

Attribution-NonCommercial-NoDerivs 4.0 International

<https://creativecommons.org/licenses/by-nc-nd/4.0/>

Access to this work was provided by the University of Maryland, Baltimore County (UMBC) ScholarWorks@UMBC digital repository on the Maryland Shared Open Access (MD-SOAR) platform.

Please provide feedback

Please support the ScholarWorks@UMBC repository by emailing scholarworks-group@umbc.edu and telling us what having access to this work means to you and why it's important to you. Thank you.

Generic Properties of Koopman Eigenfunctions for Stable Fixed Points and Periodic Orbits^{*}

Matthew D. Kvalheim^{*} David Hong^{**} Shai Revzen^{***}

^{*} Department of Electrical and Systems Engineering, University of Pennsylvania, Philadelphia, PA 19104 USA (e-mail: kvalheim@seas.upenn.edu).

^{**} Department of Statistics, University of Pennsylvania, Philadelphia, PA 19104 USA (e-mail: dahong67@wharton.upenn.edu).

^{***} Department of Electrical Engineering and Computer Science, Ecology and Evolutionary Biology Department, Robotics Institute, University of Michigan, Ann Arbor, MI 48109 (e-mail: shrevzen@umich.edu)

Abstract: Our recent work established existence and uniqueness results for \mathcal{C}^k (actually $\mathcal{C}_{\text{loc}}^{k,\alpha}$) linearizing semiconjugacies for \mathcal{C}^1 flows defined on the entire basin of an attracting hyperbolic fixed point or periodic orbit (Kvalheim and Revzen, 2019). Applications include (i) improvements, such as uniqueness statements, for the Sternberg linearization and Floquet normal form theorems, and (ii) results concerning the existence, uniqueness, classification, and convergence of various quantities appearing in the “applied Koopmanism” literature, such as principal eigenfunctions, isostables, and Laplace averages.

In this work we consider the broadness of applicability of these results with an emphasis on the Koopmanism applications. In particular we show that, for the flows of “typical” \mathcal{C}^∞ vector fields having an attracting hyperbolic fixed point or periodic orbit with a fixed basin of attraction, the \mathcal{C}^∞ Koopman eigenfunctions can be completely classified, generalizing a result known for analytic eigenfunctions of analytic systems.

Copyright © 2021 The Authors. This is an open access article under the CC BY-NC-ND license (<https://creativecommons.org/licenses/by-nc-nd/4.0/>)

Keywords: Koopman operator, eigenfunctions, generic properties, isostables, periodic orbits
AMS subject classification: 37C10, 37C15, 37C20

1. INTRODUCTION

Linear dynamical systems and control systems are very well understood, in contrast with their nonlinear counterparts. Most models of real-world systems are, unfortunately, nonlinear. Thus, any means for applying linear systems techniques to the analysis and synthesis of nonlinear systems is of general interest in both scientific and engineering applications.

A common approach is to *approximate* a nonlinear system as a linear system near some nominal trajectory and apply linear systems techniques to the approximation (Khalil, 2002, Sec. 4.3, 12.2). While this approach works well in many situations, it is inherently *local* and often fails if the system is sufficiently far from the nominal trajectory. A recent alternative approach seeks linear representations of nonlinear systems that are instead *global* and *exact*. This is the approach taken in the applied Koopman operator

theory literature, initiated largely by Mezić (1994); Mezić and Banaszuk (2004); Mezić (2005), around 70 years after Koopman’s seminal work (Koopman, 1931).

The Koopman operator of a (nonlinear) dynamical system is an infinite-dimensional linear operator that acts on scalar-valued functions of state, or *observables*, by evolving them via the underlying dynamics. Since this operator is linear, one can discuss its spectral theory, and its spectral objects often have dynamical relevance. In particular, Koopman eigenfunctions are observables that evolve linearly under the dynamics; the dynamics and control of observables spanned by Koopman eigenfunctions is thus governed by linear systems theory. We emphasize that such a reduction is both *exact* and *global*; furthermore, given enough independent eigenfunctions one obtains an exact, global change of coordinates transforming the nonlinear system into a linear one.

Thus, methods to identify Koopman eigenfunctions are of interest, and the numerical computation of such eigenfunctions is an active research area.¹ The body of work most relevant to the present paper concerns the numerical com-

^{*} Kvalheim and Revzen were supported by ARO award W911NF-14-1-0573 to Revzen and by the ARO under the Multidisciplinary University Research Initiatives (MURI) Program, award W911NF-17-1-0306 to Revzen. Kvalheim was also supported by the ARO under the SLICE MURI Program, award W911NF-18-1-0327. Hong was supported in part by the Dean’s Fund for Postdoctoral Research of the Wharton School.

¹ Due to space constraints we mention only the review Budišić et al. (2012) here; see the references in Kvalheim and Revzen (2019) for many additional examples of this literature.

putation of *isostables* (Mauroy et al., 2013) and *isostable coordinates* (Wilson and Moehlis, 2016a; Shirasaka et al., 2017; Wilson and Ermentrout, 2018; Monga et al., 2019) for dynamical systems having an asymptotically stable equilibrium or limit cycle; these objects can be expressed in terms of Koopman eigenfunctions as discussed in Kvalheim and Revzen (2019). Isostables and isostable coordinates are useful tools for nonlinear model reduction, and it has been proposed that these objects could prove useful in real-world applications such as treatment design for Parkinson's disease, migraines, cardiac arrhythmias (Wilson and Moehlis, 2016b), and jet lag (Wilson and Moehlis, 2014).

In analyzing the theoretical properties of any algorithm for computing some quantity, it is desirable to know whether the computation is well-posed (Hadamard, 1902), and in particular whether the quantity in question *exists* and is *uniquely determined*. An existence and uniqueness theory for isostables, isostable coordinates, and more general Koopman eigenfunctions is thus desirable. In the context of attracting equilibria and limit cycles, some existence results can be obtained by invoking Hartman-Grobman type linearization theorems (Lan and Mezić, 2013; Eldering et al., 2018). On the other hand, it seems that uniqueness was less well understood, with an exception for the case of analytic eigenfunctions for analytic dynamical systems having a nonresonant linearization (Mauroy et al., 2013; Mezić, 2019).

Our recent work (Kvalheim and Revzen, 2019) filled much of the gap by establishing existence and uniqueness results for $\mathcal{C}_{\text{loc}}^{k,\alpha}$ linearizing semiconjugacies for \mathcal{C}^1 dynamical systems, of which Koopman eigenfunctions are a special case. In particular, we obtained uniqueness results for \mathcal{C}^k Koopman eigenfunctions; we also obtained \mathcal{C}^k existence results that, to the best of our knowledge, are stronger than those appearing elsewhere in the literature for $2 \leq k \leq \infty$. We obtained a particularly strong result for the case $k = \infty$: the \mathcal{C}^∞ eigenfunctions admit a complete classification for \mathcal{C}^∞ dynamical systems satisfying a nondegeneracy condition. The conclusion of this classification result yields much information about the eigenfunctions, so one would naturally like to understand how often its hypotheses hold.

The contribution of the present work is to show that the classification, existence, and uniqueness results for \mathcal{C}^∞ eigenfunctions in Kvalheim and Revzen (2019) in fact hold for “typical” \mathcal{C}^∞ vector fields having an asymptotically stable equilibrium or periodic orbit with a fixed basin of attraction, where “typical” means for sets of \mathcal{C}^∞ vector fields which are open and dense in suitable topologies.

The remainder of the paper is organized as follows. In §2 we prove our main result, Theorem 5, after some preliminary definitions and lemmas. In §3 we discuss the implications of Theorem 5 for the results of Kvalheim and Revzen (2019) relevant to Koopman eigenfunctions. Finally, Appendix A contains background on symmetric polynomials for the convenience of the reader.

2. MAIN RESULTS

This section contains our main result, Theorem 5. But first, we need some preliminary definitions and lemmas.

The following definition is Kvalheim and Revzen (2019, Def. 1) and is essentially an asymmetric version of definitions appearing in Sternberg (1957); Sell (1985). By some abuse of notation we also apply this definition to real matrices by viewing them as complex matrices with real entries; when discussing eigenvalues and eigenvectors of a linear self-map or matrix in this work, we always mean eigenvalues and eigenvectors of its complexification. By a further abuse of notation we also apply this definition to (the complexifications of) general linear self-maps of finite-dimensional vector spaces.

Definition 1. (((X, Y) k -nonresonance). Let $X \in \mathbb{C}^{d \times d}$ and $Y \in \mathbb{C}^{n \times n}$ be matrices with eigenvalues μ_1, \dots, μ_d and $\lambda_1, \dots, \lambda_n$, respectively, repeated with multiplicities. For any $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$, we say that (X, Y) is k -nonresonant if, for any $i \in \{1, \dots, d\}$ and any $m = (m_1, \dots, m_n) \in \mathbb{N}_{\geq 0}^n$ satisfying $2 \leq m_1 + \dots + m_n < k + 1$,

$$\mu_i \neq \lambda_1^{m_1} \dots \lambda_n^{m_n}. \quad (1)$$

(Note that this condition vacuously holds if $k = 1$; i.e., any two matrices are 1-nonresonant.)

For $n \in \mathbb{N}_{\geq 1}$ let $\mathcal{N}_n \subset \mathbb{R}^{n \times n}$ be the set of $n \times n$ real matrices A with *distinct eigenvalues* such that (A, A) is ∞ -nonresonant; by abuse of notation we also apply this definition to linear self-maps of general n -dimensional real vector spaces. Denoting by $\text{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n \times n}$ the invertible matrices, it follows from Def. 1 that $\mathcal{N}_n \subset \text{GL}(n, \mathbb{R})$ (since $0 = 0^m$ for all $m \in \mathbb{N}$). Below we use the notation $\exp: \mathbb{R}^{n \times n} \rightarrow \text{GL}(n, \mathbb{R})$, $\exp(A) := e^A$, when convenient.

Lemma 2. $\mathbb{R}^{n \times n} \setminus \mathcal{N}_n$ and $\mathbb{R}^{n \times n} \setminus \exp^{-1}(\mathcal{N}_n)$ both have Lebesgue measure zero.

Proof. From Def. 1, matrices in $\mathbb{R}^{n \times n} \setminus \mathcal{N}_n$ have eigenvalues $\lambda_1, \dots, \lambda_n$ that satisfy: (i) $\lambda_j = \lambda_k$ for some $j \neq k$ or (ii) $\lambda_i = \lambda_1^{m_1} \dots \lambda_n^{m_n}$ for some $i \in \{1, \dots, n\}$ and $(m_1, \dots, m_n) \in \mathcal{M}_n$ where

$$\mathcal{M}_n := \{(m_1, \dots, m_n) \in \mathbb{N}_{\geq 0}^n : m_1 + \dots + m_n \geq 2\}.$$

Condition (i) is equivalent to

$$0 = f(\lambda_1, \dots, \lambda_n) := \prod_{j \neq k} (\lambda_j - \lambda_k),$$

and condition (ii) is equivalent to

$$\exists m \in \mathcal{M}_n : 0 = g_m(\lambda_1, \dots, \lambda_n),$$

where

$$g_m(\lambda_1, \dots, \lambda_n) := \prod_i \prod_{\sigma \in S_n} (\lambda_i - \lambda_1^{m_{\sigma(1)}} \dots \lambda_n^{m_{\sigma(n)}})$$

and S_n is the group of permutations σ of $\{1, \dots, n\}$. Since f and g_m (for any m) are symmetric polynomials in the eigenvalues, they are also expressible as polynomials in the matrix entries. This follows from the fundamental theorem of symmetric polynomials and Vieta's theorem by recalling that the eigenvalues are roots of the characteristic polynomial whose coefficients are polynomials in the matrix entries (for more details see Appendix A). None of them are identically zero since, e.g.,

$$F(\text{diag}(\lambda_1, \dots, \lambda_n)) \equiv f(\lambda_1, \dots, \lambda_n) \neq 0,$$

and likewise for each G_m . As a result,

$$\mathbb{R}^{n \times n} \setminus \mathcal{N}_n = F^{-1}(0) \cup \bigcup_{m \in \mathcal{M}_n} G_m^{-1}(0)$$

is a countable union of measure zero sets and so is also measure zero. In more detail: each set in the union

is measure zero since (i) polynomials are real analytic functions and (ii) the zero set of a real analytic function which is not identically zero has measure zero (Mityagin, 2015). Defining the real analytic functions $\tilde{F} := F \circ \exp$ and $\tilde{G}_m := G_m \circ \exp$,

$$\mathbb{R}^{n \times n} \setminus \exp^{-1}(\mathcal{N}_n) = \tilde{F}^{-1}(0) \cup \bigcup_{m \in \mathcal{M}_n} \tilde{G}_m^{-1}(0)$$

is measure zero by the same reasoning. \square

Let $\mathcal{S}_n \subset \bar{\mathcal{S}}_n \subset \mathbb{R}^{n \times n}$ denote the sets of $n \times n$ real matrices whose eigenvalues belong to the open and closed unit disks in \mathbb{C} , respectively. Given any matrix $X \in \mathbb{C}^{n \times n}$, we define its *spectral radius* $\rho(X) := \max_{\mu \in \text{spec}(X)} |\mu|$, where $\text{spec}(X) \subset \mathbb{C}$ denotes the set of eigenvalues of a matrix X . By abuse of notation we also apply the preceding two definitions to linear self-maps of general n -dimensional real vector spaces; there is no ambiguity since eigenvalues do not depend on a choice of basis.

Lemma 3. $\mathcal{S}_n \cap \mathcal{N}_n$ is open in \mathcal{S}_n and \mathcal{S}_n is open in $\mathbb{R}^{n \times n}$.

Proof. Fix any $A \in \mathcal{S}_n \cap \mathcal{N}_n \subset \text{GL}(n, \mathbb{R})$. Since $A \in \mathcal{S}_n$, there exists $k \in \mathbb{N}_{\geq 2}$ such that $\rho(A^{-1})\rho(A)^{k+1} < 1$. It follows from Def. 1 that ∞ -nonresonance of (B, B) is implied by (i) k -nonresonance of (B, B) , which implies that B is invertible, and (ii) $\rho(B^{-1})\rho(B)^{k+1} < 1$. Since the inverse and eigenvalues of a matrix depend continuously on the matrix (Palis and De Melo, 1982, p. 53), the set of matrices satisfying each of these two conditions is open in \mathcal{S}_n . Similarly, the set of matrices having distinct eigenvalues is also open in \mathcal{S}_n . Hence A has a neighborhood in \mathcal{S}_n contained in \mathcal{N}_n ; since A was arbitrary, $\mathcal{S}_n \cap \mathcal{N}_n$ is open in \mathcal{S}_n . Continuity of eigenvalues also directly implies openness of \mathcal{S}_n in $\mathbb{R}^{n \times n}$. \square

Lemma 4. $\exp^{-1}(\mathcal{S}_n \cap \mathcal{N}_n)$ is dense in $\exp^{-1}(\bar{\mathcal{S}}_n) \subset \mathbb{R}^{n \times n}$.

Proof. Note that $\exp^{-1}(\mathcal{S}_n) \subset \exp^{-1}(\bar{\mathcal{S}}_n)$ are the sets of matrices having only eigenvalues with negative and nonpositive real parts, respectively. Examination of the real canonical form of matrices $A \in \exp^{-1}(\bar{\mathcal{S}}_n)$ reveals that $\exp^{-1}(\mathcal{S}_n)$ is dense in $\exp^{-1}(\bar{\mathcal{S}}_n)$. Lem. 3 and continuity of \exp imply that $\exp^{-1}(\mathcal{S}_n)$ is open in $\mathbb{R}^{n \times n}$, so Lem. 2 implies that $\exp^{-1}(\mathcal{S}_n \cap \mathcal{N}_n)$ is dense in $\exp^{-1}(\mathcal{S}_n)$ and thus (by the preceding sentence) also in $\exp^{-1}(\bar{\mathcal{S}}_n)$. \square

Recall that a \mathcal{C}^k ($k \in \mathbb{N}_{\geq 1}$) flow on a smooth manifold Q is a \mathcal{C}^k map $\Phi: Q \times \mathbb{R} \rightarrow Q$ satisfying $\Phi^0 = \text{id}_Q$ and $\Phi^{t+s} = \Phi^t \circ \Phi^s$ for all $t, s \in \mathbb{R}$, where $\Phi^t := \Phi(\cdot, t)$. (A \mathcal{C}^k map is one which has continuous mixed partial derivatives up to order k in local coordinates.) As a typical example, an ordinary differential equation (ODE)

$$\frac{dx}{dt} = f(x) \quad (2)$$

defined by a \mathcal{C}^k vector field which is *complete* (Lee, 2013, p. 215) generates a unique \mathcal{C}^k flow Φ , where $t \mapsto \Phi^t(x_0)$ is the unique solution to (2) with initial condition $\Phi^0(x_0) = x_0$. (Any \mathcal{C}^1 vector field is complete when restricted to the basin of attraction of a compact asymptotically stable set.)

We use the following notation in the remainder of this paper. Given a differentiable map $F: M \rightarrow N$ between smooth manifolds, $D_x F$ denotes the derivative of F at the point $x \in M$. (Recall that $D_x F: T_x M \rightarrow T_{F(x)} N$ is a linear map between tangent spaces (Lee, 2013), which can

be identified with the Jacobian of F evaluated at x in local coordinates.) In particular, given a \mathcal{C}^1 flow $\Phi: Q \times \mathbb{R} \rightarrow Q$ and fixed $t \in \mathbb{R}$, we write $D_x \Phi^t: T_x Q \rightarrow T_{\Phi^t(x)} Q$ for the derivative of the time- t map $\Phi^t: Q \rightarrow Q$ at the point x .

We need some additional notation for our main result. Let Q be a smooth n -dimensional manifold with $n \geq 1$. Let $\mathfrak{X}_{\text{fix}}^\infty(Q)$ and $\mathfrak{X}_{\text{per}}^\infty(Q)$ be the sets of \mathcal{C}^∞ vector fields f whose flows possess an asymptotically stable fixed point x_f with basin Q and asymptotically stable nonstationary periodic orbit Γ_f with basin Q , respectively. For the case of nonstationary periodic orbits we assume that $\dim(Q) \geq 2$. We use the notation Φ_f for the flow of such a vector field f . Given $f \in \mathfrak{X}_{\text{per}}^\infty$, we let $x_f \in \Gamma_f$ be an arbitrary point and $\tau_f > 0$ be the period of Γ_f ; if Γ_f is hyperbolic, we let $E_{x_f}^s$ be the unique $D_{x_f} \Phi_f^{\tau_f}$ -invariant complement to $\text{span}(\{f(x_f)\})$. Let $\mathcal{G}_{\text{fix}} \subset \mathfrak{X}_{\text{fix}}^\infty$ and $\mathcal{G}_{\text{per}} \subset \mathfrak{X}_{\text{per}}^\infty$ denote the “good” vector fields such that every $f \in \mathcal{G}_{\text{fix}}$ satisfies $e^{D_{x_f} f} = D_{x_f} \Phi_f^1 \in \mathcal{S}_n \cap \mathcal{N}_n$ and such that the periodic orbit for each $g \in \mathcal{G}_{\text{per}}$ is hyperbolic and satisfies $D_{x_g} \Phi_g^{\tau_g}|_{E_{x_g}^s} \in \mathcal{S}_{n-1} \cap \mathcal{N}_{n-1}$.

The theorem below is our main result. We refer the reader to Hirsch (1994, Ch. 2) for the definitions of the \mathcal{C}^k Whitney (strong) and compact-open (weak) topologies, but the theorem’s effective meaning is clear from its proof.

Theorem 5. \mathcal{G}_{fix} (resp. \mathcal{G}_{per}) is open in $\mathfrak{X}_{\text{fix}}^\infty(Q)$ (resp. $\mathfrak{X}_{\text{per}}^\infty(Q)$) with respect to the \mathcal{C}^1 compact-open topology and dense in $\mathfrak{X}_{\text{fix}}^\infty(Q)$ (resp. $\mathfrak{X}_{\text{per}}^\infty(Q)$) with respect to the \mathcal{C}^∞ Whitney topology.

Remark 6. Many results proved in Kvalheim and Revzen (2019), including those recapitulated in the following §3, hold for flows of \mathcal{C}^∞ vector fields belonging to \mathcal{G}_{fix} or \mathcal{G}_{per} . Thus, a “typical” \mathcal{C}^∞ vector field in $\mathfrak{X}_{\text{fix}}^\infty(Q)$ or $\mathfrak{X}_{\text{per}}^\infty(Q)$ satisfies the hypotheses of those results.

Remark 7. Despite the suggestive statement of Lem. 2, we have not attempted to formalize “typical” in a measure-theoretic sense in Theorem 5 due to the apparent lack of natural definitions of “measure zero” subsets of $\mathfrak{X}_{\text{fix}}^\infty$ and $\mathfrak{X}_{\text{per}}^\infty$. In this direction, it would be interesting to know whether “typical” could be interpreted in a stronger sense using the framework of *prevalence* (Ott and Yorke, 2005).

Proof. We prove the theorem for $\mathfrak{X}_{\text{fix}}^\infty$; the case of $\mathfrak{X}_{\text{per}}^\infty$ is handled similarly using Floquet theory. The $\mathfrak{X}_{\text{fix}}^\infty$ statements hold vacuously if $\mathfrak{X}_{\text{fix}}^\infty(Q) = \emptyset$; if $\mathfrak{X}_{\text{fix}}^\infty(Q) \neq \emptyset$ then Q is diffeomorphic to \mathbb{R}^n (Wilson, 1967), so we may henceforth assume that $Q = \mathbb{R}^n$ and that $x_f = 0$.

Density — Let $f \in \mathfrak{X}_{\text{fix}}^\infty(\mathbb{R}^n)$ be arbitrary and let $U \subset Q$ be a precompact open neighborhood of $0 (= x_f)$. Let $\varphi: \mathbb{R}^n \rightarrow [0, \infty)$ be a \mathcal{C}^∞ function equal to 1 on a neighborhood of 0 and having support contained in U . Since 0 is asymptotically stable, $D_0 f \in \exp^{-1}(\bar{\mathcal{S}}_n)$. Lem. 4 implies the existence of a sequence $(A_n)_{n \in \mathbb{N}}$ of matrices with $A_n \rightarrow D_0 f$ and with $e^{A_n} \in \mathcal{S}_n \cap \mathcal{N}_n$ for all n . We now define a sequence $(g_n)_{n \in \mathbb{N}}$ of \mathcal{C}^∞ vector fields with $D_0 \Phi_{g_n}^1 = e^{A_n} \in \mathcal{S}_n \cap \mathcal{N}_n$ via

$$g_n(x) := f(x) + \varphi(x)(A_n - D_0 f) \cdot x.$$

All derivatives of the g_n converge uniformly to those of f on U , and g_n is equal to f on $\mathbb{R}^n \setminus U$, so g_n converges to f in the \mathcal{C}^∞ Whitney topology. Note that $g_n(0) = f(0) = 0$ for all n . It remains only to prove that $g_n \in \mathfrak{X}_{\text{fix}}^\infty(\mathbb{R}^n)$ for all

n sufficiently large, i.e., that 0 is globally asymptotically stable for g_n for large n ; this follows from a general result of Smith and Waltman (1999, Thm 2.2).

Openness — Fix any vector field $f \in \mathcal{G}_{\text{fix}} \subset \mathfrak{X}_{\text{fix}}^\infty(\mathbb{R}^n)$, so that $D_0\Phi_f^1 \in \mathcal{S}_n \cap \mathcal{N}_n$. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of vector fields in $\mathfrak{X}_{\text{fix}}^\infty(\mathbb{R}^n)$ converging to f in the \mathcal{C}^1 compact-open topology; i.e., g_n and Dg_n converge to f and Df uniformly on compact sets. Since the \mathcal{C}^1 compact-open topology can be given the structure of a Banach space, the (Banach space version of the) implicit function theorem implies that $x_{g_n} \rightarrow 0$ and hence $D_{x_{g_n}}\Phi_{g_n}^1 = e^{D_{x_{g_n}}g_n} \rightarrow e^{D_0f} = D_0\Phi_f^1$. It follows from Lem. 3 that $D_{x_{g_n}}\Phi_{g_n}^1 \in \mathcal{S}_n \cap \mathcal{N}_n$ and hence $g_n \in \mathcal{G}_{\text{fix}}$ for all n sufficiently large. Since $\mathfrak{X}_{\text{fix}}^\infty$ with the \mathcal{C}^1 compact-open topology is first countable, this implies the desired openness statement and completes the proof. \square

3. IMPLICATIONS FOR THE EXISTENCE AND UNIQUENESS OF KOOPMAN EIGENFUNCTIONS

The remainder of this paper describes the implications of Theorem 5 for the results of Kvalheim and Revzen (2019, Sec. 3.2–3.3). Those results were stated in terms of $\mathcal{C}_{\text{loc}}^{k,\alpha}$ functions; here we discuss only the simpler case $\mathcal{C}^k = \mathcal{C}_{\text{loc}}^{k,0}$.

3.1 Koopman eigenfunctions

Given a \mathcal{C}^1 flow $\Phi: Q \times \mathbb{R} \rightarrow Q$, where Q is a smooth manifold, we say that $\psi: Q \rightarrow \mathbb{C}$ is a *Koopman eigenfunction* with *eigenvalue* $\mu \in \mathbb{C}$ if ψ is not identically zero and

$$\forall t \in \mathbb{R}: \psi \circ \Phi^t = e^{\mu t} \psi. \quad (3)$$

The following generalizes the definitions for linear systems given in Mohr and Mezić (2016, Def. 2.2–2.3).

Definition 8. If Q is the basin of an asymptotically stable fixed point $x_0 \in Q$ for Φ , we say that an eigenfunction $\psi \in \mathcal{C}^1(Q, \mathbb{C})$ is a *principal eigenfunction* if $\psi(x_0) = 0$ and $D_{x_0}\psi \neq 0$. If instead Q is the basin of an asymptotically stable periodic orbit with image $\Gamma \subset Q$ for Φ , we say that an eigenfunction $\psi \in \mathcal{C}^1(Q, \mathbb{C})$ is a *principal eigenfunction* if $\psi(x_0) = 0$ and $D_{x_0}\psi \neq 0$ for all $x_0 \in \Gamma$.

3.2 Principal eigenfunctions for fixed points and periodic orbits

Given a (real or complex) linear self-map $Y: V \rightarrow V$, we say that a linear map $w: V \rightarrow \mathbb{C}$ is a *left eigenvector* of Y with *eigenvalue* $\lambda \in \mathbb{C}$ if $wY = \lambda w$. Differentiating (3) and using the chain rule immediately yields Prop. 9 and 10, which have appeared in the literature (see, e.g., the proof of (Mauroy and Mezić, 2016, Prop. 2)).

Proposition 9. Let x_0 be an asymptotically stable fixed point of the flow of a \mathcal{C}^1 vector field f with basin Q . If $\psi \in \mathcal{C}^1(Q, \mathbb{C})$ is a principal Koopman eigenfunction for the flow of f with eigenvalue $\mu \in \mathbb{C}$, then $D_{x_0}\psi$ is a left eigenvector of $D_{x_0}f$ with eigenvalue μ .

Proposition 10. Let Γ be the image of an asymptotically stable τ -periodic orbit of the \mathcal{C}^1 flow Φ with basin Q .² If $\psi \in \mathcal{C}^1(Q, \mathbb{C})$ is a principal Koopman eigenfunction for Φ with eigenvalue $\mu \in \mathbb{C}$, then for any $x_0 \in \Gamma$, $D_{x_0}\psi$ is a left eigenvector of $D_{x_0}\Phi^\tau$ with eigenvalue $e^{\mu\tau}$.

² We always mean that τ is the *minimal* period in “ τ -periodic orbit”.

The following result follows from Kvalheim and Revzen (2019, Rem. 3, Ex. 1, Prop. 6). The condition “ $D_{x_0}^i R = 0$ for all $0 \leq i < k$ ” should be interpreted to mean that, in local coordinates, R and all of its mixed partial derivatives of order less than k vanish at x_0 . This does not depend on the choice of local coordinates; see (Kvalheim and Revzen, 2019, Sec. 1.3.3). The \mathcal{C}^k compact-open (weak) topology (Hirsch, 1994, Ch. 2) on functions referred to below is the topology of \mathcal{C}^k -uniform convergence on compact subsets.

Proposition 11. Let f be a \mathcal{C}^1 vector field on Q with Q the basin of an attracting hyperbolic equilibrium $x_0 \in Q$ for the flow of f , where $n := \dim(Q) \geq 1$. Fix $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ and assume the spectral radius $\rho(e^{D_{x_0}f}) < 1$ satisfies

$$|e^\mu| > (\rho(e^{D_{x_0}f}))^k$$

in all of the following statements (with $(\rho(e^{D_{x_0}f}))^\infty := 0$).

Uniqueness of Koopman eigenvalues and principal eigenfunctions. Let $\psi_1 \in \mathcal{C}^k(Q, \mathbb{C})$ be any Koopman eigenfunction with eigenvalue μ .

- (1) Then there exists $m = (m_1, \dots, m_n) \in \mathbb{N}_{\geq 0}^n$ such that

$$\mu = m \cdot \lambda,$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $D_{x_0}f$ repeated with multiplicities and $\lambda := (\lambda_1, \dots, \lambda_n)$.

- (2) Assume that ψ_1 is a principal eigenfunction so that $\mu \in \text{spec}(D_{x_0}f)$, and assume that $\mu \neq m \cdot \lambda$ for any $m \in \mathbb{N}_{\geq 0}^n$ with $2 \leq \sum_i m_i \leq k$. Then ψ_1 is uniquely determined by $D_{x_0}\psi_1$, and if μ and $D_{x_0}\psi_1$ are real, then $\psi_1: Q \rightarrow \mathbb{R} \subset \mathbb{C}$ is real. In particular, if μ is an algebraically simple eigenvalue of (the complexification of) $D_{x_0}f$ and if ψ_2 is any other principal eigenfunction with eigenvalue μ , then there exists $c \in \mathbb{C} \setminus \{0\}$ such that

$$\psi_1 = c\psi_2.$$

Existence of principal eigenfunctions. Assume $f \in \mathcal{C}^k$ and that $\mu \neq m \cdot \lambda$ for any $m \in \mathbb{N}_{\geq 0}^n$ with $2 \leq \sum_i m_i \leq k$. Let w be a left eigenvector of $D_{x_0}f$ with eigenvalue μ .

- (1) Then there exists a unique principal eigenfunction $\psi \in \mathcal{C}^k(Q, \mathbb{C})$ with eigenvalue μ satisfying $D_{x_0}\psi = w$.
- (2) In fact, if Φ is the flow of f and $P \in \mathcal{C}^k(Q, \mathbb{C})$ is any “approximate eigenfunction” satisfying

$$P \circ \Phi^1 = e^\mu P + R \quad (4)$$

with $D_{x_0}^i R = 0$ for all $0 \leq i < k$ and $D_{x_0}P = w$, then

$$\psi = \lim_{t \rightarrow \infty} e^{-\mu t} P \circ \Phi^t \quad (5)$$

with convergence in the \mathcal{C}^k compact-open topology.

Remark 12. (the \mathcal{C}^∞ case). In the case $k = \infty$, the *spectral spread* hypothesis $|e^\mu| > (\rho(e^{D_{x_0}f}))^\infty := 0$ is automatically satisfied, so no assumption is needed on the spectral spread in Prop. 11 (and similarly for Prop. 13 below). We need only assume that $\mu \neq m \cdot \lambda$ for any m with $\sum_i m_i \geq 2$, and this is implied by ∞ -nonresonance of $(e^\mu, e^{D_{x_0}f})$ (to see this, take the logarithm of (1)). Therefore, Theorem 5 and Prop. 11 imply that, for a “typical” vector field $f \in \mathfrak{X}_{\text{fix}}^\infty$, a unique principal eigenfunction exists for every left eigenvector of $D_{x_0}f$, and these eigenfunctions are given by a limiting procedure. Similar remarks for $f \in \mathfrak{X}_{\text{per}}^\infty$ follow from Theorem 5 and Prop. 13 below.

The following is Kvalheim and Revzen (2019, Prop. 7).

Proposition 13. Fix $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ and let $\Phi: Q \times \mathbb{R} \rightarrow Q$ be a \mathcal{C}^k flow with Q the basin of an attracting hyperbolic nonstationary τ -periodic orbit with image $\Gamma \subset Q$, where $n+1 := \dim(Q) \geq 2$. Fix $x_0 \in \Gamma$ and let $E_{x_0}^s$ be the unique $D_{x_0}\Phi^\tau$ -invariant subspace complementary to $T_{x_0}\Gamma$. Assume the spectral radius $\rho(D_{x_0}\Phi^\tau|_{E_{x_0}^s}) < 1$ satisfies

$$|e^{\mu\tau}| > \left(\rho(D_{x_0}\Phi^\tau|_{E_{x_0}^s})\right)^k$$

in all of the following statements.

Uniqueness of Koopman eigenvalues. Let $\psi_1 \in \mathcal{C}^k(Q, \mathbb{C})$ be any Koopman eigenfunction with eigenvalue $\mu \in \mathbb{C}$. Then there exists $m = (m_1, \dots, m_n) \in \mathbb{N}_{\geq 0}^n$ such that

$$e^{\mu\tau} = e^{(m \cdot \lambda)\tau},$$

where $e^{\lambda_1\tau}, \dots, e^{\lambda_n\tau}$ are the eigenvalues of $D_{x_0}\Phi^\tau|_{E_{x_0}^s}$ repeated with multiplicities and $\lambda := (\lambda_1, \dots, \lambda_n)$.

Existence and uniqueness of principal eigenfunctions. Assume that $(e^{\mu\tau}, D_{x_0}\Phi^\tau|_{E_{x_0}^s})$ is k -nonresonant. Let $w: E_{x_0}^s \rightarrow \mathbb{C}$ be a left eigenvector of $D_{x_0}\Phi^\tau|_{E_{x_0}^s}$ with eigenvalue $e^{\mu\tau}$. Then there exists a unique principal eigenfunction $\psi \in \mathcal{C}^k(Q, \mathbb{C})$ for Φ with eigenvalue μ satisfying $D_{x_0}\psi|_{E_{x_0}^s} = w$. Additionally, if μ and w are real, then $\psi: Q \rightarrow \mathbb{R} \subset \mathbb{C}$ is real.

Remark 14. The uniqueness statements of Prop. 11 and 13 are fairly sharp; see Kvalheim and Revzen (2019, Ex. 2).

Remark 15. Prop. 11 can be used to guarantee convergence of Laplace averages (Mauroy et al., 2013); see Kvalheim and Revzen (2019, Rem. 14). Kvalheim and Revzen (2019, Rem. 15) relates Prop. 11 and 13 to the literature on isostables and isostable coordinates. Kvalheim and Revzen (2019, Rem. 16) relates Prop. 11 to the principal eigenfunctions of Mohr and Mezić (2016).

3.3 Classification of all \mathcal{C}^∞ Koopman eigenfunctions

To improve the readability of Theorems 16 and 18 below, we introduce the following multi-index notation. We define an n -dimensional multi-index to be an n -tuple $i = (i_1, \dots, i_n) \in \mathbb{N}_{\geq 0}^n$ of nonnegative integers, and define its sum to be $|i| := i_1 + \dots + i_n$. For a multi-index $i \in \mathbb{N}_{\geq 0}^n$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we define $z^{[i]} := z_1^{i_1} \dots z_n^{i_n}$. Given a \mathbb{C}^n -valued function $\psi = (\psi_1, \dots, \psi_n): Q \rightarrow \mathbb{C}^n$, we define $\psi^{[i]}: Q \rightarrow \mathbb{C}$ via $\psi^{[i]}(x) := (\psi(x))^{[i]}$ for all $x \in Q$. We also define the complex conjugate of $\psi = (\psi_1, \dots, \psi_n)$ element-wise: $\bar{\psi} := (\bar{\psi}_1, \dots, \bar{\psi}_n)$. The following result follows from Kvalheim and Revzen (2019, Rem. 3, Ex. 1, Prop. 2, Thm 3).

Theorem 16. (Classification for a point attractor). Let f be a \mathcal{C}^∞ vector field on Q with Q the basin of an attracting hyperbolic equilibrium $x_0 \in Q$ for the flow of f , where $n := \dim(Q) \geq 1$. Assume that $D_{x_0}f$ is diagonalizable over \mathbb{C} with eigenvalues $\lambda := (\lambda_1, \dots, \lambda_n)$ repeated with multiplicities and that $\lambda_j \neq m \cdot \lambda$ for all j and $m \in \mathbb{N}_{\geq 0}$ with $|m| \geq 2$. It follows that there exists an n -tuple $\psi = (\psi_1, \dots, \psi_n)$ of \mathcal{C}^∞ principal eigenfunctions such that $\psi: Q \rightarrow \psi(Q) \subset \mathbb{C}^n$ is a diffeomorphism onto an \mathbb{R} -linear subspace of \mathbb{C}^n , and every \mathcal{C}^∞ Koopman eigenfunction φ is a finite linear combination of products of the ψ_i and their complex conjugates $\bar{\psi}_i$:

$$\varphi = \sum_{|\ell|+|m| \leq k} c_{\ell,m} \psi^{[\ell]} \bar{\psi}^{[m]} \quad (6)$$

for some $k \in \mathbb{N}_{\geq 0}$ and some coefficients $c_{\ell,m} \in \mathbb{C}$.

Remark 17. Theorem 16 goes beyond Prop. 11 by completely classifying all \mathcal{C}^∞ eigenfunctions rather than just the principal ones. On the other hand, Theorem 16 requires the stronger hypothesis that $D_{x_0}\Phi^1$ is diagonalizable over \mathbb{C} . However, Theorem 5 still implies that a “typical” vector field in $\mathfrak{X}_{\text{fix}}^\infty$ satisfies the hypotheses of Theorem 16 (because ∞ -nonresonance of $(e^{D_0 f}, e^{D_0 f})$ implies the condition involving “ $\lambda_j \neq m \cdot \lambda$ ”; cf. Rem. 12). Similarly, Theorem 18 below yields a complete classification of all \mathcal{C}^∞ eigenfunctions for “typical” vector fields in $\mathfrak{X}_{\text{per}}^\infty$.

Consider a \mathcal{C}^∞ flow with Q the basin of an attracting hyperbolic nonstationary τ -periodic orbit. As discussed in Kvalheim and Revzen (2019), there exists a \mathcal{C}^∞ Koopman eigenfunction ψ_θ with eigenvalue $\mu = i \frac{2\pi}{\tau}$, where $i = \sqrt{-1}$, and this eigenfunction is unique modulo scalar multiplication (cf. Mauroy and Mezić (2012)). The following result, which follows from Kvalheim and Revzen (2019, Prop. 3, Thm 4), involves ψ_θ .

Theorem 18. (Classification for a limit cycle attractor). Let $\Phi: Q \times \mathbb{R} \rightarrow Q$ be the flow of a \mathcal{C}^∞ vector field with Q the basin of an attracting hyperbolic nonstationary τ -periodic orbit with image $\Gamma \subset Q$, where $n+1 := \dim(Q) \geq 2$. Fix $x_0 \in \Gamma$ and denote by $E_{x_0}^s$ the unique τ -invariant subspace complementary to $T_{x_0}\Gamma$. Assume that $D_{x_0}\Phi^\tau$ is diagonalizable over \mathbb{C} and that $(D_{x_0}\Phi^\tau|_{E_{x_0}^s}, D_{x_0}\Phi^\tau|_{E_{x_0}^s})$ is ∞ -nonresonant. It follows that there exists an n -tuple $\psi = (\psi_1, \dots, \psi_n)$ of \mathcal{C}^∞ principal eigenfunctions such that $(\psi, \psi_\theta): Q \rightarrow \psi(Q) \subset \mathbb{C}^{n+1}$ is a diffeomorphism onto a \mathcal{C}^∞ properly embedded submanifold of \mathbb{C}^{n+1} , and every \mathcal{C}^∞ Koopman eigenfunction φ is a finite linear combination of products of ψ_θ with products of the ψ_i and conjugates $\bar{\psi}_i$:

$$\varphi = \sum_{|\ell|+|m| \leq k} c_{\ell,m} \psi^{[\ell]} \bar{\psi}^{[m]} \psi_\theta^{j_{\ell,m}} \quad (7)$$

for some $k \in \mathbb{N}_{\geq 0}$, coefficients $c_{\ell,m} \in \mathbb{C}$, and $j_{\ell,m} \in \mathbb{Z}$.

ACKNOWLEDGEMENTS

We thank George Haller, Hoon Hong, Alexandre Mauroy, Igor Mezić, Jeff Moehlis, Ryan Mohr, Corbinian Schlosser, and Dan Wilson for useful discussions related to this work.

REFERENCES

- Blum-Smith, B. and Coskey, S. (2017). The fundamental theorem on symmetric polynomials: History's first whiff of Galois theory. *The College Mathematics Journal*, 48(1), 18–29. doi:10.4169/college.math.j.48.1.18.
- Budišić, M., Mohr, R., and Mezić, I. (2012). Applied Koopmanism. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 22(4), 047510.
- Eldering, J., Kvalheim, M., and Revzen, S. (2018). Global linearization and fiber bundle structure of invariant manifolds. *Nonlinearity*, 31(9), 4202–4245.
- Hadamard, J. (1902). Sur les problèmes aux dérivées partielles et leur signification physique. *Princeton University Bulletin*, 49–52.
- Hirsch, M.W. (1994). *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York. Corrected reprint of the 1976 original.

- Khalil, H.K. (2002). *Nonlinear systems*. Prentice Hall.
- Koopman, B.O. (1931). Hamiltonian systems and transformation in Hilbert space. *Proceedings of the National Academy of Sciences of the United States of America*, 17(5), 315.
- Kvalheim, M.D. and Revzen, S. (2019). Existence and uniqueness of global Koopman eigenfunctions for stable fixed points and periodic orbits. *arXiv preprint arXiv:1911.11996*.
- Lan, Y. and Mezić, I. (2013). Linearization in the large of nonlinear systems and Koopman operator spectrum. *Phys. D*, 242, 42–53. doi:10.1016/j.physd.2012.08.017. URL <http://dx.doi.org/10.1016/j.physd.2012.08.017>.
- Lee, J.M. (2013). *Introduction to smooth manifolds*. Springer, 2 edition. doi:10.1007/978-1-4419-9982-5.
- Mauroy, A. and Mezić, I. (2012). On the use of Fourier averages to compute the global isochrons of (quasi) periodic dynamics. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 22(3), 033112.
- Mauroy, A. and Mezić, I. (2016). Global stability analysis using the eigenfunctions of the Koopman operator. *IEEE Transactions on Automatic Control*, 61(11), 3356–3369.
- Mauroy, A., Mezić, I., and Moehlis, J. (2013). Isostables, isochrons, and Koopman spectrum for the action–angle representation of stable fixed point dynamics. *Physica D: Nonlinear Phenomena*, 261, 19–30.
- Mezić, I. (1994). *On the geometrical and statistical properties of dynamical systems: theory and applications*. Ph.D. thesis, California Institute of Technology.
- Mezić, I. (2005). Spectral properties of dynamical systems, model reduction and decompositions. *Nonlinear Dynamics*, 41(1-3), 309–325.
- Mezić, I. (2019). Spectrum of the Koopman operator, spectral expansions in functional spaces, and state space geometry. *arXiv preprint arXiv:1702.07597*.
- Mezić, I. and Banaszuk, A. (2004). Comparison of systems with complex behavior. *Physica D: Nonlinear Phenomena*, 197(1-2), 101–133.
- Mityagin, B. (2015). The zero set of a real analytic function. *arXiv preprint arXiv:1512.07276*.
- Mohr, R. and Mezić, I. (2016). Koopman principle eigenfunctions and linearization of diffeomorphisms. *arXiv preprint arXiv:1611.01209*.
- Monga, B., Wilson, D., Matchen, T., and Moehlis, J. (2019). Phase reduction and phase-based optimal control for biological systems: a tutorial. *Biological Cybernetics*, 113(1-2), 11–46.
- Ott, W. and Yorke, J. (2005). Prevalence. *Bulletin of the American Mathematical Society*, 42(3), 263–290.
- Palis, J. and De Melo, W. (1982). *Geometric theory of dynamical systems: an introduction*. Springer Science & Business Media.
- Sell, G.R. (1985). Smooth linearization near a fixed point. *American Journal of Mathematics*, 1035–1091.
- Shirasaka, S., Kurebayashi, W., and Nakao, H. (2017). Phase-amplitude reduction of transient dynamics far from attractors for limit-cycling systems. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 27(2), 023119. doi:10.1063/1.4977195.
- Smith, H.L. and Waltman, P. (1999). Perturbation of a globally stable steady state. *Proceedings of the American Mathematical Society*, 127(2), 447–453.
- Sternberg, S. (1957). Local contractions and a theorem of Poincaré. *American Journal of Mathematics*, 79(4), 809–824.
- Wilson, D. and Ermentrout, B. (2018). Greater accuracy and broadened applicability of phase reduction using isostable coordinates. *Journal of Mathematical Biology*, 76(1-2), 37–66.
- Wilson, D. and Moehlis, J. (2014). An energy-optimal approach for entrainment of uncertain circadian oscillators. *Biophysical Journal*, 107(7), 1744–1755.
- Wilson, D. and Moehlis, J. (2016a). Isostable reduction of periodic orbits. *Physical Review E*, 94(5), 052213.
- Wilson, D. and Moehlis, J. (2016b). Isostable reduction with applications to time-dependent partial differential equations. *Physical Review E*, 94(1), 012211.
- Wilson, Jr., F.W. (1967). The structure of the level surfaces of a Lyapunov function. *J. Differential Equations*, 3, 323–329. doi:10.1016/0022-0396(67)90035-6. URL [http://dx.doi.org/10.1016/0022-0396\(67\)90035-6](http://dx.doi.org/10.1016/0022-0396(67)90035-6).

Appendix A. SOME BACKGROUND ON SYMMETRIC POLYNOMIALS

For the reader's convenience, this appendix reviews some facts about symmetric polynomials relevant to Lemma 2. See, e.g., Blum-Smith and Coskey (2017) for additional background. The first fact relates a polynomial's coefficients to certain symmetric polynomials of its roots.

Theorem 19. (Vieta's theorem). If $f \in \mathbb{R}[x]$ is a degree $n \geq 1$ monic polynomial with roots $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ repeated with multiplicity, then

$$f(x) = x^n - e_1(\alpha_1, \dots, \alpha_n)x^{n-1} + e_2(\alpha_1, \dots, \alpha_n)x^{n-2} - \dots + (-1)^n e_n(\alpha_1, \dots, \alpha_n),$$

where

$$\begin{aligned} e_1(\alpha_1, \dots, \alpha_n) &:= \alpha_1 + \dots + \alpha_n, \\ e_2(\alpha_1, \dots, \alpha_n) &:= \alpha_1\alpha_2 + \alpha_1\alpha_3 + \dots + \alpha_{n-1}\alpha_n, \\ &\vdots \\ e_n(\alpha_1, \dots, \alpha_n) &:= \alpha_1 \cdots \alpha_n, \end{aligned} \tag{A.1}$$

are the n elementary symmetric polynomials in $\alpha_1, \dots, \alpha_n$.

Since the eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ of $A \in \mathbb{R}^{n \times n}$ are the roots of the characteristic polynomial $\lambda \mapsto \det(\lambda I_n - A)$, it follows that its coefficients, which are polynomials in the matrix entries, give the elementary symmetric polynomials in the eigenvalues. Thus, elementary symmetric polynomials in the eigenvalues can be obtained directly as polynomials in the matrix entries *without* computing the eigenvalues. The fundamental theorem of symmetric polynomials (FTSP) extends this conclusion to *all symmetric polynomials* in the eigenvalues, i.e., to $f \in \mathbb{R}[\lambda_1, \dots, \lambda_n]$ for which

$$\forall \sigma \in S_n : f(\lambda_1, \dots, \lambda_n) \equiv f(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)}),$$

where S_n is the group of permutations σ of $\{1, \dots, n\}$.

Theorem 20. (FTSP). If $f \in \mathbb{R}[\alpha_1, \dots, \alpha_n]$ is symmetric, there exists a (unique) polynomial $f_e \in \mathbb{R}[e_1, \dots, e_n]$ with

$$f(\alpha_1, \dots, \alpha_n) \equiv f_e(e_1(\alpha_1, \dots, \alpha_n), \dots, e_n(\alpha_1, \dots, \alpha_n)).$$

To summarize, any symmetric polynomial in the eigenvalues is expressible as a polynomial in the matrix entries.