) + D\left(\partial_x^5\left[(w_{xx}^n)^2 w_x^n\right], \partial_x^4 w_{tt}^n\right) - D\left(\partial_x^6\left[w_{xx}^n (w_x^n)^2\right], \partial_x^4 w_{tt}^n\right)$$
$$- \left(\partial_x^6 w^n \int_x^L u_{tt}^n, \partial_x^4 w_{tt}^n\right) + 5\left(\partial_x^5 w^n u_{tt}^n, \partial_x^4 w_{tt}^n\right) - 6\left(\partial_x^4 u_{tt}^n, w_{xxt}^n w_{xxxt}^n\right) - 2\left(\partial_x^4 u_{tt}^n, w_{xt}^n \partial_x^4 w_t^n\right),$$

where for these terms, it is clear that

$$|T_1| \leq h_2\left(p, \partial_x^i w^n, \partial_x^j w_t^n, u_{tt}^n\right) + \varepsilon_1 ||\partial_x^4 w_{tt}^n||^2 + \delta_1 ||\partial_x^4 u_{tt}^n||^2, \quad i, j = 1, 2, \ldots, 8, \tag{80}$$

where h_2 depends on ε_1, δ_1 and is polynomial in its slots. ***Type 2*** is next:

$$T_2 \equiv 10\left(\partial_x^4 w^n u_{xtt}^n, \partial_x^4 w_{tt}^n\right) + 10\left(w_{xxx}^n u_{xxtt}^n, \partial_x^4 w_{tt}^n\right) + 5\left(w_{xx}^n u_{xxxtt}^n, \partial_x^4 w_{tt}^n\right). \tag{81}$$

For this category we will exploit the fact that $||\partial_x^4 u_{tt}^n||^2$ appears in the LHS and that $\{u_{tt}^n(0)\}_{n=1}^\infty$ is bounded in $L^2(0, L)$ as shown in (67) which will be used in interpolation for the terms $\partial_x^i u_{tt},\ i = 1, 2, 3$. We show how to control one of the terms appearing in (81).

$$\left|\left(w_{xx}^n u_{xxxtt}^n, \partial_x^4 w_{tt}^n\right)\right| \leq C_\varepsilon ||w_{xxx}^n||^2 ||u_{xxxtt}^n||^2 + \varepsilon ||\partial_x^4 w_{tt}^n||^2 \leq C_\varepsilon ||w_{xxx}^n||^{10} + C_\varepsilon ||u_{xxxtt}^n||^{5/2} + \varepsilon ||\partial_x^4 w_{tt}^n||^2,$$

where we used Young's inequality with $p = 5$ and $q = 5/4$. Then we use interpolation for $||u_{xxxtt}^n||^{5/2}$:

$$||u_{xxxtt}^n||^{5/2} \leq ||u_{tt}^n||^{5/8} ||\partial_x^4 u_{tt}^n||^{15/8} \leq C_{\varepsilon_p} ||u_{tt}^n||^{10} + \varepsilon_p ||\partial_x^4 u_{tt}^n||^2,$$

where employed Young's inequality once again with $p = 16$ and $q = 16/15$.

Remark 5.3. We can see from the explicit expression of $\partial_x^i u_{tt}^n$, $i = 0, 1, 2, 3$, that

$$u_{tt}^n(0) = u_{xtt}^n(0) = u_{xxtt}^n(L) = u_{xxxtt}^n(L) = 0.$$

Hence, Poincaré's Inequality guarantees that $||u_{tt}^n||_i \sim ||\partial_x^i u_{tt}^n||$ for $i = 1, 2, 3$.

The remaining ***Type 2*** are bounded analogously, yielding:

$$|T_2| \leq h_3\left(\partial_x^i w^n, u_{tt}^n\right) + \varepsilon_2||\partial_x^4 w_{tt}^n||^2 + \delta_2||\partial_x^4 u_{tt}^n||^2, \quad i=1,2,\ldots,5. \tag{82}$$

Finally, we have ***Type 3***:

$$T_3 \equiv -\left(\partial_x^4 u_{tt}^n, \partial_x^4 w^n w_{xtt}^n\right) - 3\left(\partial_x^4 u_{tt}^n, w_{xxx}^n w_{xxtt}^n\right) - 3\left(\partial_x^4 u_{tt}^n, w_{xx}^n w_{xxxtt}^n\right).$$

For this category, we interpolate the terms $\partial_x^i w_{tt},\ i=1,2,3$, exploiting the fact that $\{w_{tt}^n(0)\}_{n=1}^{\infty}$ is bounded in $L^2(0,L)$ as shown in (67). We omit these details, as the calculations are identical to those described for ***Type 2***. We obtain the bound:

$$|T_3| \leq h_4\left(\partial_x^i w^n, w_{tt}^n\right) + \varepsilon_3||\partial_x^4 w_{tt}^n||^2 + \delta_3||\partial_x^4 u_{tt}^n||^2, \quad i=1,2,\ldots,5. \tag{83}$$

Combining (80), (82) and (83), absorbing with $\varepsilon_k,\ \delta_k$ small, and taking the (valid on approximants) time trace at $t=0$, we produce the following estimate:

$$||\partial_x^4 w_{tt}^n(0)||^2 + ||\partial_x^4 u_{tt}^n(0)||^2 \leq h\left(p(0), \partial_x^i w^n(0), \partial_x^j w_t^n(0), w_{tt}^n(0), u_{tt}^n(0)\right), \quad i,j=1,2,\ldots,8.$$

By combining (60) and (67), we can *finally* write (79) as:

$$||w_{ttt}^n(0)||^2 + ||u_{ttt}^n(0)||^2 \leq C\left(p(0), p_t(0), ||w_0||_{\mathcal{D}(\mathcal{A}^2)}, ||w_1||_{\mathcal{D}(\mathcal{A}^2)}\right). \tag{84}$$

Step 7 - Energy Level 3: With the initial jerk bounded, we proceed with the higher energy estimate corresponding to two time differentiations of the equation. The formal identity (applying ∂_t^2 to (14) and multiplying by w_{ttt}) is:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[||w_{ttt}||^2 + D||w_{xxtt}||^2 + D||w_x w_{xxtt}||^2 + D||w_{xx}w_{xtt}||^2\right] + k_2||w_{xxttt}||^2 \\
&\quad = D(w_x w_{xt}, w_{xxtt}^2) + D(w_{xx}w_{xxt}, w_{xtt}^2) - 4D(w_{xx}w_{xxt}w_{xt}, w_{xttt}) - 2D(w_x w_{xxt}^2, w_{xttt}) \\
&\qquad - 2D(w_x w_{xx} w_{xxtt}, w_{xttt}) - 4D(w_{xxt}w_x w_{xt}, w_{xxttt}) - 2D(w_{xx}w_{xt}^2, w_{xxttt}) - 2D(w_{xx}w_x w_{xtt}, w_{xxttt}) \\
&\qquad - \left(\partial_{xtt}\left[w_x \int_x^L u_{tt}\right], w_{ttt}\right).
\end{aligned}$$

We bound the RHS, in line with previous sections, using the Sobolev embeddings and Young's; the estimates from stiffness terms are straightforward. Inertia is handled as in previous estimates. After two temporal differentiation we have:

$$\begin{aligned}
\left(\partial_{xtt}\left[w_x \int_x^L u_{tt}\right], w_{ttt}\right) &= -\left(\partial_{tt}\left[w_x \int_x^L u_{tt}\right], w_{xttt}\right) \\
&= -\left(w_{xtt}\int_x^L u_{tt}, w_{xttt}\right) - 2\left(w_{xt}\int_x^L u_{ttt}, w_{xttt}\right) - \left(w_x \int_x^L u_{tttt}, w_{xttt}\right) \\
&\equiv \qquad \mathcal{L}_1 \qquad\qquad + \qquad\qquad \mathcal{L}_2 \qquad\qquad + \qquad\qquad \mathcal{L}_3.
\end{aligned}$$

We bound $\mathcal{L}_1$ and $\mathcal{L}_2$ as:

1. $|\mathcal{L}_1| \lesssim ||w_{xtt}||_{L^\infty}||u_{tt}||\ ||w_{xttt}|| \leq C_{\varepsilon_7}||u_{tt}||^4 + C_{\varepsilon_7}||w_{xxtt}||^4 + \varepsilon_7||w_{xxttt}||^2$
2. $|\mathcal{L}_2| \lesssim ||w_{xt}||_{L^\infty}||u_{ttt}||\ ||w_{xttt}|| \leq C_{\varepsilon_8}||w_{xxt}||^4 + C_{\varepsilon_8}||u_{ttt}||^4 + \varepsilon_8||w_{xxttt}||^2.$

The term $\mathcal{L}_3$ creates the desired conserved quantity (again using the explicit representation of u_{ttt}):

$$\mathcal{L}_3 = (u_{tttt}, u_{ttt}) + 3\left(u_{tttt}, \int_0^x w_{xt}w_{xtt}\right) = \frac{1}{2}\frac{d}{dt}||u_{ttt}||^2 + 3\left(\int_x^L u_{tttt}, w_{xt}w_{xtt}\right).$$

The additional term that was produced above can be manipulated as follows:

$$\begin{aligned}\left(u_{tttt}, \int_0^x w_{xt}w_{xtt}\right) =& \frac{d}{dt}\left(u_{ttt}, \int_0^x w_{xt}w_{xtt}\right) - \left(u_{ttt}, \int_0^x w_{xtt}^2\right) - \left(u_{ttt}, \int_0^x w_{xt}w_{xttt}\right)\\ \equiv& \qquad \frac{d\mathcal{M}_1}{dt} \qquad + \qquad \mathcal{M}_2 \qquad + \qquad \mathcal{M}_3.\end{aligned}$$

Now, $\mathcal{M}_2$ and $\mathcal{M}_3$ will be moved to the right hand side and estimated as follows:

1. $|\mathcal{M}_2| \lesssim ||u_{ttt}||^2 + ||w_{xtt}^2||^2 \lesssim ||u_{ttt}||^2 + ||w_{xtt}||^2_{L^\infty}||w_{xtt}||^2 \lesssim ||u_{ttt}||^2 + ||w_{xxtt}||^4$

2. $|\mathcal{M}_3| = \left|\left(w_{xt}\int_x^L u_{ttt}, w_{xttt}\right)\right| \leq ||w_{xt}||_{L^\infty}||u_{ttt}||\ ||w_{xttt}|| \ \leq C_{\varepsilon_9}||w_{xxt}||^4 + C_{\varepsilon_9}||u_{ttt}||^4 + \varepsilon_9||w_{xxttt}||^2.$

The $\mathcal{M}_1$ is more delicate, since it must be absorbed by conservative quantities:

$$\begin{aligned}|\mathcal{M}_1| \leq \varepsilon||u_{ttt}||^2 + C_\varepsilon||w_{xtt}||^2_{L^\infty}||w_{xt}||^2 &\leq \varepsilon||u_{ttt}||^2 + C_\varepsilon||w_{xtt}||^{9/4}_{L^\infty} + C_\varepsilon||w_{xt}||^{18}\\ &\leq \varepsilon||u_{ttt}||^2 + C_{\varepsilon,\varepsilon_p}||w_{tt}||^{18} + C_\varepsilon\varepsilon_p||w_{xxtt}||^2 + C_\varepsilon||w_{xt}||^{18},\end{aligned}$$

accomplished as in (75).

Moving on, we compile the above calculations into an energy estimate, taking

$$E_3(t) = \frac{1}{2}\left[||w_{ttt}||^2 + D||w_{xxtt}||^2 + D||w_x w_{xxtt}||^2 + D||w_{xx}w_{xtt}||^2\right] \quad \text{and} \quad I_3(t) = \frac{1}{2}||u_{ttt}||^2,$$

and subsequently $\mathcal{E}_3(t) = E_3(t) + I_3(t)$. Thus, by invoking (60) and (84) to guarantee the uniform boundedness of $\mathcal{E}_3^n(0)$, we obtain:

$$\begin{aligned}\mathcal{E}_3^n(t) + k_2\int_0^t ||w^n_{xxttt}||d\tau \leq\ & g_5\left(p, p_t, p_{tt}, ||w_0||_{\mathcal{D}(\mathcal{A}^2)}, ||w_1||_{\mathcal{D}(\mathcal{A}^2)}\right) + g_6\left(p, ||w_0||_{\mathcal{D}(\mathcal{A})}, ||w_1||_{H^2_*}, M_2^*\right)t\\ &+ \sum_{j=1}^{9}\varepsilon_j\int_0^t ||w^n_{xxttt}||^2 d\tau + C\int_0^t \left[\mathcal{E}_3^n(\tau)\right]^2 d\tau \quad \text{for all } t \in [0,T],\end{aligned}$$

where $T < T_1^*$ and $M_2^*(T)$ are as in (77). In addition, $C > 0$ *does not depend* on w_0, w_1 or p. Absorbing the damping terms, we finally obtain through another application nonlinear Grönwall:

$$\mathcal{E}_3^n(t) \leq \frac{g_5 + g_6 t}{1 - C\left[g_5 t + g_6 t^2\right]} \quad 0 \leq t < T_2^* \quad \text{where} \quad T_2^* = \min_t\left(\sup_t\left\{C\left[g_3 t + g_4 t^2\right] < 1\right\}, T_1^*\right). \tag{85}$$

As before, this yields a uniform-in-n a priori bound on solutions in the topology corresponding to $\mathcal{E}_3$ on any $[0,T]$ for $T < T_2^*$. We remark once again that the regularity of p considered in theorem (5.1) is necessary for ensuring that the functions $g_1, g_2, \ldots, g_6$ are continuous functions in time, as required by the version of the Grönwall lemma we employ.

Step 8 - Sufficient Regularity for w_t:

Regularity for the damping (with smooth data) proceeds standardly, through the equation:

$$||\partial_x^4 w_t^n|| \lesssim ||p|| + ||w_{tt}^n|| + ||w_{xxx}^n||\ ||\partial_x^4 w^n||^2 + ||w_{xx}^n||\ ||w_{xxx}^n||^2 + \left(1 + ||w_{xx}^n||^2\right) ||\partial_x^4 w^n|| + ||w_{xx}^n||\ ||u_{tt}^n||.$$

Using (77) we can deduce that

$$||\partial_x^4 w_t^n|| \text{ is bounded in } L^\infty(0,T;L^2(0,L)), \tag{86}$$

for any $T < T_2^*$. Thus, combining (77) and (85) and (86), we can finally obtain a priori bounds:

$$||w^n||_{L^\infty(0,T;\mathcal{D}(\mathcal{A}))} + ||w_t^n||_{L^\infty(0,T;\mathcal{D}(\mathcal{A}))} + ||w_{tt}^n||_{L^\infty(0,T;H_*^2)} \leq C(\text{data}, T), \tag{87}$$

(among other controlled norms), where "data" indicates dependence on (w_0, w_1) measured in norms up to that of $\mathcal{D}(\mathcal{A}^2)^2$.

Step 9 - Limit Passage and Weak Solution: With our a priori bounds in hand for smooth data $w_0 \in \mathcal{D}(\mathcal{A}^2),\ w_1 \in \mathcal{D}(\mathcal{A}^2)$, we proceed to pass with the limit and construct a weak solution satisfying (26) with $\sigma = \iota = 1$ and $k_2 > 0$. The boundedness of the terms in (87) yields to the existence of a subsequence $\{w^{n_k}\}_{k=1}^\infty$ and a limit point $w \in H^1\left(0,T;\mathcal{D}(\mathcal{A})\right) \cap H^2\left(0,T;H_*^2\right)$, such that

$$w^{n_k} \rightharpoonup w \in L^2\left(0,T;\mathcal{D}(\mathcal{A})\right); \quad w_t^{n_k} \rightharpoonup w_t \in L^2\left(0,T;\mathcal{D}(\mathcal{A})\right); \quad w_{tt}^{n_k} \rightharpoonup w_{tt} \in L^2\left(0,T;H_*^2\right).$$

We must show that w satisfies the weak form (26), in this case with $\sigma = \iota = 1$ and $k_2 > 0$. The details corresponding to limit point identification for [**NL Stiffness**] are identical to those in **Step 6** of Section 4.3, thus we focus on [**NL Inertia**] terms.

We first show that $u_{tt}^{n_k} \to u_{tt}$ in $L^2\left(0,T;L^2(0,L)\right)$. To that end, we consider the differences:

$$||u_{tt}^{n_k} - u_{tt}|| \leq \left|\left|\int_x^L \left([w_{xt}^{n_k}]^2 - w_{xt}^2\right)\right|\right|^2 + \left|\left|\int_x^L \left(w_x^{n_k} w_{xtt}^{n_k} - w_x w_{xtt}\right)\right|\right|^2 \equiv \mathcal{Y}_1 + \mathcal{Y}_2.$$

We will show that both $\mathcal{Y}_1$ and $\mathcal{Y}_2$ go to zero as $k \to \infty$.

$$\mathcal{Y}_1 \lesssim ||w_{xt}^{n_k} + w_{xt}||_{L^\infty}^2 ||w_{xt}^{n_k} - w_{xt}||^2 \lesssim ||w_{xxt}||^2 ||w_{xt}^{n_k} - w_{xt}||^2 \quad \to 0 \text{ as } k \to \infty.$$

$$\begin{aligned}
\mathcal{Y}_2 \leq ||w_x^{n_k}(w_{xtt}^{n_k} - w_{xtt})||^2 + ||w_{xtt}(w_x^{n_k} - w_x)||^2 &\leq ||w_x^{n_k}||_{L^\infty}^2 ||w_{xtt}^{n_k} - w_{xtt}||^2 + ||w_{xtt}||_{L^\infty}^2 ||w_x^{n_k} - w_x||^2 \\
&\lesssim ||w_{xx}||^2 ||w_{xtt}^{n_k} - w_{xtt}||^2 + ||w_{xxtt}||^2 ||w_x^{n_k} - w_x||^2 \\
&\quad \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Now, in order to pass to the limit for the [**NL Inertia**] term we need to show that

$$\left(w_x^{n_k} \int_x^L u_{tt}^{n_k}, \phi_x\right) \to \left(w_x \int_x^L u_{tt}, \phi_x\right) \qquad \text{for all } \phi \in H_*^2.$$

$$\begin{aligned}
\left|\left(w_x^{n_k} \int_x^L u_{tt}^{n_k} - w_x \int_x^L u_{tt}, \phi_x\right)\right| &\leq \left|\left(w_x^{n_k} \int_x^L [u_{tt}^{n_k} - u_{tt}], \phi_x\right) + \left((w_x^{n_k} - w_x) \int_x^L u_{tt}, \phi_x\right)\right| \\
&\leq ||\phi_x||_{L^\infty} ||w_x^{n_k}||\ ||u_{tt}^{n_k} - u_{tt}|| + ||\phi_x||_{L^\infty} ||u_{tt}||\ ||w_x^{n_k} - w_x|| \\
&\leq ||\phi_x||_{L^\infty} ||w_x||\ ||u_{tt}^{n_k} - u_{tt}|| + ||\phi_x||_{L^\infty} ||u_{tt}||\ ||w_x^{n_k} - w_x|| \\
&\quad \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Hence, w satisfies the weak formulation (26) with $\sigma = \iota = 1$ and $k_2 > 0$. With a weak solution $w(x,t)$ in hand corresponding to smooth data, we have by Definition 4 that the solution is strong, via the estimate (87) that provides the necessary regularity for $w,\ w_t,\ w_{tt}$.

And thus we have proven theorem 5.1.

Corollary 5.2. *Strong solutions w, described in Definition 4, satisfy equation* (14) *with $\sigma = \iota = 1$ and $k_2 > 0$ in the sense of $L^2(0,T;L^2(0,L))$. Additionally, they satisfy $w_{xx}(L,t) = w_{xxx}(L,t) = 0$ for all $0 \le t \le T$.*

Proof of Corollary 5.2. The weak form is now satisfied by the constructed limit:

$$(w_{tt},\phi) + D(w_{xx},\phi_{xx}) + k_2(w_{xxt},\phi_{xx}) + D(w_{xx}^2 w_x,\phi_x) + D(w_x^2 w_{xx},\phi_{xx}) - \left(w_x \int_x^L u_{tt},\phi_x\right) = (p,\phi), \quad \forall \phi \in H_*^2, \quad a.e.\ t. \tag{88}$$

Reversing integration by parts, yields (on test functions):

$$\left(w_{tt} + D\partial_x^4 w + k_2\partial_x^4 w_t - D\partial_x\left[w_{xx}^2 w_x\right] + D\partial_{xx}\left[w_{xx}w_x^2\right] + \partial_x\left[w_x \int_x^L u_{tt}\right] - p,\phi\right) = 0, \quad \forall \phi \in C_0^\infty(0,L).$$

By density, we have the equation holding in $L^2(0,L)$, as desired:

$$w_{tt} + D\partial_x^4 w + k_2\partial_x^4 w_t - D\partial_x\left[w_{xx}^2 w_x\right] + D\partial_{xx}\left[w_{xx}w_x^2\right] + \partial_x\left[w_x \int_x^L u_{tt}\right] = p \quad a.e.\ x, \quad a.e.\ t. \tag{89}$$

The solution resides in H_*^2, but we must show the natural boundary conditions $w_{xx}(L,t) = w_{xxx}(L,t) = 0$. The argument proceeds as before, invoking (89) and yielding, upon integration by parts:

$$\phi_x(L)\left((1+w_x^2(L))w_{xx}(L) + w_{xxt}(L)\right) - \phi(L)\left((1+w_x^2(L))w_{xxx}(L) + w_x(L)w_{xx}^2(L) + w_{xxxt}(L)\right) = 0, \quad \forall \phi \in H_*^2, \tag{90}$$

where we interpret the time derivatives above distributionally. Considering $\phi \in H_0^1 \cap H_*^2 \subseteq H_*^2$, we see $\phi_x(L)\left((1+w_x^2(L))w_{xx}(L) + w_{xxt}(L)\right) = 0$. There exists one such function so that $\phi_x(L) \ne 0$, and thus

$$w_{xxt}(L) + \left(1 + w_x^2(L)\right) w_{xx}(L) = 0.$$

Now, since $w \in H^1(0,T;\mathcal{D}(\mathcal{A}))$ for smooth solutions, we have $w \in C([0,T];\mathcal{D}(\mathcal{A}))$. Hence $w_{xx}(L,t)$, $w_{xxx}(L,t)$ are continuous functions of time, so we have a linear ODE of the form $f'(t) + g(t)f(t) = 0$, with classical solution

$$w_{xx}(L,t) = w_{xx}(L,0)e^{-\int_0^t(1+w_x^2(L,s))ds}.$$

As $w_0 \in \mathcal{D}(\mathcal{A})$, $w_{xx}(L,0) = 0$ and thus $w_{xx}(L,t) = 0$ for all $t \in (0,T)$.

The same argument now applies for $\phi \in H_*^2$, yielding

$$w_{xxxt}(L) + \left(1 + w_x^2(L)\right) w_{xxx}(L) = 0,$$

from which we deduce that $w_{xxx}(L,t) = 0$ for all $t \in (0,T)$. □

5.3.2 Uniqueness and Continuous Dependence

Consider w and v to be two strong solutions of (14) with $\sigma = \iota = 1$ and $k_2 > 0$ and let $z \equiv w - v$. Using the multiplier z_t on (14) we obtain:

$$\frac{1}{2}\frac{d}{dt}\left[||z_t||^2 + D||z_{xx}||^2 + D||w_x z_{xx}||^2 + D||w_{xx}z_x||^2\right] + k_2||z_{xxt}||^2 + \left(\partial_x\left[w_x \int_x^L \overline{u}_{tt} - v_x \int_x^L \hat{u}_{tt}\right], z_t\right)$$
$$= D\left(w_x w_{xt}, z_{xx}^2\right) - D\left(v_{xx}\left[w_x + v_x\right], z_x z_{xxt}\right) + D\left(w_{xx}w_{xxt}, z_x^2\right) - D\left(v_x\left[w_{xx} + v_{xx}\right], z_{xx}z_{xt}\right), \tag{91}$$

where

$$\overline{u}_{tt}(x) = -\int_0^x \left[w_{xt}^2 + w_x w_{xtt}\right] d\xi \quad \text{ and } \quad \hat{u}_{tt}(x) = -\int_0^x \left[v_{xt}^2 + v_x v_{xtt}\right] d\xi.$$

The presence of strong damping allows us to estimate the RHS in a straightforward manner (without the subtlety needed in Section 4.3.2):

1. $D\left|\left(w_x w_{xt}, z_{xx}^2\right)\right| \lesssim ||w_{xx}||\ ||w_{xxt}||\ ||z_{xx}||^2$
2. $D\left|\left(v_{xx}\left[w_x + v_x\right], z_x z_{xxt}\right)\right| \leq C_{\varepsilon_1}||v_{xx}||^2||w_{xx} + v_{xx}||^2||z_{xx}||^2 + \varepsilon_1||z_{xxt}||^2$
3. $D\left|\left(w_{xx} w_{xxt}, z_x^2\right)\right| \lesssim ||w_{xx}||\ ||w_{xxt}||\ ||z_{xx}||^2$
4. $D\left|\left(v_x\left[w_{xx} + v_{xx}\right], z_{xx} z_{xt}\right)\right| \leq C_{\varepsilon_2}||v_{xx}||^2||w_{xxx} + v_{xxx}||^2||z_{xx}||^2 + \varepsilon_2||z_{xxt}||^2.$

For the inertial term, we have:

$$\begin{aligned}\left(\partial_x\left[w_x\int_x^L \overline{u}_{tt} - v_x\int_x^L \hat{u}_{tt}\right], z_t\right) &= -\left((w_x - v_x)\int_x^L \overline{u}_{tt}, z_{xt}\right) - \left(v_x\int_x^L\left[\overline{u}_{tt} - \hat{u}_{tt}\right], z_{xt}\right)\\ &\equiv \qquad\qquad \mathcal{N} \qquad\qquad + \qquad\qquad \mathcal{O}.\end{aligned}$$

Firstly:

$$|\mathcal{N}| = \left|\left(\overline{u}_{tt}, \int_0^x z_x z_{xt}\right)\right| \lesssim ||z_{xt}||_{L^\infty}||\overline{u}_{tt}||\ ||z_x|| \leq \varepsilon_3||z_{xxt}||^2 + c_{\varepsilon_3}||\overline{u}_{tt}||^2||z_{xx}||^2.$$

The second term $\mathcal{O}$ yields:

$$\begin{aligned}\mathcal{O} = -\left(\overline{u}_{tt} - \hat{u}_{tt}, \int_0^x v_x z_{xt}\right) &= \left(\int_0^x\left[w_{xt}^2 - v_{xt}^2\right], \int_0^x v_x z_{xt}\right) + \left(\int_0^x\left[w_x w_{xtt} - v_x v_{xtt}\right], \int_0^x v_x z_{xt}\right)\\ &\equiv \qquad\qquad \mathcal{O}_1 \qquad\qquad + \qquad\qquad \mathcal{O}_2,\end{aligned}$$

where

$$\begin{aligned}\mathcal{O}_2 = \left(\int_0^x\left[z_x w_{xtt} + v_x z_{xtt}\right], \int_0^x v_x z_{xt}\right) &= \frac{1}{2}\frac{d}{dt}\left|\left|\int_0^x v_x z_{xt}\right|\right|^2 - \left(\int_0^x v_{xt} z_{xt}, \int_0^x v_x z_{xt}\right)\\ &\quad + \left(\int_0^x z_x w_{xtt}, \int_0^x v_x z_{xt}\right).\end{aligned}$$

The conserved d/dt quantity will remain on the LHS, with the rest moved to the RHS and estimated:

$$\begin{aligned}\left|\left(\int_0^x v_{xt} z_{xt}, \int_0^x v_x z_{xt}\right)\right| &\leq \varepsilon_4||v_{xt}||_{L^\infty}^2||z_{xt}||^2 + c_{\varepsilon_4}\left|\left|\int_0^x v_x z_{xt}\right|\right|^2\\ &\leq \varepsilon_4||v_{xxt}||^2\ ||z_{xxt}||^2 + c_{\varepsilon_4}\left|\left|\int_0^x v_x z_{xt}\right|\right|^2\\ \left|\left(\int_0^x z_x w_{xtt}, \int_0^x v_x z_{xt}\right)\right| &\lesssim ||w_{xtt}||^2||z_{xx}||^2 + \left|\left|\int_0^x v_x z_{xt}\right|\right|^2.\end{aligned}$$

Remark 5.4. The above calculation demonstrates the necessity of forming an energy identity (namely the **Energy Level 3** formed in the proof of *Theorem* 5.1) that provides higher spatial regularity for w_{tt}.

Lastly we have:

$$|\mathcal{O}_1| \le \left| \left(\int_0^x (w_{xt}+v_{xt})\, z_{xt}, \int_0^x v_x z_{xt} \right) \right| \le \varepsilon_5 \left(||w_{xxt}+v_{xxt}|| \right)^2 ||z_{xxt}||^2 + c_{\varepsilon_5} \left|\left| \int_0^x v_x z_{xt} \right|\right|^2 .$$

Defining:

$$\mathcal{E}(t) = \frac{1}{2} \left[||z_t||^2 + D||z_{xx}||^2 + D||w_x z_{xx}||^2 + D||w_{xx} z_x||^2 + \left|\left| \int_0^x v_x z_{xt} \right|\right|^2 \right],$$

combining estimates for $\mathcal{N}$ and $\mathcal{O}$, and recalling $\overline{u}_{tt}(x) = -\int_0^x \left[w_{xt}^2 + w_x w_{xtt} \right]$, we obtain:

$$\mathcal{E}(t) + k_2 \int_0^t ||z_{xxt}||^2 \le \mathcal{E}(0) + \int_0^t K(w,v)\mathcal{E}(\tau)d\tau + \sum_{i=1}^{5} \varepsilon_i \int_0^t C(w,v)||z_{xxt}||^2, \tag{92}$$

where the above dependencies are of the following sense:

$$K(w,v) = K\left(||w||_3^2, ||w_t||_2^2, ||v||_3^2, ||w_{tt}||_1^2 \right) \quad \text{and} \quad C(w,v) = \left(||w||_2^2, ||w_t||_2^2 \right).$$

The regularity of strong solutions in the inertial case with data in $\mathcal{D}(\mathcal{A}^2)^2$ (see e.g., (85)) provide $w_{ttt} \in L^\infty(0,T; L^2(0,T))$ (with a bound in terms of the data), and with $w_{tt} \in L^\infty(0,T; H_*^2)$; thus we have $w_{tt} \in C([0,T]; H^1(0,L))$. From this, and the energy estimate for inertial solutions ((87) with Remark 4.7), we obtain $C([0,T])$ boundedness (for $T < T^*(\text{data}_w, \text{data}_v)$) of the quantities $C(w,v),\ K(w,v)$ the individual trajectories $(w,w_t),\ (v,v_t)$. Taking $\sup_{[0,T]}$ and choosing ε_i sufficiently small (depending on the data), we obtain:

$$\mathcal{E}(t) \le \mathcal{C}_1 \mathcal{E}(0) + \mathcal{C}_2 \int_0^t \mathcal{E}(\tau)d\tau,$$

where $t \in [0,T]$ and we have the dependencies $\mathcal{C}_i\Big(||(w_0,w_1)||_{\mathscr{H}_s^I}, ||(v_0,v_1)||_{\mathscr{H}_s^I}, ||p||_{H^2(0,T;L^2(0,L))} \Big)$. The standard Grönwall lemma yields:

$$\mathcal{E}(t) \le \mathcal{E}(0)e^{\mathcal{C}_1 t}, \quad t \in [0,T].$$

Uniqueness and continuous dependence follow as in Section 4.3.2 for stiffness-only dynamics, i.e., in the sense that $||z||_2^2 + ||z_t||^2 \lesssim \mathcal{E}(t)$.

6 Global Solutions for Sufficiently Small Data

6.1 Precise Statement of the Theorem

Theorem 6.1. *Suppose $\iota = \sigma = 1$ with $k_2 > 0$, and take $p \equiv 0$. Then there exists a number $Q > 0$ such that if $||(w_0,w_1)||_{\mathcal{D}(\mathcal{A}^2)\times\mathcal{D}(\mathcal{A}^2)} \le Q$, then the corresponding strong solution (w,w_t) of (14)–(15) has time of existence $T^*(w_0,w_1) = +\infty$ and there exist $M,\omega > 0$ depending only on Q such that*

$$||(w(t),w_t(t))||^2_{\mathcal{D}(\mathcal{A}^2)\times\mathcal{D}(\mathcal{A}^2)} \le M\exp(-\omega t).$$

We note that the above theorem will obtain unproblematically in the case of $\iota = 0$ and $k_2 > 0$, i.e., when nonlinear inertia is neglected and Kelvin-Voigt damping is included. On the other hand, it is clear the result should be possible with weaker damping. See the second point in Section 7.

6.2 Outline of Proof

We proceed as in [20, 25] to obtain global existence indirectly via the Barrier method, which exploits the superlinearity in the problem. Using the damping, we will employ stabilization type multipliers at every energy level to obtain an inequality of the form in the theorem below, which we take from [20]:

Theorem 6.2. *Suppose that $X : [0, \infty) \to [0, \infty)$ is a continuous function such that there is a $T > 0$ so that $X(T) < \infty$ and*

$$X(T) + C_1 \int_0^T X(s)ds \leq C_2 X(0) + C_3 \sum_{i=1}^{N_1} X^{\alpha_i}(0) + C_4 \sum_{i=1}^{N_2} X^{\beta_i}(T) + C_5 \sum_{i=1}^{N_3} \int_0^T X^{\gamma_i}(s)ds, \tag{93}$$

where $\alpha_i > 1,\ i = 1, 2, \ldots, N_1,\ \beta_j > 1,\ j = 1, 2, \ldots, N_2$ and $\gamma_k > 1, k = 1, 2, \ldots, N_3$. Then there exists a $\mathscr{C}$ depending on $C_i, N_i, \alpha_i, \beta_i$ so that if $X(0) \leq \epsilon \leq \mathscr{C}$, then

$$X(t) \leq \frac{\epsilon}{\mathscr{C}} \exp\left(-\mathscr{C}t\right).$$

For us, $X(t)$ will be the sum of (the majority of) the norms appearing in the formal energy identities we have constructed thus far. Any continuous function that satisfies the integral inequality has the desired property: exponential decay for sufficiently small initial conditions $X_i(0)$, which in turn yields global-in-time existence [20].

To form an inequality of the form (93) we will utilize the previous calculations we have obtained for the energy estimates in Section 5, along with additional estimates based on equipartition multipliers at each level. We will attempt to form (93) for *each* $X_i(t),\ i = 0, 1, 2, 3$ separately, where X_i's correspond to each energy level we have defined before, and sum the results. To streamline exposition of comparable calculations in the earlier sections, we demonstrate the detailed calculations for $X_0(t)$. For $X_i(T),\ i = 1, 2, 3$, we will only highlight deviations from the details in the proof of theorems 4.1 and 5.1.

6.3 Proof of Theorem 6.1

Step 1 - Inequality for X_0: Recalling the estimate for $\mathcal{E}_0$ in **Step 2** of Section 5, we redefine:

$$X_0(t) = ||w_t(t)||^2 + ||w_{xx}(t)||^2 + ||w_x w_{xx}(t)||^2 + ||u_t(t)||^2$$

and we have immediately the inequality:

$$X_0(T) + k_2 \int_0^T ||w_{xxt}||^2 \leq X_0(0). \tag{94}$$

It is crucial to retain the inertial term to appear under the integral sign, thus we augment (94) to obtain:

$$X_0(T) + \int_0^T \left[k_2 ||w_{xxt}||^2 + ||u_t||^2\right] \leq X_0(0) + \int_0^T ||u_t||^2. \tag{95}$$

The inertial term appearing on the RHS will now have to be estimated. Note that if *we bound it above by* $X_0(0)$, it will appear under the time integral and such a bound would be inconsistent with the form of inequality (93). Rather, we estimate as:

$$||u_t||^2 \lesssim ||w_x w_{xt}||^2 \lesssim ||w_{xx}||^2 ||w_{xt}||^2 \lesssim ||w_{xx}||^6 + ||w_{xt}||^3, \tag{96}$$

where we've used Young's $p = 3$ and $q = 3/2$. We interpolate $||w_{xt}||^3$ as follows:

$$||w_{xt}||^3 \leq ||w_t||^{3/2} ||w_{xxt}||^{3/2} \leq c_{\varepsilon_1} ||w_t||^6 + \varepsilon_1 ||w_{xxt}||^2, \tag{97}$$

again using Young's with $p = 4$ and $q = 4/3$. Thus we have the ε for absorption, and we obtain:

$$X_0(T) + c_1 \int_0^T \left[||w_{xxt}||^2 + ||u_t||^2\right] \leq X_0(0) + c_2 \int_0^T \left[||w_{xx}||^6 + ||w_t||^6\right]. \tag{98}$$

Invoking norm equivalence between $H^2(0, L)$ and H^2_*, (98) becomes:

$$X_0(T) + c_1 \int_0^T \left[||w_t||^2 + ||u_t||^2\right] \leq X_0(0) + c_2 \int_0^T \left[||w_{xx}||^6 + ||w_t||^6\right]. \tag{99}$$

Now, the equipartition (stability) multiplier for this level is w; multiplying (14) by the solution and integrating by parts in space and time, we obtain:

$$\begin{aligned}\frac{k_2}{2}||w_{xx}(T)||^2 + D \int_0^T \left[||w_{xx}||^2 + 2||w_x w_{xx}||^2\right] - \int_0^T \left[||w_t||^2 + ||u_t||^2\right] \\ = \frac{k_2}{2}||w_{xx}(0)||^2 - \int_0^L w w_t\Big|_0^T - 2\int_0^L u u_t\Big|_0^T.\end{aligned}$$

We note that from $u(x,t) = -\frac{1}{2}\int_0^x w_x^2(\xi, t) d\xi$, we have:

$$||u(t)||^2 \lesssim \Big|\Big| \int_0^x w_x(\xi, t) \left[\int_0^\xi w_{xx}(\zeta, t) d\zeta\right] d\xi \Big|\Big|^2 \lesssim ||w_x(t) w_{xx}(t)||^2 \lesssim X_0(t),$$

via Fubini and Jensen's inequality, having then extended the integrals to $x \in [0, L]$. Hence, $||u(T)||^2 + ||u(0)||^2 \leq c_3 X_0(0)$. The RHS can then be estimated straightforwardly, yielding:

$$\frac{k_2}{2}||w_{xx}(T)||^2 + D \int_0^T \left[||w_{xx}||^2 + 2||w_x w_{xx}||^2\right] - \int_0^T \left[||w_{xxt}||^2 + ||u_t||^2\right] \leq c_3 X_0(0). \tag{100}$$

We then take an appropriate linear combination of (99) and (100) (with constants depending on the damping coefficient k_2), and eliminate the negative terms appearing in (100). Then, by possible adjustments of the constants, we have:

$$X_0(T) + C_1 \int_0^T X_0 \leq C_2 X_0(0) + C_3 \int_0^T X_0^6. \tag{101}$$

Step 2 - Inequality for X_1 and X_2: In this step we will proceed by forming the inequality that corresponds to $X_1 + X_2$. As we will see later, there will be terms in the X_2 estimate that will need to by absorbed by some appearing in X_1. We define:

$$X_1(t) = ||w_{tt}(t)||^2 + ||w_{xxt}(t)||^2 + ||w_{xt}(t) w_{xx}(t)||^2 + ||w_x(t) w_{xxt}(t)||^2 + ||u_{tt}(t)||^2.$$

Remark 6.1. Note X_1 does not include the quantity $\left|\left|\int_0^x w_{xt}^2\right|\right|^2$, as $\mathcal{E}_1$ does. As it can be seen from the following calculations, the aforementioned norm is not needed in obtaining (93).

Following similar calculations as in **Step 4** in the proof of Theorem 5.1, we obtain:

$$\left(\partial_{xt}\left[w_x \int_x^L u_{tt}\right], w_{tt}\right) = \frac{1}{2}\frac{d}{dt}||u_{tt}||^2 + \frac{d}{dt}\left(u_{tt}, \int_0^x w_{xt}^2\right) - 3\left(u_{tt}, \int_0^x w_{xt} w_{xtt}\right). \tag{102}$$

The conserved quantity of (102) will remain to the LHS, while the remaining terms will be moved to the RHS and be estimated, after we proceed with integration in time, as follows:

1. $\left|\left(u_{tt}(T), \int_0^x w_{xt}^2(T)\right)\right| \leq \delta_1 ||u_{tt}(T)||^2 + c_{\delta_1}||w_{xxt}(T)||^4$

2. $\left|\left(u_{tt}(0), \int_0^x w_{xt}^2(0)\right)\right| \lesssim ||u_{tt}(0)||^2 + ||w_{xxt}(0)||^4$

3. $\left|\left(u_{tt}, \int_0^x w_{xt}w_{xtt}\right)\right| \leq \delta_2 ||u_{tt}||^2 + c_{\delta_2}||w_{xxt}||^6 + c_{\delta_2,\varepsilon_1}||w_{tt}||^6 + c_{\delta_2}\varepsilon_1 ||w_{xxtt}||^2.$

In addition to the above calculations, we add the term $\int_0^T ||u_{tt}||^2$ to both sides of the inequality. On the RHS it will be estimated via:

$$||u_{tt}||^2 \lesssim ||w_{xxt}||^2||w_{xt}||^2 + ||w_{xxt}||^2||w_x||\ ||w_{xtt}|| + ||w_{xx}||^2||w_{xtt}||^2.$$

Using Young's and interpolation, as in (96) and (97), and directly invoking the stiffness calculations in **Step 4** in the proof of Theorem 4.1, we arrive at:

$$X_1(T) + \int_0^T \left[k_2||w_{xxtt}||^2 + ||u_{tt}||^2\right] \leq c_1\left(X_1(0) + X_1^2(0) + X_0^2(0) + X_0^{16}(0)\right) + c_2X_1^2(T)$$
$$+c_3\int_0^T \left[X_1^2 + X_1^3 + X_1^4 + X_0^2 + X_0^3\right] + \delta\int_0^T ||u_{tt}||^2 + \varepsilon\int_0^T ||w_{xxtt}||^2, \qquad (103)$$

where ε and δ collect the various ε_i's and δ_i's corresponding to earlier applications of Young's inequality.

For the time-differentiated version of the equations, w_t acts as the equipartition multiplier. After the appropriate calculations, and straightforward estimation, we obtain:

$$\frac{k_2}{2}||w_{xxt}(T)||^2 + D\int_0^T \left[||w_{xxt}||^2 + ||w_{xt}w_{xx}||^2 + ||w_x w_{xxt}||^2\right] - \int_0^T \left[||w_{tt}||^2 + ||u_{tt}||^2\right]$$
$$\leq c_1(X_1(0) + X_0(0)) + \varepsilon_1||w_{tt}(T)||^2 + \delta_1||u_{tt}(T)||^2 + c_2\int_0^T \left[X_1^2 + X_1^0\right] + \delta_2\int_0^T ||u_{tt}||^2. \qquad (104)$$

Now, we define:

$$X_2(t) = ||w_{xxt}(t)||^2 + ||\partial_x^4 w(t)||^2 + ||w_x(t)\partial_x^4 w(t)||^2 + ||u_{xxt}(t)||^2.$$

Then, duplicating the calculations in **Step 5** in the proof of Theorem 5.1 and adding $\int_0^T ||u_{xxt}||^2$ to both sides we have:

$$X_2(T) + \int_0^T \left[k_2||\partial_x^4 w_t||^2 + ||u_{xxt}||^2\right] \leq c_1\left(X_2(0) + X_0^9(0)\right) + c_2\int_0^T \left[X_2^2 + X_1^2 + X_0^{10/3} + X_0^4\right]$$
$$+\varepsilon_2\int_0^T ||w_{xxtt}||^2. \qquad (105)$$

Remark 6.2. The term $||w_{xxtt}||^2$ appearing on the RHS of the above inequality is the reason why we chose to have the calculations of X_1 and X_2 combined.

To complete the estimate for $X_2(t)$, we proceed by employing $\partial_x^4 w$ as a multiplier. The calculations corresponding to stiffness are described in **Step 5** in the proof of Theorem 4.1. Inertial terms are handled through differentiation and spatial integration by parts:

$$\int_0^T \left(\partial_x\left[w_x\int_x^L u_{tt}\right], \partial_x^4 w\right) = \int_0^T \left(w_{xx}\int_x^L u_{tt}, \partial_x^4 w\right) - \int_0^T \left(w_x u_{tt}, \partial_x^4 w\right)$$
$$= \int_0^T \left(w_{xx}\int_x^L u_{tt}, \partial_x^4 w\right) - \int_0^T \left(u_{tt}, w_{xx}w_{xxx}\right) - 2\int_0^T \left(u_{ttx}, w_{xx}^2\right) - \int_0^T \left(u_{ttxx}, w_x w_{xx}\right).$$

The only non-trivial term to estimate is the last; integrate by parts *in* t and note $\partial_t(-w_x w_{xx}) = u_{xxt}$:

$$-\int_0^T (u_{ttxx}, w_x w_{xx}) = -w_x w_{xx} u_{xxt}\Big|_0^T - \int_0^T ||u_{xxt}||^2.$$

Combining, we obtain:

$$\begin{aligned}\frac{k_2}{2}||\partial_x^4 w(T)||^2 + D\int_0^T \left[||\partial_x^4 w||^2 + ||w_x \partial_x^4 w||^2\right] - \int_0^T \left[||w_{xxt}||^2 + ||u_{xxt}||^2\right] \\ \le c_1\left(X_2(0) + X_1(0) + X_0(0) + X_0^2(0)\right) + \varepsilon_3||w_{xxt}(T)||^2 + \delta_3||u_{xxt}(T)||^2 \\ + c_2\int_0^T \left[X_2^2 + X_1^2 + X_0^2 + X_0^9 + X_0^{17}\right] + \varepsilon_4\int_0^T ||\partial_x^4 w||^2. \end{aligned} \tag{106}$$

As before, we add (103) to (105), and we add (104) to (106); we then choose an appropriate linear combination of the sums for absorption of negative integral terms; we then choose ε_i, δ_i appropriately, and invoke norm equivalence for H_*^2, yielding the estimate for $X_1 + X_2$:

$$\begin{aligned} X_2(T) + X_1(T) + C_1\int_0^T [X_2 + X_1] \\ \le C_2\Big(X_2(0) + X_1(0) + X_1^2(0) + X_0(0) + X_0^2(0) + X_0^9(0) + X_0^{16}(0)\Big) \\ + C_3 X_1^2(T) + C_4\int_0^T \left[X_2^2 + X_2^4 + X_1^2 + X_1^3 + X_1^4 + X_0^2 + X_0^3 + X_0^9 + X_0^{17}\right]. \end{aligned} \tag{107}$$

Step 3 - Inequality for X_3: Define:

$$X_3(t) = ||w_{ttt}(t)||^2 + ||w_{xxtt}(t)||^2 + ||w_x w_{xxtt}(t)||^2 + ||w_{xx}(t) w_{xtt}(t)||^2 + ||u_{ttt}(t)||^2.$$

The estimate corresponding to *two time* differentiations of (14) with the multiplier w_{ttt} can be directly formed from the existing calculations for **Step 7** in the proof of *Theorem* (5.1).

$$\begin{aligned} X_3(T) + c_1\int_0^T \left[||w_{ttt}||^2 + ||u_{ttt}||^2\right] \le c_2\left(X_3(0) + X_2^9(0) + X_1^9(0)\right) + c_3\left(X_2^9(T) + X_1^9(T)\right) \\ + c_4\int_0^T \left[X_3^2 + X_2^2 + X_2^4 + X_1^2 + X_1^3 + X_1^4 + X_0^2 + X_0^3 + X_0^4\right], \end{aligned} \tag{108}$$

where we added $\int_0^T ||u_{ttt}||^2$ to both sides and proceeded as in earlier estimates in this section.

In this case, w_{tt} is the equipartition multiplier, and calculations corresponding to stiffness are duplicated from **Step 7** in the proof of *Theorem* (5.1). The inertial term calls for a slightly altered approach:

$$\left(\partial_{xtt}\left[w_x\int_x^L u_{tt}\right], w_{tt}\right) = -\left(w_{xtt}\int_x^L u_{tt}, w_{xtt}\right) - 2\left(w_{xt}\int_x^L u_{ttt}, w_{xtt}\right) - \left(w_x\int_x^L u_{tttt}, w_{xtt}\right).$$

The first two terms above can be treated similarly to $\mathcal{J}_i$ of **Step 5** in the proof of *Theorem* (5.1), and for the latter we write:

$$\int_0^T\left(u_{tttt}, -\int_0^x w_x w_{xtt}\right) = \int_0^T (u_{tttt}, u_{tt}) + \int_0^T \frac{d}{dt}\left(u_{ttt}, \int_0^x w_{xt}^2\right) - 2\int_0^T\left(u_{ttt}, \int_0^x w_{xt} w_{xtt}\right).$$

The first term will be integrated by parts *in time* and the following two will be estimated as above.

Hence, assembling everything together we have:

$$\frac{k_2}{2}||w_{xxtt}(T)||^2 + c_1 \int_0^T \left[||w_{xxtt}||^2 + ||w_x w_{xxtt}||^2 + ||w_{xx} w_{xtt}||^2\right] - c_2 \int_0^T \left[||w_{ttt}||^2 + ||u_{ttt}||^2\right]$$
$$\leq c_3 \left(X_3(0) + X_1^2(0) + X_1(0)\right) + c_4 X_1(T) + \varepsilon_1 ||w_{ttt}(T)||^2 + \varepsilon_2 ||u_{ttt}(T)||^2 + c_5 X_1^2(T)$$
$$+ c_6 \int_0^T \left[X_3^2 + X_2^2 + X_2^4 + X_1^2 + X_1^4 + X_0^2 + X_0^4\right]. \tag{109}$$

Combining (108) with (109) in an appropriate linear combination, we obtain:

$$X_3(T) + C_1 \int_0^T X_3 \leq C_2 \left(X_3(0) + X_2^9(0) + X_1(0) + X_1^2(0) + X_1^9(0)\right) + C_3 \left(X_1^9(T) + X_2^9(T)\right)$$
$$+ C_4 X_1(T) + C_5 \int_0^T \left[X_3^2 + X_2^2 + X_2^4 + X_1^2 + X_1^3 + X_1^4 + X_0^2 + X_0^3 + X_0^4\right]. \tag{110}$$

Here we remark that the term $X_1(T)$ appearing above is bounded by (103).

Finally, we note that the bound above depends on the boundedness of the quantity $X_3(0)$ which contains the term $||w_{ttt}(0)||^2 + ||u_{ttt}(0)||^2$. This term does not explicitly appear as data, however, it is directly bounded by the data $||(w_0, w_1)||^2_{\mathcal{D}(\mathcal{A}^2)^2}$, which can be shown directly on approximate solutions, as was the focus of **Step 6** in Section 5.1.

Step 4 - Global Estimate: With the constituent inequalities in hand from **Steps 1–3**, we form:

$$X(T) = X_0(T) + X_1(T) + X_2(T) + X_3(T).$$

This quantity is nonnegative, and continuous due to the regularity of constructed solutions. We then add (101), (107) and (110), and with minor algebraic manipulations, we obtain:

$$X(T) + C_1 \int_0^T X(s)ds \leq C_2 X(0) + C_3 F_0 + C_4 \left(X^2(T) + X^9(T)\right) + C_5 \int_0^T F(s)ds, \tag{111}$$

where

$$F_0 = X^2(0) + X^9(0) + X^{16}(0)$$

and

$$F(s) = X^2(s) + X^3(s) + X^{10/3}(s) + X^4(s) + X^6(s) + X^9(s) + X^{17}(s).$$

This final estimate (111) is of the form in Theorem 6.2, which concludes the proof of Theorem 6.1 □

7 Comments, Open Problems, and Future Work

We briefly state and discuss some open problems and directions for future work.

- **The existence of finite energy, weak solutions** seems to be a challenging one. It is clear that additional compactness is needed for the identification of limit points associated only to the stiffness portion of the dynamics. Compensated compactness requirements (e.g., those in L^1) might be adapted to the nonlinear structures, though it is unclear if such an approach would be more expedient than the higher order energy methods employed here.

- **The elimination or weakening of damping** seems a natural course. In our estimates, it is clear that the regularizing effects of Kelvin-Voigt damping are stronger than explicitly needed in the construction of solutions and estimation of inertial terms. On the other hand, weak damping of the form $k_0 w_t$ is clearly too weak to address inertial terms. Unfortunately, for cantilevered beams, the physical interpretation of $A^{1/2}w_t \sim k_1\partial_x^2 w_t$ damping is unclear—see the discussions in [8] and [17]. Additionally, it is a question for future work to utilize weaker (than Aw_t)damping to obtain global solutions with sufficiently small data for (14) with $\sigma, k_2 = 1$ and $\iota = 0$.

- Explicit **proof of blow up** for large data in this quasilinear system would nicely complement our local existence results. Currently, numerical evidence indicates that large data quickly leads to non-physical solutions.

- **The introduction of non-conservative forces** as discussed in the introduction (e.g., with application to piezoelectric energy harvesting) is a natural next step. In fact, the earlier work [8] addresses a piston-theoretic beam, as does the more recent [27, 28]. However, exploiting the superlinearity of the nonlinear stiffness to provide a rigorous framework for long-time behavior of trajectories—or even constructing limit cycle oscillations—is a desirable future goal.

- **The 2-D cantilever model**, invoking inextensible elasticity (see the engineering references [35, 36]), is the topic of forthcoming work. This challenging mathematical problem was untouchable before establishing the theory in this treatment. Difficulties for the 2-D problem include the challenging mixed, clamped-free-type plate boundary conditions, as well as the loss of the 1-D Sobolev embeddings (which were used profusely and non-trivially) in this treatment. Closing estimates will require *even higher* differentiations of the equations, resulting in further involved calculations beyond the numerous pages here.

8 Acknowledgements

The authors are generously supported by NSF-DMS-1907620, "Experiment, Theory, and Simulation of Aeroelastic Limit Cycle Oscillations for Energy Harvesting Applications".

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