ISSN: 2148-9955

International Journal of Research in Education and Science (IJRES)

www.ijres.net

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To cite this article:

Rakes, C.R. & Ronau, R.N. (2019). Rethinking mathematics misconceptions: Using knowledge structures to explain systematic errors within and across content domains. *International Journal of Education in Mathematics, Science and Technology (IJEMST), 5*(1), 1-21.

International Journal of Research in Education and Science

Volume 5, Issue 1, Winter 2019 ISSN: 2148-9955

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Article Info

Article History

Received:
23 July 2018

Accepted:
11 September 2018

Keywords

Mathematics education
Misconceptions
Mixed methods
Structural equation modeling

Abstract

The present study examined the ability of content domain (algebra, geometry, rational number, probability) to classify mathematics misconceptions. The study was conducted with 1,133 students in 53 algebra and geometry classes taught by 17 teachers from three high schools and one middle school across three school districts in a Midwestern state. Students answered 17 multiple choice items from the National Assessment of Educational Progress. A prompt was added to each item for students to explain the reasoning for their answer. Explanations were examined qualitatively to determine whether the responses were indicative of misconceptions. The responses were then analyzed quantitatively with structural equation modeling. Several knowledge structures (variable, measurement, spatial reasoning, additive/multiplicative structures, absolute/relative comparison) were found to be foundational to understanding multiple content domains. Multiple foundational structures were found to be related to misconceptions in each content domain, and misconceptions for multiple content domains were found within each foundational structure. The quantitative analyses found that the best classification of misconceptions included correlations among the content domains and a factor independent of content domain. The qualitative analyses led to the conclusion that interactions among the foundational knowledge structures best described the factor independent of content domain.

Introduction

Addressing mathematics misconceptions appropriately is critical for fostering deep student learning (Clements & Sarama, 2014; Sztajn, Confrey, Wilson, & Edgington, 2012) and learning how to do so has been recognized as an important component of mathematics teacher preparation (e.g., Association of Mathematics Teacher Educators, 2017; American Statistical Association, 2015) and effective professional practice (National Council of Teachers of Mathematics (NCTM), 2014). Addressing mathematics misconceptions while teaching begins at the lesson planning stage, when teachers consider common patterns of student reasoning, difficulties, and mistakes for a particular topic and continues through lesson enactment and assessment (Swan, 2001). Understanding how mathematics misconceptions are related to one another across topics in a conceptual framework can be especially useful for identifying broader patterns that affect student long-term mathematical development (Bransford et al., 2000).

Study Purpose and Research Questions

The present study examined student reasoning in algebra, geometry, rational number, and probability to identify common patterns across content domains to support a conceptual framework for mathematics misconceptions. Misconceptions are often described as conceptual misunderstandings (e.g., Blanco & Garrote, 2007; Durken & Rittle-Johnson, 2015; Van Dooren, Lehtinen, & Verschaffel, 2015). The present study followed Smith, diSessa, and Roschelle (1993) by defining misconceptions as any "student conception that produces a systematic pattern of errors" (p. 119). This definition excludes errors that were produced by simple procedural errors (i.e., slips and bugs, as in VanLehn, 1983) and recognizes that the reasoning included with consistent patterns of errors may be indicative of misconceptions, consistent with Earls (2018).

This definition provided a consistent construct for examining student reasoning for incorrect responses for the purpose of identifying systematic patterns indicative of misconceptions and analyzing relationships between those patterns. The investigation was structured to address three research questions:

1. How well can student reasoning be used to identify misconceptions?
2. How well does content domain classify errors rooted in misconceptions?
3. How well does the addition of factors independent of content domain improve the classification of misconceptions?

Mixed methods are needed to fully investigate these questions. While the qualitative analysis provides insight into the types of mathematics misconceptions that lead to particular responses, the quantitative analyses provide validity to generalizations about the ways those misconceptions are interconnected and what factors might lead to particular misconceptions (Creswell & Plano Clark, 2007). Research about mathematics misconceptions is typically qualitative only or includes only descriptive quantitative analyses of correct responses, only loosely connected to the misconceptions analysis (e.g., Earls, 2018; Fisher, 1988; Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2015). In the present study, both the qualitative and quantitative analyses focus entirely on misconceptions, providing greater validity to conclusions. The three research questions work together to guide a progression of analyses that lead to the development of a conceptual framework about mathematics misconceptions. The application of structural equation modeling allows for more robust inferences about the nature of the conceptual framework (Byrne, 2012).

Significance of the Study

A framework to help teachers connect errors to underlying thinking is critical for determining appropriate interventions. Konold (1988) found that errors are reflective of underlying mathematics understanding and correcting only the incorrect statement does little to address those misunderstandings. In contrast, when students misunderstand a fundamental concept (such as proportional reasoning) that supports multiple topics (such as probability, linearity, similarity), fully addressing the fundamental misunderstanding may offer a more efficient approach to improving student understanding of the supported topics (Confrey & Krupa, 2010; Lobato, Ellis, Charles, & Zbiek, 2010; Sztajn et al., 2012). Correct answers do not always indicate correct conceptual understanding, making the assessment of mathematical understanding an even more complex undertaking (Lobato et al., 2010).

Cooper (2009) found that although pre-service teachers could identify error patterns readily, they had difficulty determining the source of the error. She found that their most common strategy for addressing errors was to re-teach material at a slower pace, focusing on breaking down procedures into smaller chunks. Such a procedure-based approach to fix an error may be successful for a specific type of task, but the underlying conceptual misunderstanding will often reassert itself in new situations (Fisher, 1988; Hiebert and Grouws, 2007; Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003).

In contrast, Hiebert and Grouws (2007), Skemp (2006), and Kieran (1989, 1992, 2007) all concluded that effective interventions with the longest lasting benefits must address important mathematical understandings that, if misunderstood, lead to errors. A framework to explain how fundamental mathematics ideas support more advanced mathematics, the focus of the present study, is therefore critical for helping teachers trace surface mistakes back to their underlying root sources (Bransford et al., 2000).

Conceptual Framework

The conceptual framework for the present study describes relationships between understanding, reasoning, misconceptions, procedural fluency, and student responses to tasks or problems. The degree to which concepts are connected will directly influence the depth of learning (Hiebert & Carpenter, 1992; Skemp, 2006). If students learn in a way that connects procedures and concepts, then conceptual understanding (ideas or connections among ideas, facts, or procedures as in Hiebert et al., 2005) may produce stronger, more consistent procedural skills, which in turn may reinforce deeper more robust conceptual understanding. Existing research has not clarified whether procedures emerge from concepts or vice versa (Rittle-Johnson, Schneider, & Star, 2015). The available evidence suggests that both directions occur but that procedures emerge from concepts more (e.g., Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Siegler, & Alibali, 2001). Relational understanding may be integrated into a student's knowledge framework for use with future unfamiliar, non-routine tasks or problems (Skemp, 2006).

Alternatively, a focus on instrumental understanding leads to the development of procedures without meaning, with incomplete or erroneous meaning, or even the lack of awareness of meaning (Skemp 2006).

Misconceptions are often the result of over- or under-generalization of properties or concepts (e.g., Chang, 2002; Falk, 1992; Fuys & Liebov, 1997; Kalchman & Koedinger, 2005; Van Dooren et al., 2003) and can lead to faulty reasoning about a particular task or problem (e.g., Falk, 1992; Kahneman & Tversky, 1972; Küchemann, 1978). Faulty reasoning about a task or problem may emerge from misunderstandings or incomplete understanding about ideas and/or connections among ideas (e.g., Clement, 1982; Kim et al., 2016). Faulty reasoning may lead to a misconception or indicate an already-existing misconception (Figure 1). Some faulty reasoning emerges from misunderstandings about the particular problem or task, not the underlying mathematical concepts or procedures (English, 2004; Kotsopoulis & Lee, 2012; Schoenfeld, 1982, 1992). Correct responses do not necessarily indicate that a student understands the mathematical concepts completely (e.g., De Bock et al., 1998).

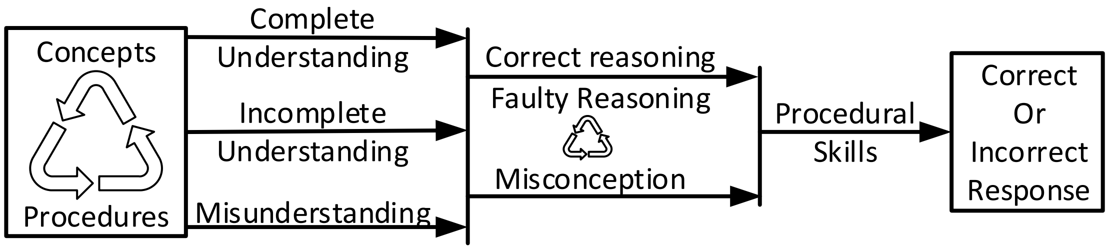

Figure 1. Pathways of understanding, reasoning, and procedural skills leading to student responses

For example, if a student believes that the absolute value of a number is just that number with a positive sign, and every absolute value example they encounter in school results in positive numbers, then their conceptual framework will be reinforced, leaving intact their incorrect/incomplete belief about absolute value (Karp, Bush, & Dougherty, 2015). Similarly, if a student believes that a variable represents a specific unknown value, and every equation they encounter results in a single numerical value answer, then their incorrect/incomplete about variable meaning will be left intact (Küchemman, 1978). A framework connecting misconceptions to both content domains and foundational knowledge structures may offer teachers insight into the nature of misconceptions.

Content Domain Misconceptions

Researchers have typically classified mathematics misconceptions by the content domain in which they are observed. For example, Earls (2018) studied misconceptions about sequences and series in calculus. Lamon (1999) and Schield (2006) considered misconceptions inherent to rational numbers while Watson and Shaughnessy (2004) focused on probability and statistics misconceptions. Chang (2002), Kalchman and Koedinger (2005), Blanco and Garrote (2007) and Socas Robayna (1997) studied algebra misconceptions. Clements and Battista (1992), Monaghan (2000), and Swindal (2000) examined geometry misconceptions. The content domains in the present study were rational number, probability, algebra, and geometry because they make up the bulk of secondary mathematics education in the U.S.

Rational Number

Rational number concepts confound student mathematical understanding more than whole numbers and integers, in part because of the major conceptual shift that is required of students when learning rational numbers and in part because of their multiple representations and uses across content domains (Fuson, Kalchman, & Bransford, 2005; Kilpatrick, Swafford, & Findell, 2001; Lamon, 2007; Moss, 2005). Misconceptions about rational numbers emerge from difficulties with fraction meaning and equivalence (Behr, Harel, Post, & Lesh, 1992; Booth & Newton, 2012). Steen (2007) described interpretation as a more fundamental problem for students than the computation of rational numbers. Understanding rational numbers, and particularly fractions, is predictive of future mathematics success (Bailey, Hoard, Nugent, & Geary, 2012; Torbeyns, Schneider, Xin, & Siegler, 2015). Booth and Newton (2012) found that understanding fraction magnitudes, especially unit fractions (numerator = 1), are strong predictors of algebra readiness.

Probability

Probability misconceptions such as representativeness and uniformity beliefs influence how students approach problem solving situations as well as their understanding of non-linearity (Falk, 1992). Agnoli and Krantz

(1989) found that students who relied on static comparisons (i.e., students depend upon their image of a typical context without regard to information presented in a problem) based their decisions on a priori knowledge rather than the meaning of the mathematical statement.

Algebra

Researchers have repeatedly found that the abstract nature of algebra poses difficult challenge for students (e.g., Carraher & Schliemann, 2007; Kieran, 1989; Welder, 2012). Specifically, abstractness poses an impediment that arises when students attempt to construct multiple representations of algebraic objects (Kieran, 1992). Understanding functional relationships also requires facility with algebraic language. Students often fail to recognize that function notation is a general form intended for all functions rather than just linear (Chang, 2002). As a result, students may develop a misconception that functions are supposed to be linear (Chang, 2002; Kalchman & Koedinger, 2005). Van Dooren et al. (2003) found that students cling to linear functions tenaciously without consideration for inconsistencies in their reasoning. Lamon (1999) suggested a deeper misunderstanding in such situations. She noted that students continue exhibiting misconceptions even when a function is linear. She found that when functions are linear, students often consider the relationship to be additive (i.e., recursive) rather than multiplicative (i.e., functional). Her study used algebraic equations that were presented as proportions, suggesting that her findings about additive and multiplicative units are applicable to both algebra and rational numbers.

Geometry

The Van Hiele framework classifies understanding of geometric shapes into five levels (described in detail by Clements & Battista, 1992; Crowley, 1987): Level 0 (visual), Level 1 (descriptive/analytical, Level 2 (abstract/relational), Level 3 (deduction), and Level 4 (rigor). Clements and Battista (1992) and Crowley (1987) pointed out that the development of spatial reasoning does not directly correspond to particular ages or grade levels: adults may operate at Level 0 while children in grade school may reach Levels 1 or 2.

The Van Hiele framework provides a structure for understanding misconceptions that develop from underdeveloped or misunderstood geometric concepts. For example, a student who believes that a square is not a square unless its base is horizontal does not associate the properties of a square to the label. Instead, such a student relies strictly on the visual orientation of a particular drawing (Van Hiele Level 0; Clements & Battista, 1992). A student who believes that the angle sum of a quadrilateral is the same as its area acknowledges that a quadrilateral has the property of a constant sum for its interior angles but confuses the meaning of an angle with the meaning of area (Van Hiele Level 1; Clements & Battista, 1992). Gal and Linchevski (2010) found that visual perception (Van Hiele Level 0) in and of itself may be worse than insufficient: without intervention, it may actually impede understanding of shape and spatial reasoning.

Content Domain Interactions

Relationships between misconceptions across content domains have also been studied. Rational number and/or probability misconceptions may hold a primary, predictive position relative to algebra and geometry misconceptions (e.g., Fuson et al., 2005; Kilpatrick et al., 2001; Lamon, 2007; Moss, 2005), but hierarchical relationships between rational number and probability misconceptions or between algebra and geometry misconceptions have not been clearly demonstrated. No studies indicated the possibility that algebra and/or geometry misconceptions hold a predictive position for rational number or probability misconceptions. Furthermore, no studies examined these relationships beyond descriptions, case studies, and small sample sizes, making valid generalizations difficult about relationships between misconceptions across content domain.

Foundational Structures

A classification system based on mathematics content domains may be insufficient for addressing the observed complexities in the development and recidivism of particular misconceptions. For example, Falk (1992), Green (1982), and Watson and Shaughnessy (2004) examined misconceptions specific to probability, but to do so, they focused on the interaction between probability concepts and rational numbers. Van Dooren et al. (2003) found that misconceptions stemming from an over-reliance on linear proportionality hampered students' ability to solve non-linear problems in algebra and geometry. They suggested that probability problems involving non-linear relationships have the potential to help address those misconceptions. The ability to solve algebraic and geometric problems and to interpret probabilities have been consistently found to be connected to concepts of proportionality and linearity (e.g., Freudenthal, 1983; Shaughnessy & Bergman, 1993; Stacey, 1989). Moss, Beatty, Barkin, and Shillolo (2008) suggested that developing stronger links between algebra and number concepts may help address misconceptions about function relationships in algebra. Moss et al. (2008) and Lamon (1999) described misconceptions about equality and linearity in algebra as arising from an inability to assign meaning to unfamiliar contexts, a necessary step in problem solving, regardless of the content.

Variable symbols and the meaning of variation may influence student capacity to understand probability concepts such as randomness (Shaughnessy, Canada, & Ciancetta, 2003). Earls (2018) traced calculus misconceptions to prerequisite algebra, geometry, and number concepts and discussed the role of hierarchical progressions of content. He noted that the presence of prerequisite knowledge does not guarantee that a misconception will not develop. Küchemann (2010) advocated for broader use of looking for generalized numerical patterns to strengthen students' understanding of algebraic concepts. Hunt (2015) noted the strengths of using ratio concepts to help students with learning disabilities understanding equivalence concepts. These examples suggest that mathematics misconceptions may not be bounded by content domain or explained by natural progressions of content.

Several overarching structures (variable, measurement, spatial reasoning, equivalence, additive and multiplicative relationships, absolute and relative comparisons) have been found to be foundational to knowledge within and across content domains (e.g., Asquith, Stephens, & Knuth, 2007; Briggs, Demana, & Osborne, 1986; Cheng & Mix, 2014; Kilpatrick et al., 2001; Lamon, 1999; Siegler, Thompson, & Schneider, 2011; Zhang et al., 2014). By *foundational*, we mean that these structures are necessary building blocks of a cognitive framework with which an individual reasons and makes sense of mathematics. Such foundational structures are essential for developing relational understanding and mathematical reasoning (English, 2004). For example, Ball and Bass (2002) pointed out that relationships among representations and equivalence are necessary components for understanding subtraction.

Variable

Understanding the meaning of variable is fundamental to the transition from arithmetic to algebra (Briggs et al., 1986; Edwards, 2000; Graham & Thomas, 2000; Kalchman & Koedinger, 2005). But variable meaning goes beyond algebra and is interwoven throughout mathematics content and influences learning of geometry, statistics, and probability concepts (Edwards & Phelps, 2008; Shaughnessy et al., 2003). Variable interpretation and representing variables symbolically is especially challenging for students (Knuth, Alibali, McNeil, Weinberg, & Stevens, 2005; Swafford & Langrall, 2000). Students generally have difficulty recognizing the systemic consistency in the multiple uses of variables (MacGregor & Stacey, 1997), especially functional relationships (Küchemann, 1978; Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2015).

Measurement

Measurement instruction is often interwoven with geometry, but measurement is integral to every mathematics content and provides straightforward connections between mathematics content domains (Hurrell, 2015). Measurement also connects mathematics to other content areas such as science and social studies (Foley, 2010) and the real world (Fuson, Clements, & Beckmann Kazez, 2010). Measurement misconceptions often arise when students misunderstand how to count units or which units to count as well as misinterpreting the meaning of the measurement value (McCool & Holland, 2012; Parmer, Garrison, Clements, & Sarama, 2011).

Spatial Reasoning

Spatial reasoning refers to spatial movements, symmetries, and transformations (Fuson et al., 2010). Spatial reasoning ability in the early grades (i.e. grades K-2) has been shown to be a strong predictor of student achievement in later grades (e.g., Casey et al., 2015; Cheng & Mix, 2014; Garderen, 2006; Hegarty & Kozhevnikov, 1999; LeFevre et al., 2010; Satyam, 2015; and Zhang et al., 2014). Mix and Cheng (2012) noted that children and adults who perform better on spatial tasks also perform better on mathematics ability assessments and that the relationship between spatial ability and mathematics ability remains consistent from early childhood through middle age. They pointed out that spatial ability is strongly connected to long-term success in STEM courses and careers and therefore presents a leverage point for education interventions. Spatial ability and mathematics share cognitive processes beginning early in development, and spatial ability is related to mathematics ability throughout development, including the early elementary grades (Cheng & Mix, 2014; Cross, Woods, & Schweingruber, 2009).

Equivalence

Equality is a key concept that influences understanding in every content domain (e.g., Asquith et al., 2007; Griffin, 2016). Understanding equivalence is foundational for success in algebra, geometry, and other higher-level mathematics (Asghari, 2009; McNeil, 2014). For example, equivalence is integral to solving equations (Baratta, 2011) and comparing algebraic expressions (Solares & Kieran, 2011). Equivalence relationships are also critical to learning geometry (Freudenthal, 1983) and statistics (Albright, Reeve, Reeve, & Kisamore, 2105).

Additive and Multiplicative Relationships

Distinguishing between contexts that require multiplicative rather than additive relationships is an important mathematical understanding throughout elementary and secondary education (e.g., arithmetic, fractions, proportional reasoning, and linear functions; Strom, 2008; Vergnaud, 1994). Students may manifest misconceptions by applying their understanding of linearity to a wide array of mathematical situations, including when deciding whether a context calls for an additive or multiplicative comparison of two quantities (Van Dooren et al., 2003). Lamon (1999) found that such errors are not simply about overusing linear proportions. Similarly, Moss et al. (2008) found that students who misunderstand multiplicative structures are likely to struggle with algebra problems, for example mistaking recursive patterns in sequences for a relationship between independent and dependent variables.

Absolute and Relative Comparisons

Understanding how to properly compare values influences mathematics understanding in algebra, geometry, and probability, especially if the comparisons require understanding rational numbers as an integrated value (i.e., its own value on a number line, not simply an aggregate of two whole numbers; Green, 1982; Lobato et al., 2010). Siegler et al. (2011) found that magnitude comparisons yield *distance effects* (the closer a number is to the comparison value, errors in comparisons increase). Meert, Grégoire, and Noël (2010) found that fraction magnitude comparisons were more difficult when numerators were the same rather than denominators, indicating that the transition from whole number to rational number thinking is dependent on understanding the denominator. Difficulties with number comparisons have been found to affect students all the way through high school and into adulthood (DeWolf & Vosniadou, 2015; Van Hoof, Vandewall, Vershaffel, & Van Dooren, 2015).

Method

The present study examined student response explanations to identify common patterns in reasoning that led to incorrect answers across mathematics content domains (algebra, geometry, rational number, and probability) and to determine the extent to which factors independent of content domain explain mathematics misconceptions. The study employed an exploratory sequential mixed methods design (Creswell & Plano Clark, 2007), in which the qualitative data analysis preceded and informed the quantitative data analyses.

Participants

Seventeen mathematics teachers participated from three high schools and one middle school across three school districts with 1,144 students enrolled in 53 algebra and geometry classes. Eleven (1%) did not respond to any question (unit non-response), so the final sample size was 1,133 students.

Instrumentation

Seventeen algebra, geometry, rational number, and probability multiple choice (A, B, C, D, E) items were selected from National Assessment of Educational Progress (NAEP) released items (U.S. Department of Education,1996, 2005, 2007). Twelve items were taken from the Grade 8 NAEP assessments. The four items taken from the Grade 12 assessments were rated by NAEP as having medium difficulty and low complexity. One item was taken from the Grade 4 assessments. The same items were therefore used for middle and high school students. Although all 17 items remained as given by NAEP, a prompt was included for each question asking students to explain how or why they chose their response, following the advice of Zawojewski and Shaughnessy (2000). The internal consistency was found to be adequate ($\alpha = 0.791$, 95% CI [0.773, 0.808]).

Item Response Theory (IRT) was used to verify that the items chosen for the instrument were not too difficult or easy, able to discriminate across high and low ability students, and not overly subject to guessing. The difficulty coefficients were appropriate, ranging from -1.308 (Easy) to 0.930 (Hard). Item discrimination was also appropriate, ranging from 0.428 (less differences between high and low ability students) to 1.192 (more differences between high and low ability students). The guessing coefficient was zero for all 17 items.

Data Analyses

The NAEP items were designed to measure achievement of the four content domains and capture common errors through carefully crafted distractors. The open response portion was examined for each item to identify error patterns rooted in conceptual misunderstanding, using Hiebert and Grouws' (2007) definition of conceptual understanding, "mental connections between mathematical facts, procedures, and ideas" (p. 380). The responses were then analyzed with structural equation modeling.

Qualitative Analysis

Of the 1,133 students in the sample, 295 students provided explanations for their responses. Approximately 26% of student explanations on the 17 NAEP items were completed (N = 5,008 out of 19,261 potential explanations). A qualitative sub-sample was chosen using purposive stratification across classes and item choices. Specifically, responses were organized by class and item and then chosen so that a wide array of classes were represented for each item, and each response within an item was represented. This technique resulted in a sub-sample of 72 students. The coding of explanations for each item (72 students by 17 items) was based on the operational definition of misconceptions (conceptual misunderstandings that led to systematic errors). The patterns identified in this sub-sample were applied to the full sample of 1,133 students for the subsequent quantitative analyses.

Although correct responses could emerge with misconceptions, the hypotheses focused on distractors. Student explanations of the correct responses for each item were examined for possible underlying misconceptions, but the explanations did not provide compelling evidence of systematic error patterns. A focus on misconceptions underlying correct responses might require deeper investigation than is possible with written explanations, such as diagnostic interviews (as in De Bock et al., 2002; Lobato et al., 2010).

Such interviews fell outside the scope of the present study. Peer debriefings were used to enhance the reliability of the coding (Patton, 2002). Peers consisted of colleagues who were experts in mathematics education with varying levels of experience coding mathematics misconceptions. Student written explanations along with coding notes were shared with the colleagues. Feedback consisted of discussions regarding over- and under-emphasized points and potential assumptions that affected the coding process.

Once the explanations had been coded, the data were examined for emergent patterns to identify connections that might potentially explain variance in student responses (as recommended by Creswell, 2007). The qualitative analysis was used to examine whether the foundational structures identified in the literature review (a) were present in student explanations of misconceptions across content domains and (b) were complete, that is, whether other structures not identified in the literature review were present in student explanations of misconceptions. The qualitative analysis was critical for interpreting the subsequent quantitative analysis results.

Quantitative Analysis

Upon completion of the qualitative analysis, the responses for all 1,133 students in the sample were coded dichotomously (1 = indicative of misconceptions; 0 = not indicative of misconceptions) based on the patterns identified in the qualitative analysis of the 72-student sub-sample. The full sample was randomly divided into two halves (N_1 = 567; N_2 = 566). The first half was used to produce initial results while the second half was to verify the results, thus minimizing the threat of over-fitting the models to a single sample, as recommended by Byrne (2012).All structural equation models were analyzed with MPlus 7.4 (Muthén & Muthén, 2015). Because data were dichotomous, the robust weighted least squares estimator was used (WLSMV estimator in MPlus). This estimation technique has been found to be the best approach to analyzing categorical variables when sample sizes are greater than 200 (Finney & DiStefano, 2013; Flora & Curran, 2004; Muthén, du Toit, & Spisic, 1997; Rhemtulla, Brosseau-Liard, & Savalei, 2012).

A bifactor modeling approach was used to test the degree to which the addition of factors independent of content domain improve the classification of misconceptions (Research Question 3). Bifactor modeling partitions item variance into independent sets of traits, usually with one set of correlated factors and another single factor independent of the factors in the other set (*orthogonal factor*). This modeling approach has been found to be highly useful for examining an existing instrument and deepening understanding of trait structure (Reise, 2012). For the present study, content domain represented the first set of factors, the trait structure most commonly identified with mathematics misconceptions. The single independent factor represents a single source of variance for all items in the instrument, above and beyond what is accounted for by content domain (Reise, 2012). Based on the literature review, this single independent factor was hypothesized to be foundational knowledge structures, which were found to interweave throughout the content domains. The qualitative analysis was used to determine the viability of considering the single independent factor to represent the foundational knowledge structures.

Three fit indices were used to determine the adequacy of model fit and to compare models: Comparative Fit Index (CFI), Tucker and Lewis Index (TLI; also referred to as Non-Normed Fit Index, or NNFI), and Root Mean Square Error of Approximation (RMSEA) as recommended by Schermelleh-Engel, Moosbrugger, and Müller (2003). The Weighted Root Mean Square Residual (WRMR) is specific to categorical data and was included as a fourth goodness of fit indicator.

Missing Data

Of the 1,133 students, 242 (21.4%) were missing at least one value, and all items had missing data. Little's (1987) test of missing completely at random (MCAR) indicated that the missing data patterns were statistically significant (i.e., not MCAR), χ^2 (1004) = 1103.33, p = .015.

The WLSMV estimator has been found to produce valid estimates with non-normality and missing data present (Finney & DiStefano, 2013; Muthén et al., 1997). The estimates are based on all data present (i.e., no deletion) without inflating statistical power. No imputation methods were therefore used to create complete datasets.

Results

The study investigated three research questions to identify systematic patterns in explanations of erroneous responses indicative of misconceptions and to examine classification systems of those misconceptions (i.e., content domain hierarchies, foundational structures). The explanations were examined qualitatively, and classification systems were analyzed using structural equation modeling.

Qualitative Analysis Results

The qualitative analysis addressed Research Question 1 by examining explanations of reasoning that led to erroneous responses. Prior to the analysis, hypotheses were developed about which distractors might indicate underlying misconceptions. Hypotheses were corroborated for 44 of the 68 (64.7%) distractors (*misconception not hypothesized and not found*; *misconception hypothesized and same misconception found* in Table 1). The outcome that over a third of the hypotheses (35.3%) were not corroborated by student explanations suggests that experimenter expectancy bias was not a major issue.

The analysis allowed for the emergence of misconception patterns that were not hypothesized, which accounted for 22.1% of the distractors (Table 1). For example, one item asked students, "The cost to mail a first-class letter is 33 cents for the first ounce. Each additional ounce costs 22 cents. Fractions of an ounce are rounded up to the next whole ounce. How much would it cost to mail a letter that weighs 2.7 ounces? A) 55 cents, B) 66 cents, C) 77 cents, D) 88 cents, E) 99 cents." Of 1,062 responses, 453 (42.7%) chose the correct response (Choice C). The hypothesis for this item predicted only that Choice D would indicate an additive/multiplicative structure misconception (as in Moss et al., 2008). Student explanations for Choice D confirmed this hypothesis with statements such as, "I multiplied .33¢ times 2.7" and "Because 33 + 33 = 66; 0.7 → 1 ➔ 66 + 22 = 88" (*misconception hypothesized and same misconception found* in Table 1). Choices A, B, and E were examples of unanticipated misconceptions in student explanations (*misconception not hypothesized but found* in Table 1). Students who chose E followed the same reasoning as students who chose D, with the exception that they remembered to round the 2.7 to 3. These students wrote explanations such as "33 x 3 ounces = 99 cents."

Students who chose B used two types of reasoning to arrive at their answer. Some students simply added: "33 + 22 + 11," "33 + 22 + 0.7." Another student explained, "You have 2 whole ounces, 33¢ + 22¢ = 55¢, next you have to figure out the .7. Take 22 • 70%, which is 18. So you add 12 to 55, total would be 66¢." This final statement, apart from the readily apparent calculation errors, showed the same basic reasoning as the first two—lack of understanding about when multiplication or addition should be used. This particular justification included several erasures over numbers, indicating that the student had changed numbers to arrive at the closest answer available (a phenomenon described by Clement, 1982). Another type of justification for Choice B relied on a multiplicative-only strategy, for example, "I multiplied 33 times 2." Students who chose A used similar reasoning to that used by students who chose B. These students also dropped the 0.7 or the second ounce. The justifications were variations of "33 + 22 = 55." Students who ignored the second ounce made statements such as, "First ounce is 33 cents, next 0.7 of ounce is 22, rounded." Based on these explanations, we coded choices A, B, D, and E as responses indicating an underlying misconception.

Table 1. Analysis of misconceptions from student explanations of distractors

Outcome	Distractor count	Percent of distractors[a]
Distractors were not hypothesized as indicating misconceptions, and student explanations corroborated that the distractor was not indicative of misconceptions	23	33.8%
Distractors were hypothesized as indicating particular misconceptions, and student explanations corroborated that the distractors indicated those misconceptions.	21	30.9%
Distractors were hypothesized as indicating particular misconceptions, and student explanations corroborated that the distractors indicated misconceptions, but the misconceptions were different ones than those hypothesized.	4	5.9%
Distractors were not hypothesized as indicating misconceptions, but student explanations indicated misconceptions.	15	22.1%
Distractors were hypothesized as indicating particular misconceptions, but student explanations did not indicate misconceptions.	5	7.4%

Note. Each of the 17 items had 4 distractors for a total of 68 distractors. [a]Sum is not exactly 100% due to rounding.

Another item presented a spinner divided in two halves, with one of the halves divided in half again and an arrow pointed to one of the quarter regions with a circle inside. The item asked students, "If Rose spins a spinner like the one below 300 times, about how many times should she expect it to land on the space with a circle? A) 75, B) 90, C) 100, D) 120, and E) 150." Of the 1,112 student respondents, 583 (52.4%) correctly chose A. We hypothesized that a spatial reasoning misconception would result in students deciding that the probability of the region was 1/3 instead of 1/4 (i.e., failure to recognize that the regions shown in the figure were unequal) and predicted that Choices C and D would result from students taking 1/3 of 300 or 1/3 of 360 respectively. Explanations for Choice C verified the hypothesis (i.e., *misconception hypothesized and same misconception found* in Table 1) with statements such as, "Because there's 3 spaces and 300 ÷ 3 is 100" or "I divided 300 ÷ 3 & got 100.” Students who chose D, on the other hand, did not verify our hypothesis. Instead, these students indicated that they had recognized the probability of the region as being 1/4 but made a computation error.

No one who chose D gave explanations that indicated a misconception (i.e., *misconception hypothesized but not found* in Table 1). Explanations for Choice B did, however, indicate an unanticipated misconception (*misconception hypothesized but different misconception found* in Table 1). These students recognized that the region was 1/4 of the circle and used the the number of degrees in a circle. These students made statements such as, “The circle is in an angle of 90º,” “Circle = 360, divide it by 4, you get 90,” or “Because the circle is split up into 3 parts; a circle’s measure is 360, if cut in half, each part will be 180, if one half of the split circle is cut in half again, that side is now 2 sets of 90º.” These explanations consistently indicated a lack of recognition that the ratio of interest required the numerator to be 300 rather than 360, a misconception about the rational number rather than the spatial aspects of the problem. The overall item was therefore coded as indicating misconceptions about both rational number meaning and spatial reasoning, depending on which distractor was chosen.

As a result of the analyses, the coding of all items was changed as needed to match student response patterns. These changes enhanced the ability of the coding to represent student underlying conceptual understanding (construct validity). Five of the foundational structures identified in the conceptual framework were found (Table 2).

Table 2. Classification of items by content domain and foundational structure

		Foundational Structure				
Content Domain	Item	Variable	Measurement	Spatial Reasoning	Absolute/ Relative Comparison	Additive/ Multiplicative Structures
Algebra	5		X			
	6					X
	7	X				X
	8	X				
	16	X				
Geometry	9			X		
	10[a]		X			X
	11	X				
Rational Number	12					X
	13		X			
	14				X	
	15			X		
Probability	1				X	
	2[b]					
	3[b]					
	4			X		
	17[b]					

Note. Content domains were determined by NAEP classification. Foundational structures determined by student response analysis. [a]Student responses to Item 10 indicated misconceptions about more than one foundational structure across distractors. [b]Items 2, 3, and 17 indicated misconceptions about rational numbers.

Two features of Table 2 supported further exploration via structural equation modeling. First, even with the limited pool of 17 items, which were not initially developed to detect misconceptions, student explanations for 40 of the 68 distractors (58. 8%) did indicate consistent patterns of misunderstanding about concepts rather than simple procedural errors. Multiple foundational structures were found within each content domain, and each foundational structure was found in multiple content domains. A structural equation modeling analysis offered an avenue for investigating the relationships between misconceptions across content domains and foundational structures. The testing of a bifactor model in which the single independent factor represented the foundational structures was therefore warranted.

Structural Equation Modeling Results

The structural equation modeling analyses began with an examination of the relationships between misconceptions classified by content domain. Hypothesized models were based on relationships identified in the literature review. The hypotheses were: content domain misconceptions were correlated (i.e., no potentially causal pathways; Model A, Figure 2A), probability misconceptions predict algebra, geometry, and rational number misconceptions (Model B, Figure 2B), rational number misconceptions predict algebra, geometry, and probability misconceptions (Model C, Figure 2C), and both probability and rational number misconceptions predict algebra and geometry misconceptions (Model D, Figure 2D). No evidence was found in the literature for algebra or geometry misconceptions predicting rational number or probability misconceptions.

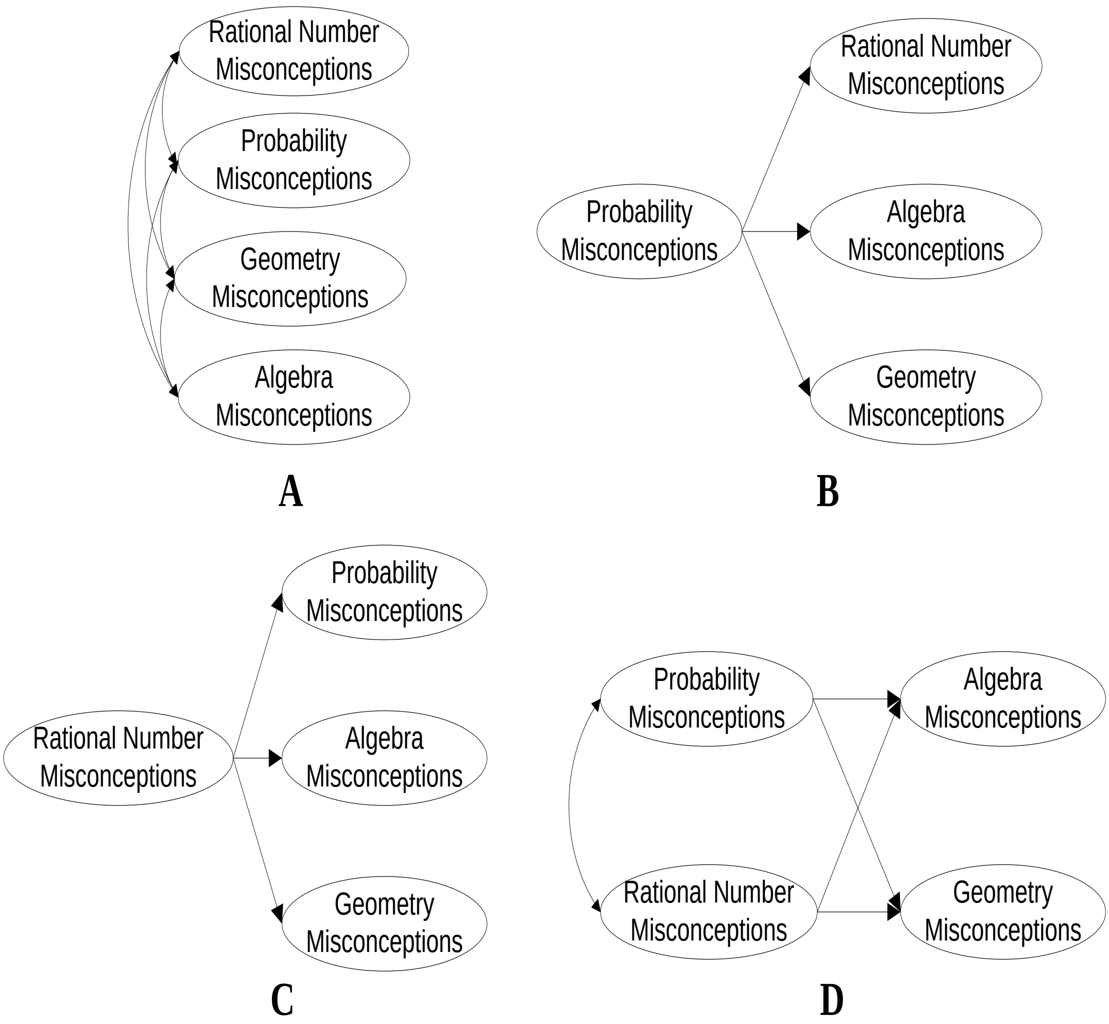

Figure 2. Content domain structural equation models. NAEP questions (not shown) were associated with each content domain according to Table 2

Based on the literature review, a model was theoretically feasible in which probability misconceptions predict rational number misconceptions, which in turn predict algebra and geometry misconceptions (Figure 3). Initial analyses revealed, however, that this model was statistically equivalent to Model C, so the more parsimoniuos Model C was the only one retained in the analyses (Hershberger & Marcoulides, 2013). Similarly, a model in which rational number misconceptions predict probability misconceptions, which in turn predict algebra and geometry misconceptions (Figure 3) was also found to be equivalent to Model B, so the more parsimonious Model B was the one retained in the analyses.

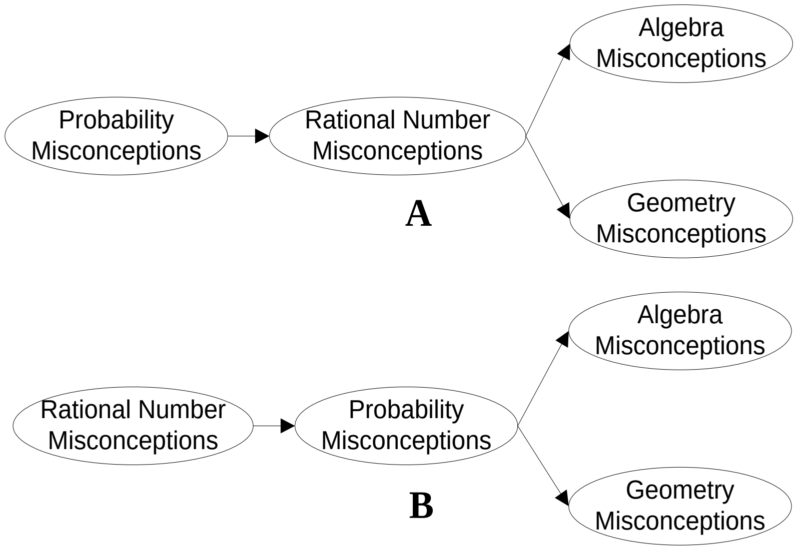

Figure 3. Alternative content domain structural equation models not included due to model equivalency with models shown in Figure 2

Once the content domain models (Models A-D; Figure 2) were analyzed, a bifactor model (Model E; Figure 4) was examined to determine the viability of the hypothesis that an underlying factor independent of content domain predicts mathematics misconceptions. Based on the qualitative results, this independent factor was considered to represent the foundational structures because they were found to interweave throughout misconceptions in different content domains.

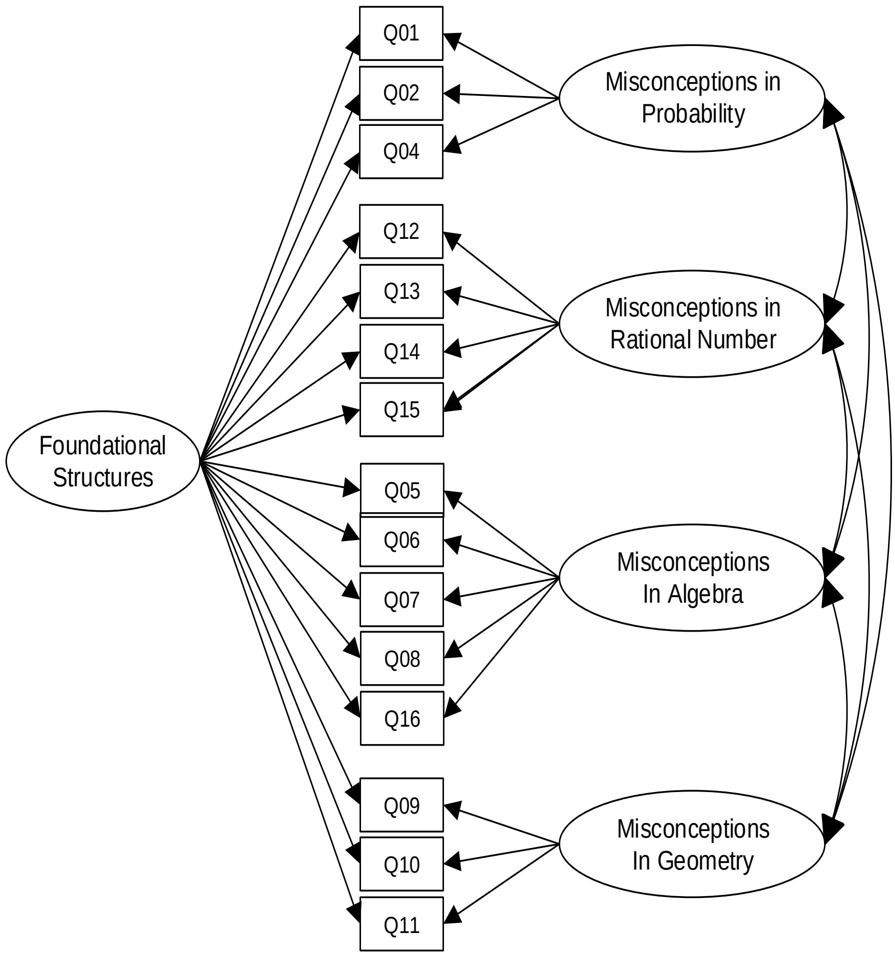

Figure 4. Model E: Single bifactor model

Model A was a one-level confirmatory factor analysis (CFA) model (Figure 2A). The RMSEA value was in the excellent range (< .05; Table 3), but CFI and TLI were below the recommended cutoff for excellent or acceptable fit (> .97 or .95, respectively; Schermelleh-Engel et al., 2003), and WRMR was above the recommended cutoff of 1.0 (Yu, 2002). One item on probability had a negative factor loading (Item 3) and another had a non-significant factor loading (Item 17). These two items were removed, and model fit improved (Model A2 in Table 3). All subsequent models excluded Items 3 and 17. No modification indices greater than 10 were produced by the model estimation. Because some fit indices remained out of recommended levels, further examination of alternative models was warranted.

Model B hypothesized that probability misconceptions predict algebra, geometry, and rational number misconceptions (Figure 2B). All fit indices indicated a slightly worse fit than Model A2. Two modification indices were produced with a value greater than 10 (i.e., likely to produce a statistically signficant reduction in chi-squared), but neither was theoretically feasible. No additional modifications were added to Model B.

Model C hypothesized that rational number misconceptions predict algebra, geometry, and probability misconceptions (Figure 2C). Some fit indices indicated that Model C produced a slightly better or comparable fit than Model A2 (CFI, TLI, RMSEA) while WRMR indicated a slightly worse fit (Table 3). No modification indices greater than 10 were produced.

Table 3. Fit indices for structural equation models

Model	Model Description	DF	CFI[a]	TLI[a]	RMSEA (90% CI)[b]	WRMR[c]
A	Content domains correlated	113	.905	.886	.042 (.034, .050)	1.142
A2	Q03 and Q17 removed from Model A	84	.935	.919	.038 (.029, .048)	1.042
B	Probability misconceptions predict algebra, geometry, and rational number misconceptions	87	.930	.916	.039 (.030, .048)	1.075
C	Rational number misconceptions predict algebra, geometry, and probability misconceptions	87	.937	.923	.037 (.028, .047)	1.051
D	Probability and rational number misconceptions predict algebra and geometry misconceptions	85	.935	.919	.038 (.029, .048)	1.050
E	Underlying factor independent of content domain predicts mathematics misconceptions (bifactor model)	69	.996	.994	.010 (.000, .027)	0.689

Note. N = 567 students. [a]Larger CFI and TLI values indicate better fitting models; Values ≥ .95 indicate acceptable fit; Values > .97indicate excellent fit (Schermelleh-Engel et al., 2003). [b]Smaller RMSEA values indicate better fitting models; RMSEA ≤ .05 indicates excellent fit (Byrne, 2012). CI = Confidence Interval. [c]Smaller WRMR values indicate better fitting models; WRMR ≤ 1.0 indicates acceptable fit (Yu, 2002).

Model D hypothesized that both rational number and probability misconceptions predict algera, geometry, and probability misconceptions (Figure 2D). All fit indices indicated that Model D produced a slightly worse or comparable fit than Model A2 (Table 3). The structural model estimates produced non-significant values for algebra and geometry misconceptions on probability misconceptions.

Model A2 was the best fitting content-based model of misconceptions. These results provided evidence against the hypotheses that some content-based misconception factors might hold a predictive, potentially causal relationship to other content-based misconception factors. The remaining mis-specification in Model A2 led to further analyses to determine if an underlying knowledge base factor might help explain some variance in misconceptions. The qualitative analyses and literature review suggest that an underlying knowledge base factor would represent the foundational structures.

A bifactor model (Model E; Figure 4) was examined to determine the viability of the hypothesis that an independent factor other than content domain predicts mathematics misconceptions. Based on the qualitative analysis, this independent (orthogonal) factor was interpreted as representing the foundational structures. Model E showed much better model fit than Model A2 (Table 3). For example, the CFI jumped from .935 in Model A2 to .995 in Model E, and the TLI jumped from .935 to .994. Both of these improvements moved the indices from moderate to excellent fit. Similarly, the WRMR dropped from 1.042 to 0.689, from above to below the threshold for acceptable fit. This result indicates that the addition of a factor independent of content domain helps explain variance in the model.

Results from the initial analyses were cross-checked with an independent subsample (N_2 = 566). Model A2 was the best fitting content domain model, and Model E was the best fitting model overall. Models A2 and E were therefore examined with the second subsample. Patterns of fit indices seen in the first subsample remained consistent in the second subsample (Table 4).

Table 4. Fit indices for structural equation models using second sub-sample

Model	Model Description	DF	CFI[a]	TLI[a]	RMSEA (90% CI)[b]	WRMR[c]
A2	Content domains correlated	84	.937	.921	.033 (.022, .043)	0.992
E	Underlying factor independent of content domain predicts mathematics misconceptions (bifactor model)	69	.950	.924	.032 (.021, .043)	0.886

Note. *N* = 566 students. [a]Larger CFI and TLI values indicate better fitting models; Values ≥ .95 indicate acceptable fit; Values > .97indicate excellent fit (Schermelleh-Engel et al., 2003). [b]Smaller RMSEA values indicate better fitting models; RMSEA ≤ .05 indicates excellent fit (Byrne, 2012). CI = Confidence Interval. [c]Smaller WRMR values indicate better fitting models; WRMR ≤ 1.0 indicates acceptable fit (Yu, 2002).

Because patterns of significance in parameter estimates differed between the first and second subsamples, final parameter estimates in Model E were computed with the full sample to avoid over-interpreting patterns unique to a particular subsample (i.e., threat to construct validity). Most parameter estimates were statistically significant for the foundational structures factor, and most were not statistically significant for the content domain factors (Table 5).

Table 5. Factor loadings in final model (Model E, Table 3, Figure 4)

Factor Loading	Estimate	S.E.	Est./S.E.	Factor Loading	Estimate	S.E.	Est./S.E.
Foundational Structures by				Probability by			
Q01	0.233	0.068	3.444**	Q01	---[a]	---	---
Q02	0.409	0.051	7.993***	Q02	0.546	0.232	2.352*
Q04	0.25	0.086	2.903**	Q04	1.667	0.5	3.332**
Q05	0.563	0.051	11.043***	Algebra by			
Q06	0.305	0.062	4.882***	Q05	---[a]	---	---
Q07	0.869	0.05	17.354***	Q06	1.608	0.952	1.69
Q08	0.101	0.07	1.436	Q07	-0.073	1.081	-0.068
Q09	0.288	0.051	5.672***	Q08	2.054	1.458	1.409
Q10	0.459	0.053	8.716***	Q16	2.345	1.274	1.841
Q11	0.51	0.067	7.630***	Geometry by			
Q12	0.462	0.077	5.982***	Q09	---[a]	---	---
Q13	0.329	0.099	3.315**	Q10	2.626	2.13	1.233
Q14	0.082	0.131	0.628	Q11	12.892	23.525	0.548
Q15	0.379	0.069	5.459***	Rational Number by			
Q16	0.509	0.07	7.317***	Q12	---[a]	---	---
				Q13	1.356	0.31	4.377***
				Q14	1.901	0.554	3.432**
				Q15	-0.173	0.342	-0.504

$^{**}p < .01$. $^{***}p < .001$. *N* = 1,133 students. [a]Fixed factor in model.

The foundational structures factor was modeled to be independent (orthogonal) to the content domain factors in Model E (covariances = 0), consistent with bifactor modeling (Reise, 2012). Covariances among content domain factors were estimated to be statistically non-significant in Model E (Table 6).

Table 6. Factor covariances in final model (Model E, Table 3, Figure 4)

Factor Covariances	Estimate	S.E.	Est./S.E.
Algebra with			
Probability	0.034	0.034	0.991
Geometry with			
Probability	0.007	0.016	0.457
Algebra	0.003	0.009	0.376
Rational Number with			
Probability	0.071	0.042	1.706
Algebra	0.057	0.055	1.042
Geometry	0.005	0.013	0.411

N = 1,133 students.

Discussion

The present study examined patterns in student explanations of error responses to determine connections to misconceptions (Research Question 1). The results of those analyses were used to examine the degree to which content domains classify misconceptions (Research Question 2) and the degree to which factors independent of content domain improve the classification of misconceptions (Research Question 3).

The qualitative analysis revealed misconceptions about multiple foundational structures within each content domain and misconceptions about multiple content domains within each foundational structure. The quantitative analyses found that the best content domain model was a correlational model (Model A2; Table 3; Figure 2A). The best overall model was a bifactor model, which included a factor independent of content domain (Model E; Table 3; Figure 4). The predominance of the bifactor model indicates stronger connections among misconceptions across content domains than can be explained by correlations alone. These results indicate that content domain misconceptions are highly interactive, which may help explain the stability and robustness of erroneous thinking resulting from misconceptions (as in Moschkovich, 1998). The literature review and the qualitative analyses indicated that this factor, independent of content domains, is best interpreted as the foundational structures. These findings provide two insights into mathematics misconceptions. First, misconceptions among content domains are not hierarchical and are better understood through an analysis of the mathematical structure among concepts (consistent with recommendations by Kieran, 2007). Second, the finding of a factor outside of content domain that contributes to misconceptions is startling because relationships among misconceptions have typically been viewed as resulting from simple correlations among content domain.

Limitations

Although the 17 NAEP item data produced robust models, additional items may have uncovered other systematic patterns of errors. The items chosen for each content domain did not include all intersections of content. For example, none of the algebra questions involved rational number concepts, and none of the probability questions involved algebra, equivalence, or measurement. The questions therefore limited the ability of the models to detect potential relationships between misconceptions across content domain.

The dichotomous coding of misconceptions introduced a large degree of non-normality in the data, which limited the degree of model complexity that could be supported in the analyses. A dataset consisting of continuous variables designed to detect misconceptions and a larger sample size may be needed to detect more than one factor independent of content domain.

Future Directions

Future research about mathematics misconceptions may focus on the development of diagnostic tools to help teachers identify gaps in foundational structures across content domains as well as the nature and impact of the foundational structures on various content domains. Although misconceptions can be inferred through an analysis of student explanations on current achievement tests, diagnostic measures targeting misconceptions to reveal where learning was derailed along a content trajectory can provide teachers with critical information for crafting classroom interventions. Shifting data collection procedures to consistently include student explanations, written work, and reasoning (as recommended by Zawojewski & Shaughnessy, 2000) will also allow for stronger connections between qualitative and quantitative analyses of mathematics misconceptions.

The development of continuous measures of misconceptions (e.g., percentages, scores) may support more robust quantitative analyses of misconceptions. The use of modeling techniques such as structural equation modeling and bifactor modeling resulted in findings that run counter to typical findings about mathematics misconceptions (i.e., misconceptions not hierarchical and not fully explained by content domain connections). Such techniques are therefore recommended to support stronger analyses of mathematics misconceptions.

Future research may also focus on professional development to enable teachers to dig deeper into student thinking and create interventions that repair, strengthen, and reinforce conceptual understanding of the foundational structures across content domains. Teaching that addresses misconceptions through both content and foundational structure lenses may strengthen a particular foundational structure and prevent future misconceptions (e.g., avoiding rules that expire, as in Karp et al., 2015). Helping students reason through incomplete information and building reasoning from prior knowledge may also provide meaningful opportunities to struggle with important mathematics and prevent future misconceptions and address currently-held misconceptions (Hiebert & Grouws, 2007; Resnick, 1983). Shifting teacher views of misconceptions to foundational structures rather than prerequisite knowledge or general relationships among content domains may result in a stronger focus on conceptual understanding and how student conceptual understanding of mathematics is integral to assessing and addressing student misconceptions. Such shifts may be necessary for students' long-term success in mathematics.

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