

This is the pre-peer reviewed version of the following article: Sousedík, B. Inexact and primal multilevel FETI-DP methods: a multilevel extension and interplay with BDDC. Int J Numer Methods Eng. 2022; 1- 15. doi:10.1002/nme.7057, which has been published in final form at <https://doi.org/10.1002/nme.7057>. This article may be used for non-commercial purposes in accordance with Wiley Terms and Conditions for Use of Self-Archived Versions

Access to this work was provided by the University of Maryland, Baltimore County (UMBC) ScholarWorks@UMBC digital repository on the Maryland Shared Open Access (MD-SOAR) platform.

Please provide feedback

Please support the ScholarWorks@UMBC repository by emailing scholarworks-group@umbc.edu and telling us what having access to this work means to you and why it's important to you. Thank you.

Inexact and primal multilevel FETI-DP methods a multilevel extension and interplay with BDDC

Bedřich Sousedík

Abstract We study a framework that allows to solve the coarse problem in the FETI-DP method approximately. It is based on the saddle-point formulation of the FETI-DP system with a block-triangular preconditioner. One of the blocks approximates the coarse problem, for which we use the multilevel BDDC method as the main tool. This strategy then naturally leads to a version of multilevel FETI-DP method, and we show that the spectra of the multilevel FETI-DP and BDDC preconditioned operators are essentially the same. The theory is illustrated by a set of numerical experiments, and we also present a few experiments when the coarse solve is approximated by algebraic multigrid.

Keywords domain decomposition · FETI-DP · BDDC · multilevel methods

Mathematics Subject Classification (2000) 65F08 · 65F10 · 65M55 · 65N55 · 65Y05

1 Introduction

The last two decades in the development of iterative substructuring class of the domain decomposition methods has been marked by a significant progress of the two most advanced methods: *Finite Element Tearing and Interconnecting - Dual, Primal* (FETI-DP) and *Balancing Domain Decomposition by Constraints* (BDDC). The FETI-DP method by Farhat et al. [8, 9] was developed from an earlier version, called FETI (or FETI-1) by Farhat and Roux [10, 11]. The methods from the FETI family are *dual* because they iterate on a system of Lagrange multipliers that enforce continuity on the interfaces. However, in

The work was supported by the U.S. National Science Foundation under award DMS1913201.

Bedřich Sousedík

Department of Mathematics and Statistics, University of Maryland, Baltimore County,
1000 Hilltop Circle, Baltimore, MD 21250, USA

Corresponding author, Tel.: +1-410-455-3298, Fax: +1-410-455-1066

E-mail: sousedik@umbc.edu

the FETI-DP some of the “coarse” variables are treated as *primal*. The BDDC method was developed by Dohrmann [4, 26], as a fully *primal* counterpart of the FETI-DP, from an earlier BDD method by Mandel [25]. Interestingly, two methods that turned out to be equivalent to BDDC were developed independently by Cros [3] and by Fragakis and Papadrakakis [12], see [29, 41] for a proof. The methods are closely related, and the essential role here is played by the coarse problem. The proof that the eigenvalues of the FETI-DP and BDDC are the same except possibly for the multiplicity of eigenvalue equal to one, provided that the coarse components of both methods are the same, was obtained by Mandel et al. [27], and simpler proofs were derived soon afterwards by Li and Widlund [24], and also by Brenner and Sung [2].

Even though both methods were originally developed for elliptic problems, they were later extended to other models see, e.g., the monographs [23, 32, 44] for an overview. From our current perspective, we find the most important extensions beyond the original two-level algorithms. The possibility for a multilevel extension of the BDDC was already mentioned by Dohrmann [4]. The theory of three-level BDDC was developed by Tu [45, 46] and extended into a general multispace BDDC framework (with multilevel BDDC as a particular instance) by Mandel et al. [30]. For further developments of multilevel BDDC see, e.g., [17, 33, 37, 40, 42, 47, 48]. Extensions of the FETI-DP are less straightforward because the inverse of the coarse problem is built directly into the system matrix, and to the best of our knowledge the literature on this topic is limited. For example, a hybrid approach to three-level FETI was proposed by Klawonn and Rheinbach [21], and a recursive application of the FETI-DP was recently proposed by Toivanen et al. [43], see also Horák [13, Chapter 11].

The FETI-DP method is, due to the use of dual variables, suitable in particular for numerical solution of certain classes of coercive variational inequalities such as the contact problems see, e.g., [5, 6, 14]. However, factorization of the coarse problem may become a computational bottleneck when the number of substructures is large. Our main goal here is to develop a variant of the FETI-DP method that would allow for an approximate solution of the coarse problem using a multilevel strategy, but we do not apply the method recursively as [43]. Instead, we keep the coarse variables *primal*. Klawonn and Rheinbach [20] studied inexact FETI-DP methods based on the saddle-point formulation of the FETI-DP which allows to move (the approximation of) the inverse of the coarse problem into the preconditioner. We also use the saddle-point formulation, and the preconditioner based on an approximation of the block inverse of the system matrix. However, while [20] applied algebraic multigrid (AMG) to the coarse problem, our main focus here is the multilevel BDDC and also our formulation of the algorithms is different since we do not rely on partial subassembly and a change of variables. This strategy then naturally leads to a version of multilevel FETI-DP method, which we refer to as *primal*. We show that the spectra of the preconditioned multilevel FETI-DP and BDDC operators are essentially the same, in the same way as in the two-level case, and we present numerical experiments that confirm the theory. We note that an alternative point of view is that our method is a variant of the

multilevel BDDC that allows to use the dual, FETI-DP formulation on the finite element level.

The paper is organized as follows. In Section 2 we introduce the model problem and substructuring components, in Section 3 we recall the FETI-DP and BDDC methods, in Section 4 we formulate the inexact and multilevel FETI-DP method, in Section 5 we relate it to the multilevel BDDC, and in Section 6 we present results of numerical experiments.

2 Model problem and substructuring components

We introduce a model problem and some standard substructuring concepts. So, let us assume we are interested in solving a system of linear equations

$$Au = f, \quad (1)$$

where $u \in U$, and A is a symmetric, positive definite matrix obtained by a discretization of an elliptic partial differential equation subject to appropriate boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3 . Let Ω be decomposed into N nonoverlapping subdomains Ω^s , $s = 1, \dots, N$, equally called substructures. Let each substructure be obtained as a union of Lagrangian P1 or Q1 finite elements with mesh size h , and let H denote the characteristic size of a subdomain. The first step in substructuring is elimination of degrees of freedom interior to each substructure, $u_I \in U_I$, by so-called *static condensation*, and the problem is reduced to substructure interfaces. To this end, we denote by W_i the space of degrees of freedom on the substructure i interface, by w_i the vector of substructure interface degrees of freedom, and let

$$W = W^1 \times \dots \times W^N.$$

Next, we denote by $S^i : W^i \rightarrow W^i$ the Schur complement of subdomain i after eliminating the interior degrees of freedom, and by g^i the substructure force vector. Also, we introduce U_Γ as the space of the global interface degrees of freedom, and denote by \widehat{W} the space of degrees of freedom continuous across the substructure interfaces. Finally, we introduce $R^i : U_\Gamma \rightarrow W^i$ as the restriction map (a zero-one matrix) of global to local degrees of freedom. Then, we denote

$$S = \begin{bmatrix} S^1 & & \\ & \ddots & \\ & & S^N \end{bmatrix}, \quad w = \begin{bmatrix} w^1 \\ \vdots \\ w^N \end{bmatrix}, \quad g = \begin{bmatrix} g^1 \\ \vdots \\ g^N \end{bmatrix}, \quad R = \begin{bmatrix} R^1 \\ \vdots \\ R^N \end{bmatrix}, \quad (2)$$

where R has a full column rank such that $R^i R^{iT} = I$, and $\widehat{W} = \text{range } R$. Then, problem (1) reduced to interfaces may be written as

$$\widehat{S}w = f_\Gamma, \quad \widehat{S} = R^T S R, \quad f_\Gamma = R^T g. \quad (3)$$

Next ingredient is the averaging operator $E = R^T D_P$, where $D_P : W \rightarrow W$ is a given weight matrix such that $ER = I$. Also, let $B : W \rightarrow \Lambda$ denote the matrix enforcing the condition $w \in \widetilde{W}$ by $Bw = 0$, that is $\text{range } R = \text{null } B$. In agreement with the block structure (2), we consider

$$B = [B^1 \cdots B^N], \quad B^i = W^i \rightarrow \Lambda, \quad \text{range } B = \Lambda.$$

Let B_D^T denote a generalized inverse of B , that is $B_D^T : \Lambda \rightarrow W$ and $BB_D^T = I_\Lambda$. In computations, the matrix B_D is constructed from B and D_P , see [22, 27], and, in particular, we will assume that

$$B_D^T B + RE = I.$$

The FETI-DP and BDDC methods are characterized by a selection of *coarse degrees of freedom* (resp. *constraints*). These can be values at corners or averages over edges or faces. Let $\widetilde{W} \subset W$ be the subspace of vectors with continuous coarse degrees of freedom across substructure interface. Specifically, suppose we are given a space X and a matrix $C : W \rightarrow X$, and define

$$\widetilde{W} = \{w \in W : C(I - RE)w = 0\}, \quad (4)$$

where each row of C defines one constraint. The values Cw are called local coarse degrees of freedom and by (4) they have zero jumps on adjacent substructures. To represent the global coarse degrees of freedom, we use a space U_c and a matrix R_c with full column rank, such that

$$\widetilde{W} = \{w \in W : \exists u_c \in U_c : Cw = R_c u_c\}.$$

Next, let Φ be a matrix that defines the basis for the coarse problem, given as

$$\Phi = \begin{bmatrix} \Phi^1 \\ \vdots \\ \Phi^N \end{bmatrix}, \quad C\Phi = R_c.$$

Here we use energy minimal coarse basis functions Ψ that minimize energy of the form $x^T \Phi^T S \Phi x$ subject to constraints $Cx = e$ for some vector e given for each basis function as a column of matrix R_c . Specifically, the basis functions are found by solving saddle-point problems

$$\begin{bmatrix} S & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \Psi \\ \Lambda \end{bmatrix} = \begin{bmatrix} 0 \\ R_c \end{bmatrix}. \quad (5)$$

In implementation, we use a decomposition of the space \widetilde{W} , see [27, Lemma 8],

$$\widetilde{W} = \widetilde{W}_\Pi \oplus \widetilde{W}_\Delta, \quad (6)$$

where $\widetilde{W}_\Pi = \text{range } \Phi$ is the *primal* space, which gives rise to the (global) coarse problem, and $\widetilde{W}_\Delta = \text{null } C$ is the *dual* space with coarse degrees of freedom equal to zero, which gives rise to independent subdomain problems.

3 FETI-DP and BDDC methods

3.1 FETI-DP

Problem (3) may be equivalently written as

$$\min_{w \in \widetilde{W}} \max_{\lambda \in \Lambda} \mathcal{L}(w, \lambda), \quad (7)$$

where $\mathcal{L}(w, \lambda)$ is the Lagrangian

$$\mathcal{L}(w, \lambda) = \frac{1}{2} w^T S w - w^T E^T f_\Gamma + w^T B^T \lambda.$$

Problem (7) is further equivalent to the dual problem $\max_{\lambda \in \Lambda} \mathcal{F}(\lambda)$, where

$$\mathcal{F}(\lambda) = \min_{w \in \widetilde{W}} \mathcal{L}(w, \lambda) \quad (8)$$

is the dual functional.

Next, we use the splitting (6) and introduce Lagrange multipliers μ to enforce the zero values of coarse degrees of freedom in \widetilde{W}_Δ . Then, functional (8) can be written with minimization over \widetilde{W} instead of \widetilde{W}_Δ as

$$\mathcal{F}(\lambda) = \min_{w_\Delta \in \widetilde{W}} \min_{w_\Pi \in \widetilde{W}_\Pi} \sup_{\mu} \mathcal{L}(w_\Delta + w_\Pi, \lambda) + w_\Delta^T C^T \mu. \quad (9)$$

Writing $w_\Pi = \Phi u_c$, and differentiating with respect to w_Δ , μ , u_c , λ , we get a linear system, with a solution equivalent to solving the dual problem, which is

$$\begin{bmatrix} S & C^T & S\Phi & B^T \\ C & 0 & 0 & 0 \\ \Phi^T S^T & 0 & \Phi^T S\Phi & \Phi^T B^T \\ B & 0 & B\Phi & 0 \end{bmatrix} \begin{bmatrix} w_\Delta \\ \mu \\ u_c \\ \lambda \end{bmatrix} = \begin{bmatrix} E^T f_\Gamma \\ 0 \\ \Phi^T E^T f_\Gamma \\ 0 \end{bmatrix}. \quad (10)$$

Eliminating w_Δ and μ from the first two equations, we derive the so-called *saddle-point* FETI-DP formulation, which can be written as

$$\begin{bmatrix} S_c & \Psi^T B^T \\ B\Psi & -BS_\Delta^{-1}B^T \end{bmatrix} \begin{bmatrix} u_c \\ \lambda \end{bmatrix} = \begin{bmatrix} \Psi^T E^T f_\Gamma \\ -BS_\Delta^{-1}E^T f_\Gamma \end{bmatrix}, \quad (11)$$

where S_Δ^{-1} corresponds to independent subdomain solves

$$S_\Delta^{-1} = \begin{bmatrix} I \\ 0 \end{bmatrix}^T \begin{bmatrix} S & C^T \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad (12)$$

so that $\Psi = (I - S_\Delta^{-1}S)\Phi$ are the energy minimizing coarse basis functions, the same as in (5), cf. [27, eqs. (22) and (25)], and S_c is the coarse matrix

$$S_c = \Phi^T S \Phi - \begin{bmatrix} S\Phi \\ 0 \end{bmatrix}^T \begin{bmatrix} S & C^T \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} S\Phi \\ 0 \end{bmatrix} = \Psi^T S \Psi, \quad (13)$$

where the second equality follows from [27, Theorem 3].

We note that eliminating u_c from (11) yields the *original* FETI-DP system

$$B(\Psi S_c^{-1} \Psi^T + S_\Delta^{-1}) B^T \lambda = B(\Psi S_c^{-1} \Psi^T + S_\Delta^{-1}) E^T f_\Gamma. \quad (14)$$

The original FETI-DP is the method of preconditioned conjugate gradients applied to (14), used typically with the so-called Dirichlet preconditioner

$$M_D = B_D S B_D^T. \quad (15)$$

3.2 BDDC

The BDDC preconditioner was originally formulated by Dohrmann [4] for problem (1). Formulations for the reduced problem (3) followed in [26, 27]. The BDDC is the method of preconditioned conjugate gradients, in which the (two-level) BDDC preconditioner for problem (3) is given as, cf. [27, Lemma 27],

$$M_{\widehat{S}} = E H E^T, \quad (16)$$

where

$$H = \begin{bmatrix} \Psi^T \\ I \end{bmatrix}^T \begin{bmatrix} S_c & 0 \\ 0 & S_\Delta \end{bmatrix}^{-1} \begin{bmatrix} \Psi^T \\ I \end{bmatrix} = \Psi S_c^{-1} \Psi^T + S_\Delta^{-1}. \quad (17)$$

The preconditioner formulated for problem (1) includes in addition a pre-processing step with a static condensation of the residual and post-processing step with recovery of the approximate solution corresponding to the subdomain interiors. Both formulations are equivalent, and in particular by [30, Theorem 14], the eigenvalues of the two preconditioned operators are the same except possibly for multiplicity of eigenvalue equal to one.

3.2.1 Multilevel BDDC

In case when the number of subdomains is large, solving the *coarse problem* becomes a bottleneck. This motivated the introduction of the multilevel extensions [30, 45, 46] that solve the coarse problem only approximately by applying the two-level BDDC recursively. Since we will apply this idea in the FETI-DP framework, we briefly recall the Multilevel BDDC from [30]. The substructuring components will be denoted by an additional subscript $_1$ as U_1 , $\widetilde{W}_{\Pi 1}$ and Ω_1^s , $s = 1, \dots, N_1$, etc., and called level 1. Level 1 coarse problem will be called level 2 problem. Because it has an analogous structure as the original problem on level 1, we put $U_2 = \widetilde{W}_{\Pi 1}$, and we introduce the substructuring components for level 2 in the same way as we introduced level 1 components. Generally, in a design of L -level method, we repeat this process recursively for levels $\ell = 1, \dots, L - 1$. On each decomposition level ℓ , we assume that the subdomains Ω_ℓ^s , $s = 1, \dots, N_\ell$, have characteristic size H_ℓ and form a conforming triangulation of the domain Ω . Level $\ell - 1$ substructures become level ℓ elements, level $\ell - 1$ coarse degrees of freedom become level ℓ degrees

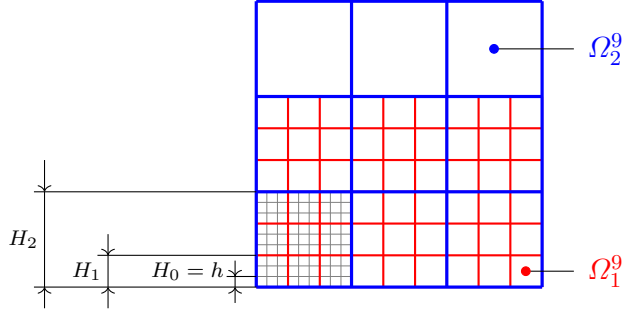


Fig. 1 An example of a uniform decomposition for a three-level method with $H_\ell/H_{\ell-1} = 3$, where $\ell = 1, 2$. Level 1 subdomains are shown in red and level 2 subdomains in blue.

of freedom, and $\Gamma_\ell \subset \Gamma_{\ell-1}$. The finite element level will be denoted as level 0 and $H_0 = h$. An example of a decomposition is shown in Figure 1.

The shape functions on level ℓ are determined by minimization of energy with respect to level $\ell - 1$ shape functions, subject to the value of exactly one level ℓ degree of freedom being one and others level ℓ degrees of freedom being zero. The minimization is done on each level ℓ element (level $\ell - 1$ substructure) separately, so the values of level $\ell - 1$ degrees of freedom are in general discontinuous between level $\ell - 1$ substructures, and only the values of level ℓ degrees of freedom between neighboring level ℓ elements coincide. The hierarchy of the spaces for L -level BDDC method can be written as

$$U_1 = \begin{matrix} U_{I1} \\ \oplus \\ \widehat{W}_1 \end{matrix} \xrightarrow{E_1} \widetilde{W}_1 = \begin{matrix} \widetilde{W}_{I1} \\ \oplus \\ \widetilde{W}_{\Delta 1} \end{matrix} = U_2 \cdots U_{L-1} = \begin{matrix} U_{IL-1} \\ \oplus \\ \widehat{W}_{L-1} \end{matrix} \xrightarrow{E_{L-1}} \widetilde{W}_{L-1} = \begin{matrix} \widetilde{W}_{I L-1} \\ \oplus \\ \widetilde{W}_{\Delta L-1} \end{matrix}.$$

The two-level BDDC preconditioner for problem (3) entails finding corrections in the spaces $\widetilde{W}_{\Delta 1}$ and \widetilde{W}_{I1} , and the multilevel BDDC for the same problem entails computing corrections in the spaces as above, that is

$$V_1 = \widetilde{W}_{\Delta 1}, \quad V_2 = U_{I2}, \quad V_3 = \widetilde{W}_{\Delta 2}, \quad \dots \quad V_{2(L-1)} = \widetilde{W}_{I L-1}, \quad (18)$$

such that

$$\widetilde{W}_1 = \left(\widetilde{W}_{\Delta 1} \oplus \widetilde{W}_{I1} \right) \subset \widetilde{V} = \sum_{k=1}^{2(L-1)} V_k \subset W. \quad (19)$$

The precise formulation of the multilevel BDDC preconditioner can be found in [30, Algorithm 17]. More generally, as an analogy to (16)–(17), we may consider an *inexact* BDDC preconditioner given as

$$\widetilde{M}_{\widehat{S}} = E \widetilde{H} E^T, \quad (20)$$

where

$$\widetilde{H} = \begin{bmatrix} \Psi^T \\ I \end{bmatrix}^T \begin{bmatrix} M_c & 0 \\ 0 & S_{\Delta}^{-1} \end{bmatrix} \begin{bmatrix} \Psi^T \\ I \end{bmatrix} = \Psi M_c \Psi^T + S_{\Delta}^{-1}, \quad (21)$$

and M_c is a preconditioner for the coarse matrix S_c . The multilevel BDDC applied to the coarse problem is then a particular example of M_c .

Finally, we note that the derivation of the FETI-DP (14) provides an alternative proof to [27, Lemma 27], which states that the preconditioned BDDC and FETI-DP operators can be written, respectively, as

$$M_{\hat{S}} \hat{S} = (EHE^T) (R^T S R), \quad (22)$$

$$M_D F = (B_D S B_D^T) (B H B^T), \quad (23)$$

which is used in the proof of equivalence of the spectra of the two operators.

Theorem 1 ([27, Theorem 26]) *The eigenvalues of the preconditioned operators of the BDDC and FETI-DP methods given, respectively, by (22) and (23) are the same except possibly for the eigenvalues equal to zero and one.*

Proof See Appendix.

Our next goal is to design a multilevel version of the FETI-DP method, based on formulation (11), and relate its spectrum to that of Multilevel BDDC.

4 Inexact and multilevel FETI-DP methods

Since the inverse of the coarse matrix S_c is built into the original FETI-DP operator in (14), its application may become a bottleneck when the number of substructures is large. Therefore, we work with the saddle-point formulation (11) that allows to solve coarse problem inexactly, which is a strategy proposed by Klawonn and Rheinbach [20].

The proposed preconditioner is based on the block inverse

$$\begin{aligned} \begin{bmatrix} X & Y^T \\ Y & -Z \end{bmatrix}^{-1} &= \begin{bmatrix} I - X^{-1} Y^T & \\ 0 & I \end{bmatrix} \begin{bmatrix} X^{-1} & 0 \\ 0 & -S_X^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -Y X^{-1} & I \end{bmatrix} = \\ &= \begin{bmatrix} I - X^{-1} Y^T & \\ 0 & I \end{bmatrix} \begin{bmatrix} X^{-1} & 0 \\ S_X^{-1} Y X^{-1} & -S_X^{-1} \end{bmatrix}, \end{aligned}$$

and $S_X = Z + Y X^{-1} Y^T$. For the inverse of matrix in (11), we then get

$$\begin{bmatrix} S_c & \Psi^T B^T \\ B \Psi & -B S_{\Delta}^{-1} B^T \end{bmatrix}^{-1} = \begin{bmatrix} I - S_c^{-1} \Psi^T B^T & \\ 0 & I \end{bmatrix} \begin{bmatrix} S_c^{-1} & 0 \\ F^{-1} B \Psi S_c^{-1} & -F^{-1} \end{bmatrix}, \quad (24)$$

and $F = B (\Psi S_c^{-1} \Psi^T + S_{\Delta}^{-1}) B^T$ is the same as the FETI-DP matrix in (14). The matrix on the right in (24) then motivates the choice of a preconditioner

$$M_F = \begin{bmatrix} M_c & 0 \\ M_D B \Psi M_c & -M_D \end{bmatrix}, \quad (25)$$

where M_D is the Dirichlet preconditioner (15), and M_c is the preconditioner for the coarse matrix S_c . In general, we may consider any suitable M_c giving

rise to *inexact* FETI-DP. For example, Klawonn and Rheinbach [20] used algebraic multigrid (AMG). Here, our main focus is on application of multilevel BDDC in place of M_c , which we then refer to as *primal multilevel* FETI-DP method. In implementation, the application of the preconditioner M_F entails one application of both M_c and M_D , which may be computed as follows.

Algorithm 2 *The preconditioner $M_F : (r_c, r_\lambda) \mapsto (u_c, \lambda)$ is applied as: First, apply the preconditioner M_c as $u_c = M_c r_c$, and compute*

$$\begin{aligned} x &= B\Psi u_c, \\ y &= x - r_\lambda. \end{aligned}$$

Then, apply the preconditioner M_D as $\lambda = M_D y$, and concatenate the results

$$\begin{bmatrix} u_c \\ \lambda \end{bmatrix}.$$

Remark 1 Algorithm 2 is closely related to the second algorithm by Klawonn and Rheinbach [20]. However, our formulation of FETI-DP does not use partial subassembly and a change of variables, and we consider multilevel BDDC besides algebraic multigrid as a preconditioner M_c for the coarse matrix S_c .

5 Analysis of the methods

The saddle-point FETI-DP matrix from (11) preconditioned by M_F from (25) is

$$\begin{bmatrix} M_c S_c & M_c \Psi^T B^T \\ M_D B \Psi (M_c S_c - I) & M_D B (\Psi M_c \Psi^T + S_\Delta^{-1}) B^T \end{bmatrix}. \quad (26)$$

First, note that with the exact coarse solve, i.e., setting $M_c = S_c^{-1}$, the matrix in position (1,1) is the identity so the matrix in position (2,1) vanishes, and the matrix in position (2,2) is the same as the preconditioned FETI-DP operator (23). Finally, multiplying the right-hand side of (11) by M_F we may check that solving the second set of equations therein is the same as solving the original FETI-DP problem (14) with the Dirichlet preconditioner M_D .

Next, let us observe that the inexact BDDC preconditioner (20) with \tilde{H} from (21) and the entry in position (2,2) of matrix (26) with the Dirichlet preconditioner M_D defined by (15) may be written, respectively, as

$$\tilde{M}_{\tilde{S}} \hat{S} = (E \tilde{H} E^T) (R^T S R), \quad (27)$$

$$M_D \tilde{F} = (B_D S B_D^T) (B \tilde{H} B^T). \quad (28)$$

That is (27)–(28) is an analogy to (22)–(23), just with \tilde{H} in place of H .

Let us now focus on the primal multilevel FETI-DP method, which combines the dual (FETI-DP type of) approach to the solve in \tilde{W}_Δ and applies the (primal) multilevel BDDC preconditioner to the coarse solve in \tilde{W}_H .

Assumption 3 *We will assume that the spaces V_k , $k = 1, \dots, 2(L-1)$, are constructed so that the associated linear systems are nonsingular, and the multilevel BDDC uses exact solves in these subspaces, so that*

$$\tilde{H}S \Big|_{\tilde{V}_{CW}} = I. \quad (29)$$

For the two-level BDDC method the Assumption 3 holds due to the exact coarse solve, see also [27, Lemma 28]. For multispace and multilevel BDDC see [30, Corollary 4, Remark 5 and Lemma 18].

Then, a variant of Theorem 1 also holds for the two operators (27)–(28).

Lemma 1 *Let \tilde{H} be defined as in (21) and so that assumption (29) is satisfied. Then the eigenvalues of the preconditioned operators given by (27) and (28) are the same except possibly for the eigenvalues equal to zero and one.*

Proof See Appendix.

Let us now use Lemma 1 to examine the eigenvalues of the preconditioned operator (26). The corresponding eigenvalue problem can be written as

$$\begin{aligned} M_c S_c u + M_c \Psi^T B^T p &= \lambda u, \\ M_D B \Psi (M_c S_c - I) u + M_D B (\Psi M_c \Psi^T + S_\Delta^{-1}) B^T p &= \lambda p. \end{aligned} \quad (30)$$

We begin by considering possible eigenvalues of $M_c S_c$. There are two possible cases for $M_c S_c u = \mu u$ where (μ, u) is the corresponding eigenpair: $\mu = 1$ or $\mu \neq 1$. If $\mu = 1$, then $M_c S_c u = u$ and, assuming $\lambda \neq 1$, we get from the first equation $u = 1/(\lambda - 1) M_c \Psi^T B^T p$. Then, the second equation of (30) simplifies to the eigenvalue problem for (28), that is $M_D \tilde{F} p = \lambda p$, which has essentially the same eigenvalues as the multilevel BDDC preconditioned operator $\tilde{M}_{\tilde{S}} \tilde{S}$ by Lemma 1. If $\mu \neq 1$, then $M_c S_c u = \mu u$ and, assuming $\lambda \neq \mu$, we get from the first equation $u = 1/(\lambda - \mu) M_c \Psi^T B^T p$. Substituting this into the second equation of (30), we obtain

$$M_D B \left(\frac{\mu - 1}{\lambda - \mu} \Psi M_c \Psi^T + \Psi M_c \Psi^T + S_\Delta^{-1} \right) B^T p = \lambda p,$$

which reduces with $\lambda = 1$ to $M_D B S_\Delta^{-1} B^T p = \lambda p$. Therefore, Theorem 1 for two-level methods translates to the multilevel BDDC and FETI-DP methods.

Theorem 4 *The eigenvalues of the preconditioned operators of the multilevel BDDC and primal multilevel FETI-DP methods are the same except possibly for the eigenvalues equal to zero and one.*

For further discussion of multilevel BDDC method we refer to [30]. It is well known that the eigenvalues of the operators provide only asymptotic insight into the convergence of GMRES, which is the main iterative method used in the numerical experiments. The bound on the error at every step involves the condition number of the operator eigenvectors see, e.g., [7, 35, 36], which is difficult to estimate. Nevertheless, the numerical experiments presented in

the next section illustrate that the convergence of GMRES applied to either multilevel FETI-DP or BDDC method is quite comparable to that of the preconditioned conjugate gradients applied to the multilevel BDDC. We also note that if the preconditioned operator is symmetric, positive definite in some inner product then it may be also used in conjugate gradient method [18, 20].

6 Numerical experiments

The methods were implemented in MATLAB. We used the right-preconditioned GMRES method with no restarts and (for multilevel BDDC also) preconditioned conjugate gradients, with relative residual tolerance 10^{-8} as the stopping criterion and zero initial guess. To estimate the largest eigenvalues of the preconditioned operators, we used the MATLAB function `eigs`.

First, we presents results of numerical experiments for the model problem corresponding to the scalar second-order elliptic problem with zero Dirichlet boundary conditions on a square domain discretized by linear quadrilateral finite elements. The domain was uniformly divided into substructures with fixed $H_\ell/H_{\ell-1}$ ratio on each level ℓ . Table 1 shows the largest eigenvalues of the preconditioned operators and the iteration counts of the multilevel FETI-DP and BDDC methods set up using corner constraints and varying numbers of levels L and coarsening ratios $H_\ell/H_{\ell-1}$, $\ell = 1, \dots, L-1$. Table 2 then shows results for the same set of problems and the two methods set up using corner constraints combined with arithmetic averages over edges. We used right-preconditioned GMRES for the FETI-DP method (11) preconditioned by M_F from (25), in which multilevel BDDC preconditioner was applied as M_C , and for the multilevel BDDC method we used also preconditioned conjugate gradients (PCG). In addition, for the FETI-DP formulation, we studied performance of a block-diagonal variant of the preconditioner M_F , that is

$$\begin{bmatrix} M_c & 0 \\ 0 & -M_D \end{bmatrix}, \quad (31)$$

which from the theory in [31] may be nearly as efficient. From the tables it can be seen the largest eigenvalues of both multilevel FETI-DP and BDDC preconditioned operators were the same. Also, the iteration counts of GMRES were essentially the same for the two methods, and the same trend can be observed for the PCG method. However, in case of the FETI-DP method with the block-diagonal preconditioner the iteration counts of GMRES are higher in all cases. Since the overhead associated with the application of M_F is small, cf. Algorithm 2, the block-diagonal preconditioner appears to be less suitable.

Figure 2 displays estimates of the largest 150 eigenvalues of the multilevel FETI-DP and BDDC preconditioned operators in the setup with three levels and coarsening ratio $H_2/H_1 = H_1/H_0 = 3$. Specifically, the left panel corresponds to the second row of Table 1 and the right panel to the second row of Table 2. In both panels all eigenvalues match as predicted by the theory.

Table 1 Numbers of GMRES iterations (and GMRES/PCG for BDDC) for the multilevel FETI-DP and BDDC methods, and for the FETI-DP with the block diagonal preconditioner (BD) from (31), with increasing number of levels L and coarsening ratios $H_\ell/H_{\ell-1}$, $\ell = 1, \dots, L-1$ with corner constraints. Here $nsub$ is the number of subdomains on each level, $ndof$ is the number of degrees of freedom, and λ_{\max} is the largest eigenvalue of the multilevel BDDC and FETI-DP preconditioned operators.

L	$nsub$	$ndof$	λ_{\max}	BD	FETI-DP	BDDC
$H_\ell/H_{\ell-1} = 3$						
2	9	100	1.8781	14	9	10/10
3	81/9	784	3.2636	23	14	15/15
4	729/81/9	6724	5.3709	29	20	21/21
5	6561/729/81/9	59,536	8.8857	35	27	28/29
$H_\ell/H_{\ell-1} = 4$						
2	16	289	2.1797	10	6	12/11
3	256/16	4225	4.1758	29	18	19/19
4	4096/256/16	66,049	7.8472	37	27	28/29
$H_\ell/H_{\ell-1} = 6$						
2	36	1369	2.7982	24	14	15/15
3	1296/36	47,089	6.0284	36	24	25/26

Table 2 Numbers of GMRES (and GMRES/PCG for BDDC) iterations for the same problems as in Table 1, but here with corner constraints combined with averages over edges.

L	$nsub$	$ndof$	λ_{\max}	BD	FETI-DP	BDDC
$H_\ell/H_{\ell-1} = 3$						
2	9	100	1.0550	8	4	4/5
3	81/9	784	1.2866	11	8	8/8
4	729/81/9	6724	1.6980	14	11	11/11
5	6561/729/81/9	59,536	2.1212	17	14	14/14
$H_\ell/H_{\ell-1} = 4$						
2	16	289	1.1094	7	4	6/6
3	256/16	4225	1.4779	13	9	9/9
4	4096/256/16	66,049	1.9393	17	13	13/13
$H_\ell/H_{\ell-1} = 6$						
2	36	1369	1.2280	13	7	7/7
3	1296/36	47,089	1.7775	17	11	11/11

Table 3 Numbers of GMRES iterations (and GMRES/PCG for BDDC) for the dam problem and the BDDC and FETI-DP methods with varying choices of constraints: corners (c), arithmetic averages over edges (e) and faces (f), and their total numbers on level 1/level 2, and also the largest eigenvalues λ_{\max} of the preconditioned operators. The inexact methods apply algebraic multigrid (AMG) to the coarse problem.

		inexact methods		three-level methods			
constraints		FETI-DP	BDDC	FETI-DP		BDDC	
type	level 1/level 2	it	it	λ_{\max}	it	λ_{\max}	it
c	11,970/99	65	71/75	99.5	66	99.5	73/87
c+e	21,180/144	105	107/125	22.0	43	21.1	42/45
c+e+f	28,002/198	134	127/140	15.1	37	14.1	33/35

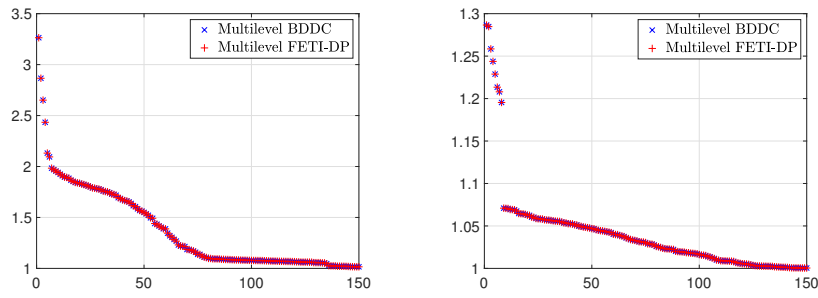


Fig. 2 The largest 150 eigenvalues of the three-level BDDC and FETI-DP preconditioned operators with coarsening ratio $H_\ell/H_{\ell-1} = 3$, where $\ell = 1, 2$, with corner constraints (left), and corners and arithmetic averages over edges (right).

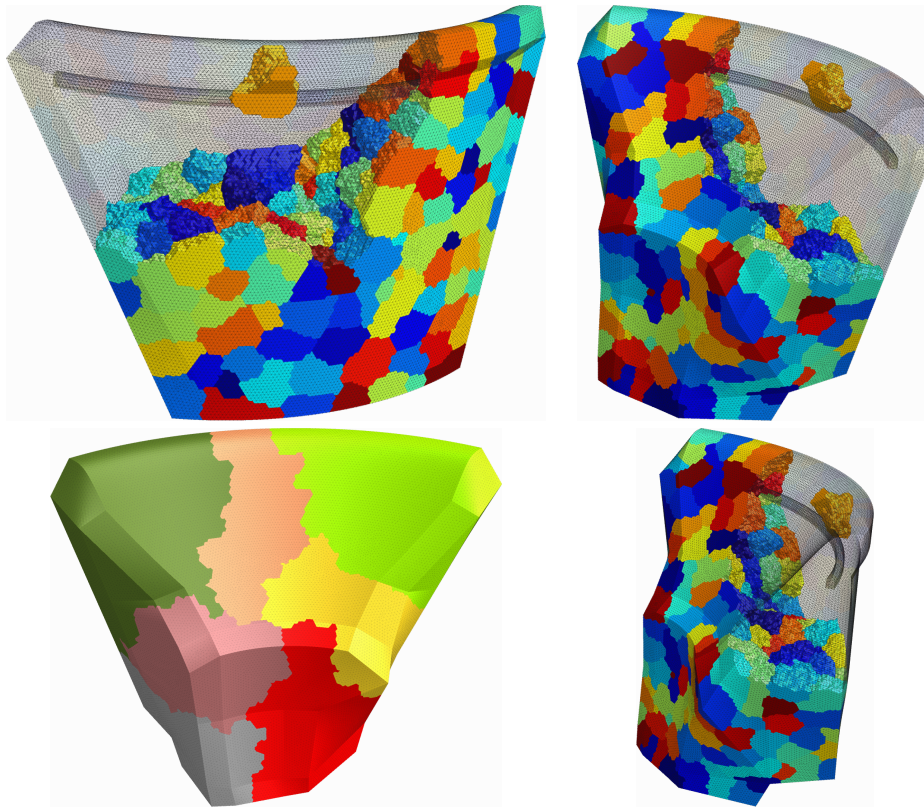


Fig. 3 Finite element discretization and substructuring of the dam consisting of 3,800,080 elements distributed into 400 subdomains in the first level (top) and 8 subdomains in the second level (bottom left). Model by courtesy of Jaroslav Kruis, images reproduced from [39].

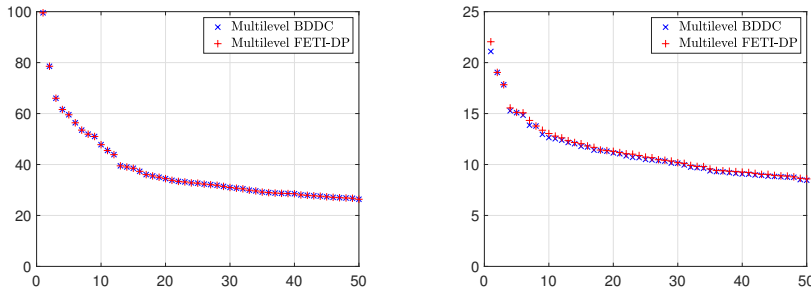


Fig. 4 The largest 50 eigenvalues of the three-level BDDC and FETI-DP preconditioned operators for the dam problem using corner constraints (left), and corners combined with arithmetic averages over edges (right).

As the second problem we used a real-world model of linear elasticity in a dam discretized using 3,800,080 finite elements with 2,006,748 degrees of freedom, distributed into 400 subdomains with 3990 corners, 3070 edges and 2274 faces. The subdomains in the first level were distributed into 8 subdomains for the three-level method. The distribution of the elements into subdomains in both substructuring levels was obtained using METIS [15,16], see Figure 3. Table 3 shows the iteration counts of GMRES (and PCG) for inexact and three-level FETI-DP and BDDC methods with varying choices of constraints. The table also shows the largest eigenvalues of the two preconditioned operators for the three-level methods. For the inexact methods we applied algebraic multigrid (AMG) as the preconditioner M_c in (25) and (20)–(21). Specifically, we used a MATLAB version of the routine HSL_MI20 [1], which performs grid coarsening using the classical algorithm by Ruge and Stüben [34]. We used the AMG routine with its default settings. It can be seen that adding more constraints as averages over the edges and faces to the coarse problem (and thus increasing its size) leads to an increase in the iteration count for the inexact methods. Nevertheless with the corner constraints, the inexact methods converge in fewer iterations than the three-level methods. On the other hand, it is expected that adding more constraints would improve convergence for the multilevel methods. We see that this happens, in particular after combining the corners with the averages over edges, and the improvement is less dramatic after adding the averages over faces. We also note that there are adaptive techniques for construction of the coarse spaces, in particular the Adaptive-Multilevel BDDC [28,42], which could be applied to the multilevel methods presented here, but this is beyond the focus of our study. Also, a thorough comparison of the proposed methods with AMG would be of independent interest. For example, in a recent study Klawonn et al. [19] found that, for the BDDC method, inexact preconditioning using AMG and the two-level BDDC applied to the coarse problem show in most situations a very similar behavior.

Figure 4 displays estimates of the 50 largest eigenvalues for the three-level FETI-DP and BDDC preconditioned operators for the setup using corner constraints (left panel), and using corners combined with arithmetic averages over edges (right panel), for the dam problem. We see that adding averages of the edges reduces the largest eigenvalue approximately four times, and even though the eigenvalues in the case of the corner constraints decrease at a somewhat faster rate, their values remain higher than those in the case when both corner and edges are used for constraints. In both panels a (close) correspondence of the eigenvalue estimates can be observed for all eigenvalues.

Finally, we note that the iteration counts of GMRES and PCG in Table 3 are close also for both inexact methods and each set of constraints. This suggests that the eigenvalues of the inexact methods may be also similar. A comparison of the eigenvalue estimates for the inexact methods is provided by Figure 5. The left panel corresponds to the problem with the same setup as in the second row of Table 1, cf. also the left panel in Figure 2. For this problem, the inexact FETI-DP converged in 11 iterations and the inexact BDDC converged in 12 iterations using both GMRES and PCG. It can be seen that in this case, with geometric discretization and uniform partitioning of the domain into subdomains, the eigenvalue estimates are the same (and real). An agreement can be observed also in the right panel, which displays 50 eigenvalues of the largest magnitude corresponding to the dam problem with corner constraints. Nevertheless, even though all eigenvalues in Figure 5 are real, we note that the eigenvalues of the inexact FETI-DP preconditioned operator are in general complex, and a strategy to make them real was provided in [20].

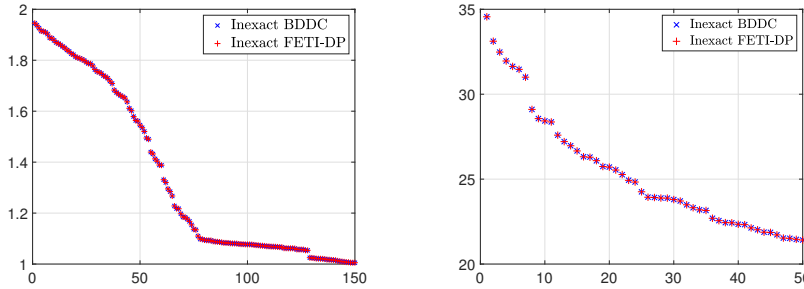


Fig. 5 The largest 150 eigenvalues of the inexact BDDC and FETI-DP preconditioned operators with geometric discretization and partitioning of the domain (left panel), and 50 eigenvalues of the largest magnitude for the dam problem (right).

Acknowledgements The author is grateful to Jaroslav Kruis for providing the discretization of the dam problem, to Marian Brezina for visualizations using his own software package *MyVis*, and to Jan Mandel for continuing support and interest in the domain decomposition methods. He would also like to thank the anonymous reviewers for comments and suggestions that helped to improve the quality of the manuscript.

References

1. Boyle, J., Mihajlović, M., Scott, J.: HSL_MI20: An efficient AMG preconditioner for finite element problems in 3D. *International Journal for Numerical Methods in Engineering* **82**(1), 64–98 (2010). URL http://www.hsl.rl.ac.uk/catalogue/hsl_mi20.html
2. Brenner, S.C., Sung, L.Y.: BDDC and FETI-DP without matrices or vectors. *Comput. Methods Appl. Mech. Engrg.* **196**(8), 1429–1435 (2007)
3. Cros, J.M.: A preconditioner for the Schur complement domain decomposition method. In: I. Herrera, D.E. Keyes, O.B. Widlund (eds.) *Domain Decomposition Methods in Science and Engineering*, pp. 373–380. National Autonomous University of Mexico (UNAM), México (2003). 14th International Conference on Domain Decomposition Methods, Cocoyoc, Mexico, January 6–12, 2002
4. Dohrmann, C.R.: A preconditioner for substructuring based on constrained energy minimization. *SIAM J. Sci. Comput.* **25**(1), 246–258 (2003)
5. Dostál, Z.: *Optimal quadratic programming algorithms: with applications to variational inequalities*. Springer-Verlag (2009)
6. Dostál, Z., Horák, D., Stefanica, D.: A scalable FETI-DP algorithm with non-penetration mortar conditions on contact interface. *J. Comput. Appl. Math.* **231**(2), 577–591 (2009)
7. Elman, H.C., Silvester, D.J., Wathen, A.J.: *Finite elements and fast iterative solvers: with applications in incompressible fluid dynamics*, second edn. Numerical Mathematics and Scientific Computation. Oxford University Press, New York (2014)
8. Farhat, C., Lesoinne, M., Le Tallec, P., Pierson, K., Rixen, D.: FETI-DP: a dual-primal unified FETI method. I. A faster alternative to the two-level FETI method. *Internat. J. Numer. Methods Engrg.* **50**(7), 1523–1544 (2001)
9. Farhat, C., Lesoinne, M., Pierson, K.: A scalable dual-primal domain decomposition method. *Numer. Linear Algebra Appl.* **7**, 687–714 (2000)
10. Farhat, C., Mandel, J., Roux, F.X.: Optimal convergence properties of the FETI domain decomposition method. *Comput. Methods Appl. Mech. Engrg.* **115**, 365–385 (1994)
11. Farhat, C., Roux, F.X.: A method of finite element tearing and interconnecting and its parallel solution algorithm. *Internat. J. Numer. Methods Engrg.* **32**, 1205–1227 (1991)
12. Fragakis, Y., Papadrakakis, M.: The mosaic of high performance domain decomposition methods for structural mechanics: Formulation, interrelation and numerical efficiency of primal and dual methods. *Comput. Methods Appl. Mech. Engrg.* **192**, 3799–3830 (2003)
13. Horák, D.: FETI based domain decomposition methods for variational inequalities. Ph.D. thesis, Department of Applied Mathematics, Faculty of Electrical Engineering and Computer Science, VŠB - Technical University Ostrava (2007)
14. Jarošová, M., Klawonn, A., Rheinbach, O.: Projector preconditioning and transformation of basis in FETI-DP algorithms for contact problems. *Math. Comput. Simulation* **82**(10), 1894–1907 (2012)
15. Karypis, G., Kumar, V.: A fast and high quality multilevel scheme for partitioning irregular graphs. *SIAM J. Sci. Comput.* **20**(1), 359–392 (1998)
16. Karypis, G., Kumar, V.: METIS: A software package for partitioning unstructured graphs, partitioning meshes, and computing fill-reducing orderings of sparse matrices, version 4.0. Technical report, Department of Computer Science, University of Minnesota (1998). URL <http://glaros.dtc.umn.edu/gkhome/views/metis>
17. Kim, H.H., Tu, X.: A three-level BDDC algorithm for mortar discretizations. *SIAM Journal on Numerical Analysis* **47**(2), 1576–1600 (2009)
18. Klawonn, A.: Block-triangular preconditioners for saddle point problems with a penalty term. *SIAM J. Sci. Comput.* **19**(1), 172–184 (1998)
19. Klawonn, A., Lanser, M., Rheinbach, O., Weber, J.: Preconditioning the coarse problem of BDDC methods - three-level, algebraic multigrid, and vertex-based preconditioners. *Electron. Trans. Numer. Anal.* **51**, 432–450 (2019)
20. Klawonn, A., Rheinbach, O.: Inexact FETI-DP methods. *International Journal for Numerical Methods in Engineering* **69**(2), 284–307 (2007)
21. Klawonn, A., Rheinbach, O.: A hybrid approach to 3-level FETI. *PAMM* **8**(1), 10,841–10,843 (2008). 79th Annual Meeting of the International Association of Applied Mathematics and Mechanics (GAMM), Bremen 2008

22. Klawonn, A., Widlund, O.B., Dryja, M.: Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients. *SIAM J. Numer. Anal.* **40**(1), 159–179 (2002)
23. Kruis, J.: Domain decomposition methods for distributed computing. Saxe-Coburg Publications, Kippen, Stirling, Scotland (2006)
24. Li, J., Widlund, O.B.: FETI-DP, BDDC, and block Cholesky methods. *Internat. J. Numer. Methods Engrg.* **66**(2), 250–271 (2006)
25. Mandel, J.: Balancing domain decomposition. *Comm. Numer. Methods Engrg.* **9**(3), 233–241 (1993)
26. Mandel, J., Dohrmann, C.R.: Convergence of a balancing domain decomposition by constraints and energy minimization. *Numer. Linear Algebra Appl.* **10**(7), 639–659 (2003)
27. Mandel, J., Dohrmann, C.R., Tezaur, R.: An algebraic theory for primal and dual substructuring methods by constraints. *Appl. Numer. Math.* **54**(2), 167–193 (2005)
28. Mandel, J., Sousedík, B.: Adaptive selection of face coarse degrees of freedom in the BDDC and the FETI-DP iterative substructuring methods. *Comput. Methods Appl. Mech. Engrg.* **196**(8), 1389–1399 (2007)
29. Mandel, J., Sousedík, B.: BDDC and FETI-DP under minimalist assumptions. *Computing* **81**, 269–280 (2007)
30. Mandel, J., Sousedík, B., Dohrmann, C.R.: Multispace and multilevel BDDC. *Computing* **83**(2-3), 55–85 (2008)
31. Murphy, M.F., Golub, G.H., Wathen, A.J.: A note on preconditioning for indefinite linear systems. *SIAM J. Sci. Comput.* **21**(6), 1969–1972 (2000)
32. Pechstein, C.: Finite and boundary element tearing and interconnecting solvers for multiscale problems, *Lecture Notes in Computational Science and Engineering*, vol. 90. Springer-Verlag (2013)
33. Peng, J., Wang, J., Shu, S.: Adaptive BDDC algorithms for the system arising from plane wave discretization of Helmholtz equations. *International Journal for Numerical Methods in Engineering* **116**(10–11), 683–707 (2018)
34. Ruge, J.W., Stüben, K.: Algebraic multigrid (AMG). In: S.F. McCormick (ed.) *Multigrid Methods, Frontiers in Applied Mathematics*, pp. 73–130. SIAM, Philadelphia (1987)
35. Saad, Y.: *Iterative methods for sparse linear systems*, second edn. Society for Industrial and Applied Mathematics (2003)
36. Simoncini, V.: Block triangular preconditioners for symmetric saddle-point problems. *Applied Numerical Mathematics* **49**(1), 63–80 (2004)
37. Šístek, J., Mandel, J., Sousedík, B., Burda, P.: Parallel implementation of Multilevel BDDC. In: A. Cangiani, R. Davidchack, E. Georgoulis, A. Gorbun, J. Levesley, M. Tretyakov (eds.) *Numerical Mathematics and Advanced Applications 2011*, pp. 681–689. Springer (2013)
38. Sousedík, B.: Comparison of some domain decomposition methods. Ph.D. thesis, Czech Technical University in Prague, Faculty of Civil Engineering, Department of Mathematics (2008)
39. Sousedík, B.: Adaptive-Multilevel BDDC. Ph.D. thesis, University of Colorado Denver, Department of Mathematical and Statistical Sciences (2010)
40. Sousedík, B.: Nested BDDC for a saddle-point problem. *Numerische Mathematik* **125**(4), 761–783 (2013)
41. Sousedík, B., Mandel, J.: On the equivalence of primal and dual substructuring preconditioners. *Electron. Trans. Numer. Anal.* **31**, 384–402 (2008)
42. Sousedík, B., Šístek, J., Mandel, J.: Adaptive-Multilevel BDDC and its parallel implementation. *Computing* **95**(12), 1087–1119 (2013)
43. Toivanen, J., Avery, P., Farhat, C.: A multilevel feti-dp method and its performance for problems with billions of degrees of freedom. *International Journal for Numerical Methods in Engineering* **116**(10–11), 661–682 (2018)
44. Toselli, A., Widlund, O.B.: *Domain Decomposition Methods—Algorithms and Theory, Springer Series in Computational Mathematics*, vol. 34. Springer-Verlag, Berlin (2005)
45. Tu, X.: Three-level BDDC in three dimensions. *SIAM J. Sci. Comput.* **29**(4), 1759–1780 (2007)
46. Tu, X.: Three-level BDDC in two dimensions. *Internat. J. Numer. Methods Engrg.* **69**(1), 33–59 (2007)

47. Tu, X.: A three-level BDDC algorithm for a saddle point problem. *Numer. Math.* **119**(1), 189–217 (2011)
48. Zampini, S., Tu, X.: Multilevel balancing domain decomposition by constraints deluxe algorithms with adaptive coarse spaces for flow in porous media. *SIAM Journal on Scientific Computing* **39**(4), A1389–A1415 (2017)

A Appendix

Theorem 1 can be found in [27, Theorem 26], see also [2, Theorem 3.9] and [38, Theorem 12]. Since its variant we use here as Lemma 1, which is virtually the same, we include the proof for completeness. The proof of Theorem 1 is obtained by dropping the tilde accent mark. That is, one would write H in place of \tilde{H} , F in place of \tilde{F} , and $M_{\tilde{S}}$ in place of $\tilde{M}_{\tilde{S}}$.

First, let us note a few properties of the operators: both ER and $B_D B^T$ are projections,

$$BR = 0, \quad (\text{from the definition}), \quad (32)$$

$$REB_D^T = (I - B_D^T B) B_D^T = B_D^T - B_D^T B B_D^T = 0, \quad (33)$$

$$EB_D^T B = E(I - RE) = E - ERE = 0. \quad (34)$$

Next, we show that

$$\begin{aligned} T_P \tilde{M}_{\tilde{S}} \tilde{S} &= M_D \tilde{F} T_P, & T_P &= B_D S R, \\ T_D M_D \tilde{F} &= \tilde{M}_{\tilde{S}} \tilde{S} T_D, & T_D &= E \tilde{H} B^T. \end{aligned}$$

The first identity is derived as

$$\begin{aligned} T_P \tilde{M}_{\tilde{S}} \tilde{S} &= (B_D S R) E \tilde{H} E^T (R^T S R) = B_D S (I - B_D^T B) \tilde{H} (RE)^T S R = \\ &= B_D S \tilde{H} (RE)^T S R - B_D S B_D^T B \tilde{H} (I - B_D^T B)^T S R = \\ &= B_D S \tilde{H} (RE)^T S R - B_D S B_D^T B \tilde{H} S R + B_D S B_D^T B \tilde{H} B^T B_D S R = \\ &= (B_D S B_D^T) B \tilde{H} B^T (B_D S R) = M_D \tilde{F} T_P, \end{aligned}$$

where some terms vanished, because using (29), (33) and (32) we get

$$\begin{aligned} B_D S \tilde{H} (RE)^T S R &= \left(R^T S R E \underbrace{\tilde{H} S}_{=I} B_D^T \right)^T = \left(R^T S \underbrace{R E B_D^T}_{=0} \right)^T = 0, \\ B_D S B_D^T B \tilde{H} S R &= B_D S B_D^T B \underbrace{\tilde{H} S}_{=I} R = B_D S B_D^T \underbrace{B R}_{=0} = 0. \end{aligned}$$

The second identity is derived as

$$\begin{aligned} T_D M_D \tilde{F} &= (E \tilde{H} B^T) B_D S B_D^T (B \tilde{H} B^T) = E \tilde{H} (I - RE)^T S B_D^T B \tilde{H} B^T = \\ &= E \tilde{H} S B_D^T B \tilde{H} B^T - E \tilde{H} (RE)^T S (I - RE) \tilde{H} B^T = \\ &= E \tilde{H} S B_D^T B \tilde{H} B^T - E \tilde{H} (RE)^T S \tilde{H} B^T + E \tilde{H} (RE)^T S R E \tilde{H} B^T = \\ &= (E \tilde{H} E^T) R^T S R (E \tilde{H} B^T) = \tilde{M}_{\tilde{S}} \tilde{S} T_D, \end{aligned}$$

where some terms vanished, because using (29), (34) and (32) we get

$$\begin{aligned} E\tilde{H}SB_D^TB\tilde{H}B^T &= E\underbrace{\tilde{H}S}_{=I}B_D^TB\tilde{H}B^T = \underbrace{EB_D^TB}_{=0}\tilde{H}B^T = 0, \\ E\tilde{H}(RE)^TS\tilde{H}B^T &= \left(B\underbrace{\tilde{H}S}_{=I}RE\tilde{H}E^T\right)^T = \left(\underbrace{BRE\tilde{H}E^T}_{=0}\right)^T = 0. \end{aligned}$$

Finally, let u_P be an eigenvector of $\widetilde{M}_{\widehat{S}}\widehat{S}$ corresponding to eigenvalue λ_P . Then $T_P u_P$ is also an eigenvector of $M_D\widetilde{F}$ provided $T_P u_P \neq 0$. So, let us assume $T_P u_P = 0$. But then,

$$\begin{aligned} 0 &= T_D T_P u_P = E\tilde{H}B^TB_D S R u_P = E\tilde{H}(I - ER)SRu_P = \\ &= E\underbrace{\tilde{H}S}_{=I}Ru_P - E\tilde{H}ERSRu_P = ERu_P - E\tilde{H}ERSRu_P = ERu_P - \widetilde{M}_{\widehat{S}}\widehat{S}u_P, \end{aligned}$$

but since ER is a projection, λ_P can be only equal to 0 or 1. Next, let u_D be an eigenvector of $M_D\widetilde{F}$ corresponding to eigenvalue λ_D . Then $T_D u_D$ is also an eigenvector of $\widetilde{M}_{\widehat{S}}\widehat{S}$ provided $T_D u_D \neq 0$. So, let us assume $T_D u_D = 0$. But then,

$$\begin{aligned} 0 &= T_P T_D u_D = B_D S R E \tilde{H} B^T u_D = B_D S (I - B_D^T B) \tilde{H} B^T u_D = \\ &= B_D \underbrace{S \tilde{H}}_{=I} B^T u_D - B_D S B_D^T B \tilde{H} B^T u_D = B_D B^T u_D - B_D S B_D^T B^T \tilde{H} B^T u_D = \\ &= B_D B^T u_D - M_D \widetilde{F} u_D, \end{aligned}$$

but since $B_D B^T$ is a projection, λ_D can be only equal to 0 or 1.