A Riesz-Thorin type interpolation theorem in Euclidean Jordan algebras

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Abstract

In a Euclidean Jordan algebra $\mathcal{V}$ of rank n which carries the trace inner product, to each element a we associate the eigenvalue vector $\lambda(a)$ in $\mathcal{R}^n$ whose components are the eigenvalues of a written in the decreasing order. For any $p \in [1,\infty]$, we define the spectral p-norm of a to be the p-norm of $\lambda(a)$ in $\mathcal{R}^n$. In a recent paper, based on the K-method of real interpolation theory and a majorization technique, we described an interpolation theorem for a linear transformation on $\mathcal{V}$ relative to the same spectral norm. In this paper, using standard complex function theory methods, we describe a Riesz-Thorin type interpolation theorem relative to two different spectral norms. We illustrate the result by estimating the norms of certain special linear transformations such as Lyapunov transformations, quadratic representations, and positive transformations.

Key Words: Euclidean Jordan algebra, Riesz-Thorin type interpolation theorem

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1 Introduction

Consider a Euclidean Jordan algebra $\mathcal{V}$ of rank n which carries the trace inner product. To each element a in $\mathcal{V}$, we associate the eigenvalue vector $\lambda(a)$ whose components are the eigenvalues of a written in the decreasing order. For any $p \in [1, \infty]$, we define the spectral p-norm on $\mathcal{V}$ by

$$||a||_p := ||\lambda(a)||_p,$$

where the right-hand side is the usual p-norm of the vector $\lambda(a)$ in $\mathcal{R}^n$. Given $r, s \in [1, \infty]$ and a linear transformation $T : \mathcal{V} \to \mathcal{V}$, we let

$$||T||_{r \to s} := \sup_{a \neq 0} \frac{||T(a)||_s}{||a||_r}.$$

In [5], based on the K-method of real interpolation theory [8], the following result was proved.

Theorem 1.1 *Suppose* $1 \leq r, s, p \leq \infty$, $0 \leq \theta \leq 1$, *and*

$$\frac{1}{p} = \frac{1-\theta}{r} + \frac{\theta}{s}. \tag{1}$$

Then, for any linear transformation $T : \mathcal{V} \to \mathcal{V}$,

$$||T||_{p \to p} \leq ||T||_{r \to r}^{1-\theta} \; ||T||_{s \to s}^{\theta}. \tag{2}$$

In particular,

$$||T||_{p \to p} \leq ||T||_{\infty \to \infty}^{1-\frac{1}{p}} \; ||T||_{1 \to 1}^{\frac{1}{p}}. \tag{3}$$

A key idea in the proof of the above result is the use of a majorization result that connects a K-functional defined on $\mathcal{V}$ with a K-functional on an L_p-space. In [5], the issue of proving an inequality of the type (2) that deals with the norm of T relative to two spectral norms (such as $||T||_{r \to s}$) was raised. In the present paper, based on standard complex function theory methods (especially, Hadamard's three lines theorem) we prove the following Riesz-Thorin type interpolation result.

Theorem 1.2 *Let* $r_0, r_1, s_0, s_1 \in [1, \infty]$ *and* $\theta \in [0, 1]$. *Consider* r_θ *and* s_θ *in* $[1, \infty]$ *defined by*

$$\frac{1}{r_\theta} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1} \quad \text{and} \quad \frac{1}{s_\theta} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1}.$$

Then, for any linear transformation T *on* $\mathcal{V}$,

$$||T||_{r_\theta \to s_\theta} \leq C \, ||T||_{r_0 \to s_0}^{1-\theta} \, ||T||_{r_1 \to s_1}^{\theta}, \tag{4}$$

where C *is a constant,* $1 \leq C \leq 4$, *that depends only on* r_0, r_1, s_0, s_1.

Illustrating this result, we estimate the norms of some special linear transformations on $\mathcal{V}$ such as Lyapunov transformations, quadratic representations, and positive transformations.

2 Preliminaries

Throughout this paper $(\mathcal{V}, \circ, \langle \cdot, \cdot \rangle)$ denotes a Euclidean Jordan algebra of rank n with unit element e [3], [7]. We let letters $a, b, c, d,$ and v denote elements of $\mathcal{V}$, x and y denote elements of $\mathcal{R}^n$, and write z for a complex variable. For $a, b \in \mathcal{V}$, we denote their Jordan product and inner product by $a \circ b$ and $\langle a, b \rangle$, respectively. It is known that any Euclidean Jordan algebra is a direct product/sum of simple Euclidean Jordan algebras and every simple Euclidean Jordan algebra is isomorphic to one of five algebras, three of which are the algebras of $n \times n$ real/complex/quaternion Hermitian matrices. The other two are: the algebra of 3×3 octonion Hermitian matrices and the Jordan spin algebra.

According to the *spectral decomposition theorem* [3], any element $a \in \mathcal{V}$ has a decomposition

$$a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n,$$

where the real numbers $a_1, a_2, \ldots, a_n$ are (called) the eigenvalues of a and $\{e_1, e_2, \ldots, e_n\}$ is a Jordan frame in $\mathcal{V}$. (An element may have decompositions coming from different Jordan frames, but the eigenvalues remain the same.) Then, $\lambda(a)$– called the *eigenvalue vector* of a– is the vector of eigenvalues of a written in the decreasing order. The *trace and spectral* p*-norm* of a are defined by

$$tr(a) := a_1 + a_2 + \cdots + a_n \quad \text{and} \quad ||a||_p := ||\lambda(a)||_p,$$

where $||\lambda(a)||_p$ denotes the usual p-norm of a vector in $\mathcal{R}^n$. An element a is said to be *invertible* if all its eigenvalues are nonzero. We note that the set of invertible elements is dense in $\mathcal{V}$. *Throughout this paper, we assume that the inner product is the trace inner product, that is,* $\langle a, b \rangle = tr(a \circ b)$.

Given a spectral decomposition $a = \sum_{j=1}^{n} a_j e_j$ and a real number $\gamma > 0$, we write

$$|a| := \sum_{j=1}^{n} |a_j| e_j, \;\; |a|^\gamma := \sum_{j=1}^{n} |a_j|^\gamma e_j \quad \text{and} \quad ||a||_1 = \sum_{j=1}^{n} |a_j| = tr(|a|). \tag{5}$$

In what follows, we say that q is the conjugate of $p \in [1, \infty]$ if $\frac{1}{p} + \frac{1}{q} = 1$ and denote the conjugate of $r \in [1, \infty]$ by r'. Also, we use the standard convention that $1/\infty = 0$.

Based on the Fan-Theobald-von Neumann type inequality [2]

$$\langle a, b \rangle \leq \langle \lambda(a), \lambda(b) \rangle \quad (a, b \in \mathcal{V})$$

and majorization techniques, the following result was proved in [5].

Theorem 2.1 *Let $p \in [1, \infty]$ with conjugate q. Then the following statements hold in $\mathcal{V}$:*

(*i*) $|\langle a, b \rangle| \leq ||a \circ b||_1 \leq ||a||_p \, ||b||_q$.

(ii) $\sup_{b \neq 0} \frac{|\langle a,b\rangle|}{||b||_q} = ||a||_p$.

3 The proof of the interpolation theorem

The Riesz-Thorin interpolation theorem, stated in the setting of L_p-spaces, is well-known in classical analysis. There is also a Riesz-Thorin type result available for linear transformations on the space of complex $n \times n$ matrices with respect to Schatten p-norms, see the interpolation theorem of Calderón-Lions ([9], Theorem IX.20). Our Theorem 1.2 is stated in the setting of Euclidean Jordan algebras relative to spectral norms. In the absence of an isomorphism type argument that immediately gives our result, we offer a proof that mimics the classical proof based on the Hadamard's three lines theorem of complex function theory ([4], Theorem 6.27). In the proof given below, we complexify the real inner product space $\mathcal{V}$ and define norms on this complexification in such a way to have a Hölder type inequality. This procedure results in a constant C in the Riesz-Thorin type inequality (4) that is different from 1. Possibly, a different argument may show that this constant can be replaced by 1.

Recall that a and b denote elements of $\mathcal{V}$ and z denotes a complex variable. For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathcal{R}^n$, we write $x + iy = (x_1 + iy_1, x_2 + iy_2, \ldots, x_n + iy_n) \in \mathbb{C}^n$. Let T be a linear transformation on $\mathcal{V}$. We consider complexifications of $\mathcal{V}$ and T:

$$\widetilde{\mathcal{V}} := \mathcal{V} + i\mathcal{V} \quad \text{and} \quad \widetilde{T}(a+ib) := T(a) + i\,T(b) \quad (a, b \in \mathcal{V}).$$

We define the inner product and spectral p-norm on $\widetilde{\mathcal{V}}$ as follows. For $a, b, c, d \in \mathcal{V}$,

$$\langle a+ib, c+id\rangle := \Big[\langle a,c\rangle + \langle b,d\rangle\Big] + i\Big[\langle b,c\rangle - \langle a,d\rangle\Big] \quad \text{and} \quad ||a+ib||_p := ||a||_p + ||b||_p.$$

It is easily seen that $\widetilde{\mathcal{V}}$ is a complex inner product space, $\widetilde{T}$ is a (complex) linear transformation on $\widetilde{\mathcal{V}}$. We state the following simple lemma.

Lemma 3.1 Consider $\widetilde{\mathcal{V}}$ and $\widetilde{T}$ as above. Let $p \in [1, \infty]$ with conjugate q, and $r, s \in [1, \infty]$. Then,

(i) $|\langle a+ib, c+id\rangle| \leq ||a+ib||_p\, ||c+id||_q$ for all $a, b, c, d \in \mathcal{V}$, and

(ii) $||\widetilde{T}||_{r\to s} = ||T||_{r\to s}$.

Proof. (i) By the definition of inner product in $\widetilde{\mathcal{V}}$ and Theorem 2.1,

$$|\langle a+ib, c+id\rangle| \leq |\langle a,c\rangle| + |\langle a,d\rangle| + |\langle b,c\rangle| + |\langle b,d\rangle| \leq ||a||_p\,||c||_q + ||a||_p\,||d||_q + ||b||_p\,||c||_q + ||b||_p\,||d||_q.$$

Since the right-hand side is $||a+ib||_p\,||c+id||_q$, the stated inequality follows.
(ii) For $a, b \in \mathcal{V}$,

$$||\widetilde{T}(a+ib)||_s = ||T(a) + i\,T(b)||_s = ||T(a)||_s + ||T(b)||_s \leq ||T||_{r\to s}(||a||_r + ||b||_r) = ||T||_{r\to s}\,||a+ib||_r.$$

This implies that $||\widetilde{T}||_{r\to s} \le ||T||_{r\to s}$. The reverse inequality holds as $\widetilde{T}$ is an extension of T to $\widetilde{\mathcal{V}}$. Hence we have (ii). $\square$

We now come to the proof of Theorem 1.2. In what follows, *for any* $p \in [1,\infty]$ *with conjugate* q, *we let*

$$C_p = \begin{cases} \sqrt{2} & \text{if } 1 \le p \le 2, \\ 2^{\frac{1}{q}} & \text{if } 2 \le p \le \infty\,. \end{cases}$$

Proof. Let the assumptions of the theorem be in place. Recalling that s' denotes the conjugate of (any) $s \in [1,\infty]$, we define

$$C := \max\{C_{r_0}C_{s_0'}, C_{r_1}C_{s_1'}\} \tag{6}$$

which is a number between 1 and 4, and depends only on r_0, r_1, s_0, s_1. We show that (4) holds for this C. Since (4) clearly holds when $\theta = 0$ or $\theta = 1$, from now on, we assume that $0 < \theta < 1$. Let

$$\alpha_j := \frac{1}{r_j},\ \beta_j := \frac{1}{s_j},\ \text{and } M_j := ||T||_{r_j \to s_j} \quad (j = 0,1),$$

$$\alpha := \frac{1}{r_\theta},\ \beta := \frac{1}{s_\theta},\ \text{and } M_\theta := ||T||_{r_\theta \to s_\theta},$$

and for a complex variable z,

$$\alpha(z) := (1-z)\alpha_0 + z\alpha_1 \quad \text{and} \quad \beta(z) := (1-z)\beta_0 + z\beta_1.$$

We show that

$$M_\theta \le C\, M_0^{1-\theta}\, M_1^{\theta}. \tag{7}$$

Now, using Theorem 2.1, Item (ii),

$$M_\theta = ||T||_{r_\theta \to s_\theta} = \sup_{0 \ne a \in \mathcal{V}} \frac{||T(a)||_{s_\theta}}{||a||_{r_\theta}} = \sup_{0 \ne a,b \in \mathcal{V}} \frac{|\langle T(a), b\rangle|}{||a||_{r_\theta}||b||_{s_\theta'}} = \sup_{||a||_{r_\theta} = 1 = ||b||_{s_\theta'}} |\langle Ta, b\rangle|.$$

To prove (7), it is enough to show that for any a and b in $\mathcal{V}$ with $||a||_{r_\theta} = 1 = ||b||_{s_\theta'}$,

$$|\langle Ta, b\rangle| \le C\, M_0^{1-\theta}\, M_1^{\theta}. \tag{8}$$

By continuity, it is enough to prove this for a and b invertible (that is, with all their eigenvalues nonzero). We fix such a and b and write their spectral decompositions:

$$a = \sum_{j=1}^{n} |a_j|\varepsilon_j e_j \quad \text{and} \quad b = \sum_{j=1}^{n} |b_j|\delta_j f_j,$$

where $\{e_1, e_2, \ldots, e_n\}$ and $\{f_1, f_2, \ldots, f_n\}$ are Jordan frames, $\varepsilon_j, \delta_j \in \{-1, 1\}$ for all j, and a_js are the eigenvalue of a, etc. Now, with the observation that $0 < \alpha, \beta < 1$, we define two elements in $\widetilde{\mathcal{V}}$:

$$a_z := \sum_{j=1}^{n} |a_j|^{\frac{\alpha(z)}{\alpha}} \varepsilon_j e_j \quad \text{and} \quad b_z := \sum_{j=1}^{n} |b_j|^{\frac{1-\beta(z)}{1-\beta}} \delta_j f_j,$$

where we consider only the principal values while defining the exponentials. Then the function

$$\phi(z) := \langle \widetilde{T}(a_z), b_z \rangle$$

is continuous on the strip $\{z : 0 \leq Re(z) \leq 1\}$ and analytic in its interior.
We estimate $|\phi(z)|$ on the lines $Re(z) = 0$ and $Re(z) = 1$ and then apply Hadamard's three lines theorem ([4], Theorem 6.27). First, suppose $Re(z) = 0$. Let

$$|a_j|^{\frac{\alpha(z)}{\alpha}} = x_j + i\, y_j,\ x := (x_1, x_2, \ldots, x_n) \in \mathcal{R}^n, \text{ and } y := (y_1, y_2, \ldots, y_n) \in \mathcal{R}^n.$$

Then, $|x_j + iy_j| = \left| |a_j|^{\frac{\alpha(z)}{\alpha}} \right| = |a_j|^{\frac{\alpha_0}{\alpha}}$. When $r_0 = \infty$, that is, when $\alpha_0 = 0$, $|x_j + iy_j| = 1$ for all j and hence (in $\mathbb{C}^n$), $||x + iy||_{r_0} = 1$. When, $r_0 < \infty$, $|x_j + iy_j|^{r_0} = |a_j|^{r_\theta}$. So, because $||a||_{r_\theta} = 1$, we have $||x + iy||_{r_0}^{r_0} = \sum_{j=1}^{n} |x_j + iy_j|^{r_0} = \sum_{j=1}^{n} |a_j|^{r_\theta} = 1$. Thus, in both cases,

$$||x + iy||_{r_0} = 1. \tag{9}$$

Now, $a_z = \sum_{j=1}^{n} (x_j + iy_j) \varepsilon_j e_j = (\sum_{j=1}^{n} x_j \varepsilon_j e_j) + i(\sum_{j=1}^{n} y_j \varepsilon_j e_j)$ and so,

$$||a_z||_{r_0} = ||\sum_{j=1}^{n} x_j \varepsilon_j e_j||_{r_0} + ||\sum_{j=1}^{n} y_j \varepsilon_j e_j||_{r_0} = ||x||_{r_0} + ||y||_{r_0}.$$

In view of (9), from Proposition 4.1 in the Appendix, we have,

$$||a_z||_{r_0} \leq C_{r_0}.$$

Similarly, $||b_z||_{s_0'} \leq C_{s_0'}$. Hence, when $Re(z) = 0$, Lemma 3.1 gives

$$|\phi(z)| \leq ||\widetilde{T}(a_z)||_{s_0}\, ||b_z||_{s_0'} \leq ||\widetilde{T}||_{r_0 \to s_0}\, ||a_z||_{r_0} ||b_z||_{s_0'} \leq ||T||_{r_0 \to s_0}\, C_{r_0}\, C_{s_0'} = C_{r_0} C_{s_0'}\, M_0.$$

A similar computation shows that

$$Re(z) = 1 \Rightarrow |\phi(z)| \leq C_{r_1} C_{s_1'} M_1.$$

By Hadamard's three lines theorem,

$$|\phi(\theta)| \leq \left(C_{r_0} C_{s_0'}\, M_0 \right)^{1-\theta} \left(C_{r_1} C_{s_1'}\, M_1 \right)^{\theta}.$$

We recall that $C = \max\{C_{r_0} C_{s_0'}, C_{r_1} C_{s_1'}\}$. Now, $a_\theta = a$ and $b_\theta = b$, and so, $\phi(\theta) = \langle T(a), b \rangle$. Hence,

$$|\langle T(a), b \rangle| \leq C\, M_0^{1-\theta}\, M_1^{\theta}.$$

This gives (8) and the proof is complete. $\square$

Remarks. Instead of the constant C defined in (6), one may consider a slightly better constant, namely, $\max\{(C_{r_0}C_{s_0'})^{1-\theta},(C_{r_1}C_{s_1'})^{\theta}\}$. However, this constant depends on θ.

We now consider the problem of estimating the norms of certain special linear transformations on $\mathcal{V}$ relative to spectral norms. First, we make two observations. Writing T^* for the adjoint of a linear transformation T on $\mathcal{V}$, we note, thanks to Theorem 2.1, that

$$||T^*||_{r\to s}=||T||_{s'\to r'},$$

where r' denotes the conjugate of r, etc. Also, knowing the norms $||T||_{1\to 1}$, $||T||_{\infty\to\infty}$, $||T||_{1\to p}$, and $||T||_{p\to 1}$, etc., one can estimate $||T||_{r\to s}$ for various r and s. When $r=s$, (3) gives such an estimate. In the result below, we consider the case $r\neq s$.

Corollary 3.2 *Let $1\leq r\neq s\leq\infty$. Then, for any linear transformation $T:\mathcal{V}\to\mathcal{V}$,*

$$||T||_{r\to s}\leq\begin{cases}2\sqrt{2}\,||T||_{\infty\to\infty}^{1-\frac{1}{r}}\,||T||_{1\to\frac{s}{r}}^{\frac{1}{r}} & \text{if } r<s,\\ 2\sqrt{2}\,||T||_{\infty\to\infty}^{1-\frac{1}{s}}\,||T||_{\frac{r}{s}\to 1}^{\frac{1}{s}} & \text{if } r>s\,.\end{cases}$$

Proof. The stated inequalities are obtained by specializing Theorem 1.2. When $r<s$, we let

$$r_0=\infty,\ s_0=\infty,\ r_1=1,\ s_1=\frac{s}{r},\ r_\theta=r,\ s_\theta=s,\text{ and }\theta=\frac{1}{r}.$$

In this case, $C=\max\{C_{r_0}C_{s_0'},C_{r_1}C_{s_1'}\}=2\sqrt{2}$. When $r>s$, we let

$$r_0=\infty,\ s_0=\infty,\ r_1=\frac{r}{s},\ s_1=1,\ r_\theta=r,\ s_\theta=s,\text{ and }\theta=\frac{1}{s}.$$

In this case also, $C=2\sqrt{2}$. $\square$

Remarks. In the result above, by considering $\max\{(C_{r_0}C_{s_0'})^{1-\theta},(C_{r_1}C_{s_1'})^{\theta}\}$, one can replace the constant $2\sqrt{2}$ by the following:

$$(2\sqrt{2})^{\max\{1-\frac{1}{r},\frac{1}{r}\}}\text{ when } r<s\text{ and }(2\sqrt{2})^{\max\{1-\frac{1}{s},\frac{1}{s}\}}\text{ when } r>s.$$

We now illustrate our results via some examples. For any $a\in\mathcal{V}$, consider the *Lyapunov transformation* L_a and the *quadratic representation* P_a defined by

$$L_a(v):=a\circ v\quad\text{and}\quad P_a(v):=2a\circ(a\circ v)-a^2\circ v\quad(v\in\mathcal{V}).$$

These self-adjoint linear transformations appear prominently in the study of Euclidean Jordan algebras. The norms of these transformations relative to some spectral norms have been described in [5]. For $r,s\in[1,\infty]$, we have (see [5])

$$||a||_\infty\leq||L_a||_{r\to s}\quad\text{and}\quad||a^2||_\infty=||a||_\infty^2\leq||P_a||_{r\to s}.$$

Additionally, for any $p\in[1,\infty]$ with conjugate q,

- $||L_a||_{p\to p} = ||L_a||_{p\to\infty} = ||L_a||_{1\to q} = ||a||_\infty$ and $||L_a||_{p\to 1} = ||L_a||_{\infty\to q} = ||a||_q$,
- $||P_a||_{p\to p} = ||P_a||_{p\to\infty} = ||P_a||_{1\to q} = ||a||_\infty^2$ and $||P_a||_{p\to 1} = ||P_a||_{\infty\to q} = ||a^2||_q$.

We now come to the estimation of $||L_a||_{r\to s}$ and $||P_a||_{r\to s}$ for $r \neq s$. First suppose $1 \le r < s \le \infty$. Then, using the above properties and the fact that for any $x \in \mathcal{R}^n$, $||x||_p$ is a decreasing function of p over $[1,\infty]$, we have

$$||a||_\infty \le ||L_a||_{r\to s} = \sup_{0\neq v\in\mathcal{V}} \frac{||L_a(v)||_s}{||v||_r} \le \sup_{0\neq v\in\mathcal{V}} \frac{||L_a(v)||_r}{||v||_r} = ||L_a||_{r\to r} = ||a||_\infty.$$

Thus,

$$||L_a||_{r\to s} = ||a||_\infty \;\; (1 \le r < s \le \infty).$$

A similar argument shows that

$$||P_a||_{r\to s} = ||a||_\infty^2 \;\; (1 \le r < s \le \infty).$$

When $1 \le s < r \le \infty$, Corollary 3.2 yields the following estimate:

$$||L_a||_{r\to s} \le 2\sqrt{2}\, ||a||_{(\frac{r}{s})'}.$$

For the same s and r, we can get a different estimate

$$||L_a||_{r\to s} \le 2\, C_q\, ||a||_p, \tag{10}$$

where $\frac{1}{p} = \frac{1}{s} - \frac{1}{r}$ (so that $p = s(\frac{r}{s})'$) and q is the conjugate of p. To see this, we apply Theorem 1.2 with

$$r_0 = \infty,\; s_0 = p,\; r_1 = q,\; s_1 = 1,\; r_\theta = r,\; s_\theta = s, \text{ and } \theta = \frac{q}{r}.$$

Then,

$$||L_a||_{r\to s} \le C\, ||L_a||_{\infty\to p}^{1-\theta}\, ||L_a||_{q\to 1}^{\theta} = C\, ||a||_p,$$

where $C = \max\{C_{r_0} C_{s_0'},\, C_{r_1} C_{s_1'}\} = 2\, C_q$. To see an interesting consequence of (10), let $1 \le r, s, p \le \infty$ with $r \neq s$ and $\frac{1}{s} = \frac{1}{p} + \frac{1}{r}$. Then, using the inequality $||a \circ b||_s \le ||L_a||_{r\to s}\, ||b||_r$, the estimate (10) leads to

$$||a \circ b||_s \le 2\, C_q\, ||a||_p\, ||b||_r \quad (a, b \in \mathcal{V}),$$

which can be regarded as a *generalized Hölder type inequality.* We remark that the special case $s = 1$ was already covered in Theorem 2.1 with 1 in place of $2C_q$. It is very likely that the inequality $||a \circ b||_s \le ||a||_p\, ||b||_r$ holds in the general case as well.

Analogous to the above norm estimates of L_a, we can estimate $||P_a||_{r\to s}$ when $r > s$ (with p and q defined above):

$$||P_a||_{r\to s} \le 2\sqrt{2}\, ||a^2||_{(\frac{r}{s})'} \quad \text{and} \quad ||P_a||_{r\to s} \le 2\, C_q\, ||a^2||_p.$$

We now consider a *positive linear transformation* P on $\mathcal{V}$, which is a linear transformation on $\mathcal{V}$ satisfying the condition

$$a \geq 0 \Rightarrow P(a) \geq 0,$$

where $a \geq 0$ means that a belongs to the symmetric cone of $\mathcal{V}$ (or, equivalently, it is the square of some element of $\mathcal{V}$). Examples of such transformations include:

- Any nonnegative matrix on the algebra $\mathcal{R}^n$.
- Any quadratic representation P_a on $\mathcal{V}$ [3].
- The transformation P_A defined on $\mathcal{S}^n$ (the algebra of $n \times n$ real symmetric matrices) by $P_A(X) = AXA^T$, where $A \in \mathcal{R}^{n \times n}$.
- The transformation $P = L^{-1}$ on $\mathcal{V}$, where $L : \mathcal{V} \to \mathcal{V}$ is linear, positive stable (which means that all eigenvalues of L have positive real parts) and satisfies the Z-property:

  $$a \geq 0, b \geq 0, \langle a, b \rangle = 0 \Rightarrow \langle L(a), b \rangle \leq 0.$$

  In particular, on the algebra $\mathcal{H}^n$ (of $n \times n$ complex Hermitian matrices), $P = L_A^{-1}$, where A is a complex $n \times n$ positive stable matrix and $L_A(X) := AX + XA^*$.
- Any *doubly stochastic transformation* on $\mathcal{V}$ [6]: It is a positive linear transformation P with $P(e) = e = P^*(e)$.

For any positive linear transformation P on $\mathcal{V}$, and $p \in [1, \infty]$ with conjugate q, we have the following from [5]:

(i) $||P||_{\infty \to p} = ||P(e)||_p$ and $||P||_{p \to 1} = ||P^*(e)||_q$.

(ii) $||P||_{p \to \infty} \leq ||P(e)||_\infty$ and $||P||_{1 \to p} \leq ||P^*(e)||_\infty$.

(iii) $||P||_{p \to p} \leq ||P(e)||_\infty^{1-\frac{1}{p}} \, ||P^*(e)||_\infty^{\frac{1}{p}}$.

So, for a positive P, an application of Corollary 3.2 gives the following inequalities:

(i) $||P||_{r \to s} \leq 2\sqrt{2}||P(e)||_\infty^{1-\frac{1}{r}} \, ||P^*(e)||_\infty^{\frac{1}{r}}$ when $r < s$.

(ii) $||P||_{r \to s} \leq 2\sqrt{2}||P(e)||_\infty^{1-\frac{1}{s}} \, ||P^*(e)||_{(\frac{r}{s})'}^{\frac{1}{s}}$ when $r > s$.

Additionally, when P is also self-adjoint and $r > s$, analogous to (10), one can get the following estimate:

$$||P||_{r \to s} \leq 2\, C_q \, ||P(e)||_p.$$

4 Appendix

Proposition 4.1 *Given $p \in [1,\infty]$ with conjugate q, consider the following real valued functions defined over $\mathcal{R}^n \times \mathcal{R}^n$, $n \geq 2$:*

$$f(x,y) = ||x||_p + ||y||_p \quad \textit{and} \quad g(x,y) = ||x+iy||_p.$$

Then,

$$\max\Big\{f(x,y) : \, g(x,y) = 1\Big\} = C_p, \tag{11}$$

where

$$C_p = \begin{cases} \sqrt{2} & \text{if } 1 \leq p < 2, \\ 2^{\frac{1}{q}} & \text{if } 2 \leq p \leq \infty\,. \end{cases}$$

Proof. By the continuity of f and g, and the compactness of the constraint set, the maximum in (11) is attained.
It is easy to see that the pair $(\overline{x}, \overline{y})$ with $\overline{x} = 2^{-\frac{1}{p}}(1,0,0\ldots,0)$ and $\overline{y} = 2^{-\frac{1}{p}}(0,1,0\ldots,0)$ satisfies the (constraint) equation $g(x,y) = 1$. Hence

$$C_p \geq f(\overline{x}, \overline{y}) = 2^{\frac{1}{q}}. \tag{12}$$

Consider any pair $(x,y) \in \mathcal{R}^n \times \mathcal{R}^n$ with $g(x,y) = 1$. Writing $x = (x_1, x_2, \ldots, x_n)$, etc., by Hölder's inequality, we have

$$||x||_p + ||y||_p \leq 2^{\frac{1}{q}} \Big(||x||_p^p + ||y||_p^p \Big)^{\frac{1}{p}} = 2^{\frac{1}{q}} \Big(\sum_{j=1}^{n} (|x_j|^p + |y_j|^p) \Big)^{\frac{1}{p}}. \tag{13}$$

We consider three cases.
Case 1: $p = \infty$. By (12), $C_\infty \geq 2^{\frac{1}{q}} = 2$ (as $q = 1$). Since $|x_j + iy_j| \leq 1$ for all j (from our constraint), we get $||x||_\infty, ||y||_\infty \leq 1$; hence $C_\infty \leq 2$. We conclude that $C_\infty = 2$.

Case 2: $2 \leq p < \infty$.
In this case, we use the well-known Clarkson inequality for complex numbers z and w (see [1], page 163):

$$2(|z|^p + |w|^p) \leq |z+w|^p + |z-w|^p.$$

Then, for each j, with $z = x_j$ and $w = iy_j$, we have

$$2\big(|x_j|^p + |y_j|^p\big) \leq |x_j + iy_j|^p + |x_j - iy_j|^p.$$

Summing over j and noting $|x_j + iy_j| = |x_j - iy_j|$, we get

$$\sum_{j=1}^{n} \big(|x_j|^p + |y_j|^p\big) \leq \sum_{j=1}^{n} |x_j + iy_j|^p = g(x,y)^p = 1.$$

It follows from (13) that $||x||_p + ||y||_p \le 2^{\frac{1}{q}}$. As this holds for all (x, y) with $g(x, y) = 1$, we have $C_p \le 2^{\frac{1}{q}}$. From (12) we conclude that $C_p = 2^{\frac{1}{q}}$.

Case 3: $1 \le p < 2$.
Let $\delta := n^{-\frac{1}{p}} 2^{-\frac{1}{2}}$. It is easy to see that the pair $(\overline{x}, \overline{y})$ with $\overline{x} = \delta(1, 1, \ldots, 1) = \overline{y}$ satisfy the constraint equation $g(x, y) = 1$. As $f(\overline{x}, \overline{y}) = \sqrt{2}$ we have, $C_p \ge \sqrt{2}$.
Now, as $1 \le p < 2$, we use a refined version of Clarkson inequality presented in [1], Theorem 2.3:

$$2^{p-1}(|z|^p + |w|^p) + (2 - 2^{\frac{p}{2}}) \min\{|z + w|^p, |z - w|^p\} \le |z + w|^p + |z - w|^p.$$

Then, for each j, with $z = x_j$ and $w = iy_j$, we have

$$2^{p-1}\big(|x_j|^p + |y_j|^p\big) + (2 - 2^{\frac{p}{2}}) \min\{|x_j + iy_j|^p, |x_j - iy_j|^p\} \le |x_j + iy_j|^p + |x_j - iy_j|^p.$$

Simplifying this expression and summing over j, we get

$$\sum_{j=1}^{n} \big(|x_j|^p + |y_j|^p\big) \le 2^{1-\frac{p}{2}} \Big(\sum_{j=1}^{n} |x_j + iy_j|^p\Big) = 2^{1-\frac{p}{2}} g(x, y)^p = 2^{1-\frac{p}{2}}.$$

This leads, via (13), to

$$||x||_p + ||y||_p \le 2^{\frac{1}{q}} \, (2^{1-\frac{p}{2}})^{\frac{1}{p}} = \sqrt{2}.$$

Now, taking the maximum of $||x||_p + ||y||_p$ over (x, y), we get $C_p \le \sqrt{2}$. Thus, when $1 \le p < 2$,

$$C_p = \sqrt{2}.$$

This completes our proof. $\square$

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