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THE CONE OF \mathcal{Z} -TRANSFORMATIONS ON THE LORENTZ CONE*

SÁNDOR ZOLTÁN NÉMETH[†] AND MUDDAPPA SEETHARAMA GOWDA[‡]

Abstract. In this paper, the structural properties of the cone of \mathcal{Z} -transformations on the Lorentz cone are described in terms of the semidefinite cone and copositive/completely positive cones induced by the Lorentz cone and its boundary. In particular, its dual is described as a slice of the semidefinite cone as well as a slice of the completely positive cone of the Lorentz cone. This provides an example of an instance where a conic linear program on a completely positive cone is reduced to a problem on the semidefinite cone.

Key words. \mathcal{Z} -transformation, Lorentz cone, Semidefinite cone, Copositive cone, Completely positive cone.

AMS subject classifications. 90C33, 15A48.

1. Introduction. Given a proper cone \mathcal{K} in a finite dimensional real Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, a linear transformation $A : \mathcal{H} \rightarrow \mathcal{H}$ is said to be a \mathcal{Z} -transformation on \mathcal{K} if

$$[x \in \mathcal{K}, y \in \mathcal{K}^*, \text{ and } \langle x, y \rangle = 0] \Rightarrow \langle Ax, y \rangle \leq 0,$$

where \mathcal{K}^* denotes the dual of \mathcal{K} in \mathcal{H} . Such transformations appear in various areas including economics, dynamical systems, optimization, see e.g., [2, 3, 9, 12] and the references therein. When \mathcal{H} is \mathbb{R}^n and \mathcal{K} is the nonnegative orthant, \mathcal{Z} -transformations become \mathcal{Z} -matrices, which are square matrices with nonpositive off-diagonal entries.

The set $\mathcal{Z}(\mathcal{K})$ of all \mathcal{Z} -transformations on \mathcal{K} is a closed convex cone in the space of all (bounded) linear transformations on \mathcal{H} . Given their appearance and importance in various areas, describing/characterizing elements of $\mathcal{Z}(\mathcal{K})$ and its interior, boundary, dual, etc., is of interest. An early result of Schneider and Vidyasagar [16] asserts that A is a \mathcal{Z} -transformation on \mathcal{K} if and only if $e^{-tA}(\mathcal{K}) \subseteq \mathcal{K}$ for all $t \geq 0$; consequently,

$$(1.1) \quad \mathcal{Z}(\mathcal{K}) = \overline{\mathbb{R}I - \pi(\mathcal{K})},$$

where $\pi(\mathcal{K})$ denotes the set of all linear transformations that leave \mathcal{K} invariant, I denotes the identity transformation, and overline denotes the closure. To see another description of $\mathcal{Z}(\mathcal{K})$, let $\text{LL}(\mathcal{K}) := \mathcal{Z}(\mathcal{K}) \cap -\mathcal{Z}(\mathcal{K})$ denote the lineality space of $\mathcal{Z}(\mathcal{K})$, the elements of which are called Lyapunov-like transformations. Then the inclusions

$$\mathbb{R}I - \pi(\mathcal{K}) \subseteq \text{LL}(\mathcal{K}) - \pi(\mathcal{K}) \subseteq \mathcal{Z}(\mathcal{K}) = \overline{\mathbb{R}I - \pi(\mathcal{K})}$$

imply that

$$\mathcal{Z}(\mathcal{K}) = \overline{\text{LL}(\mathcal{K}) - \pi(\mathcal{K})}.$$

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As the cones $\mathcal{Z}(\mathcal{K})$, $\pi(\mathcal{K})$, and $\text{LL}(\mathcal{K})$ are generally difficult to describe for an arbitrary proper cone \mathcal{K} , we consider special cases. When \mathcal{K} is the nonnegative orthant, $\mathcal{Z}(\mathcal{K})$ consists of square matrices with nonpositive off-diagonal entries, $\pi(\mathcal{K})$ consists of nonnegative matrices, and $\text{LL}(\mathcal{K})$ consists of diagonal matrices. In this paper, we focus on the Lorentz cone (also called the ice-cream cone or the second-order cone as it is induced by the 2-norm) in the Euclidean space \mathbb{R}^n , $n > 1$, defined by:

$$(1.2) \quad \mathcal{L} := \{(t, u)^\top : t \in \mathbb{R}, u \in \mathbb{R}^{n-1}, t \geq \|u\|\}.$$

This, being an example of a symmetric cone, appears prominently in conic optimization [1]. For this cone, Stern and Wolkowicz [17] have shown that $A \in \mathcal{Z}(\mathcal{L})$ if and only if for some real number γ , the matrix $\gamma J - (JA + A^\top J)$ is positive semidefinite, where J is the diagonal matrix $\text{diag}(1, -1, -1, \dots, -1)$. Another result of Stern and Wolkowicz ([18], Theorem 4.2) asserts that

$$(1.3) \quad \mathcal{Z}(\mathcal{L}) = \text{LL}(\mathcal{L}) - \pi(\mathcal{L}).$$

(Going in the reverse direction, in a recent paper, Kuzma et al., [13] have shown that for an irreducible symmetric cone \mathcal{K} , the equality $\mathcal{Z}(\mathcal{K}) = \text{LL}(\mathcal{K}) - \pi(\mathcal{K})$ holds only when \mathcal{K} is isomorphic to \mathcal{L} .) Characterizations of $\pi(\mathcal{L})$ and $\text{LL}(\mathcal{L})$ appear, respectively, in [14] and [20].

In this paper, we describe $\mathcal{Z}(\mathcal{L})$ and its interior, boundary, and dual in terms of the semidefinite cone and the so-called copositive and completely positive cones induced by \mathcal{L} (or its boundary $\partial(\mathcal{L})$), see below for the definitions. In particular, we describe the dual of $\mathcal{Z}(\mathcal{L})$ as a slice of the semidefinite cone and also of the completely positive cone of \mathcal{L} . This provides an example of an instance where a conic linear optimization problem over a completely positive cone is reduced to a semidefinite problem. To elaborate, consider \mathbb{R}^n , the Euclidean n -space of (column) vectors with the usual inner product, $\mathbb{R}^{n \times n}$, the space of all real $n \times n$ matrices with the inner product $\langle X, Y \rangle = \text{tr}(X^\top Y)$, and \mathcal{S}^n , the subspace of all real $n \times n$ symmetric matrices in $\mathbb{R}^{n \times n}$. Corresponding to a closed cone \mathcal{C} (which is not necessarily convex) in \mathbb{R}^n , let

$$\mathcal{E}_{\mathcal{C}} := \text{copos}(\mathcal{C}) := \{A \in \mathcal{S}^n : x^\top A x \geq 0 \text{ for all } x \text{ in } \mathcal{C}\}$$

denote the *copositive cone of \mathcal{C}* and

$$\mathcal{K}_{\mathcal{C}} := \text{compos}(\mathcal{C}) := \left\{ \sum_{u \in U} uu^\top : U \text{ is a finite subset of } \mathcal{C} \right\}$$

denote the *completely positive cone of \mathcal{C}* . When $\mathcal{C} = \mathbb{R}^n$, these two cones coincide with the *semidefinite cone \mathcal{S}_+^n* (consisting of all real $n \times n$ symmetric positive semidefinite matrices); when $\mathcal{C} = \mathbb{R}_+^n$, these reduce, respectively, to the (standard) copositive cone and completely positive cone. All these cones appear prominently in conic optimization. A result of Burer [5] (see also, [4, 7]) says that any nonconvex quadratic programming problem over a closed cone with additional linear and binary constraints can be reformulated as a linear program over a suitable completely positive cone. For this and other reasons, there is a strong interest in understanding copositive and completely positive cones. For the closed convex cones $\mathcal{E}_{\mathcal{C}}$ and $\mathcal{K}_{\mathcal{C}}$, various structural properties (such as the interior, boundary) as well as duality, irreducibility, and homogeneity properties, have been investigated in the literature, see for example, [19, 6, 8, 11]. Taking \mathcal{C} to be one of \mathbb{R}^n , \mathcal{L} , or $\partial(\mathcal{L})$, we show that

$$(1.4) \quad \mathcal{Z}(\mathcal{L})^* = \{B \in \mathbb{R}^{n \times n} : \langle B, I \rangle = 0, -JB \in \mathcal{K}_{\mathcal{C}}\}$$

and deduce the equality of slices

$$(1.5) \quad \{X \in \mathbb{R}^{n \times n} : \langle J, X \rangle = 0, X \in \mathcal{S}_+^n\} = \{X \in \mathbb{R}^{n \times n} : \langle J, X \rangle = 0, X \in \mathcal{K}_{\mathcal{C}}\}.$$

2. Preliminaries. In a (finite dimensional real) Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, a nonempty set \mathcal{C} is said to be a *closed cone* if it is closed and $tx \in \mathcal{C}$ whenever $x \in \mathcal{C}$ and $t \geq 0$ in \mathbb{R} . Throughout this paper, \mathcal{C} denotes a closed cone.

A nonempty set \mathcal{K} is said to be a *closed convex cone* if it is a closed cone which is also convex. Such a cone is said to be *proper* if $\mathcal{K} \cap -\mathcal{K} = \{0\}$ and has nonempty interior. Corresponding to a closed convex cone \mathcal{K} , we define its dual in \mathcal{H} as the set

$$\mathcal{K}^* = \{x \in \mathcal{H} : \langle x, y \rangle \geq 0, \forall y \in \mathcal{K}\}.$$

We say that a linear transformation $A : \mathcal{H} \rightarrow \mathcal{H}$ is *copositive* on \mathcal{K} if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{K}$. We also let $\pi(\mathcal{K}) = \{A : A(\mathcal{K}) \subseteq \mathcal{K}\}$, where A denotes a linear transformation on \mathcal{H} . For a set S in \mathcal{H} , we denote the closure, interior, and the boundary by \bar{S} , S° , and $\partial(S)$, respectively.

We will be considering closed convex cones in the space $\mathcal{H} = \mathbb{R}^n$ which carries the usual inner product and in the space $\mathbb{R}^{n \times n}$ which carries the inner product $\langle X, Y \rangle := \text{tr}(X^\top Y)$, where the trace of a square matrix is the sum of its diagonal entries. In $\mathbb{R}^{n \times n}$, \mathcal{S}^n denotes the subspace of all symmetric matrices and \mathcal{A}^n denotes the subspace of all skew-symmetric matrices. We note that $\mathbb{R}^{n \times n}$ is the orthogonal direct sum of \mathcal{S}^n and \mathcal{A}^n .

We recall some (easily verifiable) properties of the Lorentz cone \mathcal{L} given by (1.2). \mathcal{L} is a self-dual cone in \mathbb{R}^n , that is, $\mathcal{L}^* = \mathcal{L}$; its interior and boundary are given, respectively, by

$$\mathcal{L}^\circ = \{(t, u)^\top : t > \|u\|\},$$

$$\partial(\mathcal{L}) = \{(t, u)^\top : t = \|u\|\} = \{\alpha(1, u)^\top : \alpha \geq 0, \|u\| = 1\}.$$

We also have

$$(2.1) \quad [0 \neq x, y \in \mathcal{L}, \langle x, y \rangle = 0] \Rightarrow x = \alpha(1, u)^\top \text{ and } y = \beta(1, -u)^\top, \\ \text{for some } \alpha, \beta > 0 \text{ and } \|u\| = 1.$$

For a closed cone \mathcal{C} in \mathbb{R}^n , we consider the copositive cone $\mathcal{E}_{\mathcal{C}}$ and the completely positive cone $\mathcal{K}_{\mathcal{C}}$ (defined in the introduction). Note that these are cones of symmetric matrices.

In the Hilbert space \mathcal{S}^n (which carries the inner product from $\mathbb{R}^{n \times n}$), the following hold.

- (1) $\mathcal{K}_{\mathcal{C}}$ is the dual cone of $\mathcal{E}_{\mathcal{C}}$ [19].
- (2) When $\mathcal{C} - \mathcal{C} = \mathbb{R}^n$, both $\mathcal{E}_{\mathcal{C}}$ and $\mathcal{K}_{\mathcal{C}}$ are proper cones ([10], Proposition 2.2). In particular, this holds when \mathcal{C} is one of \mathbb{R}^n , \mathcal{L} , or $\partial(\mathcal{L})$.
- (3) We have $\mathcal{S}_+^n = \mathcal{E}_{\mathbb{R}^n} \subset \mathcal{E}_{\mathcal{L}} \subset \mathcal{E}_{\partial(\mathcal{L})}$, or equivalently, $\mathcal{K}_{\partial(\mathcal{L})} \subset \mathcal{K}_{\mathcal{L}} \subset \mathcal{K}_{\mathbb{R}^n} = \mathcal{S}_+^n$.

3. Main results. In this section, we provide a closure-free description of $\mathcal{Z}(\mathcal{L})$ and, additionally, describe the dual, interior, and the boundary of $\mathcal{Z}(\mathcal{L})$. We recall that $J = \text{diag}(1, -1, -1, \dots, -1)$ and \mathcal{A}^n denotes the set of all skew-symmetric matrices in $\mathbb{R}^{n \times n}$.

THEOREM 3.1. *Let \mathcal{C} denote one of \mathbb{R}^n , \mathcal{L} , or $\partial(\mathcal{L})$. Then,*

$$(3.1) \quad \mathcal{Z}(\mathcal{L}) = \mathbb{R}I - J(\mathcal{E}_{\mathcal{C}} + \mathcal{A}^n).$$

Proof. Let $A \in \mathcal{Z}(\mathcal{L})$. From the result of Stern and Wolkowicz [17] mentioned in the introduction, we have

$$2\gamma J - (JA + A^\top J) = 2P$$

for some $\gamma \in \mathbb{R}$ and $P \in \mathcal{S}_+^n$. Hence, $JA + (JA)^\top = 2(\gamma J - P)$, which implies

$$(3.2) \quad 2JA = JA + (JA)^\top - [(JA)^\top - JA] = 2(\gamma J - P) - 2Q,$$

where $2Q = (JA)^\top - JA$ is skew-symmetric. Since $J^2 = I$, this leads to

$$A = \gamma I - J(P + Q),$$

where $P \in \mathcal{S}_+^n$ and $Q \in \mathcal{A}^n$. As $\mathcal{S}_+^n \subset \mathcal{E}_{\mathcal{L}} \subset \mathcal{E}_{\partial(\mathcal{L})}$, this proves that

$$(3.3) \quad \mathcal{Z}(\mathcal{L}) \subseteq \mathbb{R}I - J(\mathcal{S}_+^n + \mathcal{A}^n) \subseteq \mathbb{R}I - J(\mathcal{E}_{\mathcal{L}} + \mathcal{A}^n) \subseteq \mathbb{R}I - J(\mathcal{E}_{\partial(\mathcal{L})} + \mathcal{A}^n).$$

Now, to see the reverse inclusions, suppose $A = \gamma I - J(P + Q)$ for some $\gamma \in \mathbb{R}$, $P \in \mathcal{E}_{\partial(\mathcal{L})}$, and Q skew-symmetric. Let $0 \neq x, y \in \mathcal{L}$ with $\langle x, y \rangle = 0$. By (2.1), x and y are in $\partial(\mathcal{L})$, and Jy is a positive multiple of x . Hence, $\langle Px, Jy \rangle \geq 0$ as $P \in \mathcal{E}_{\partial(\mathcal{L})}$ and $\langle Qx, Jy \rangle = 0$ as Q is skew-symmetric. Thus,

$$\langle Ax, y \rangle = \gamma \langle x, y \rangle - \langle JPx, y \rangle + \langle JQx, y \rangle = -\langle Px, Jy \rangle + \langle Qx, Jy \rangle \leq 0.$$

This shows that $A \in \mathcal{Z}(\mathcal{L})$ and so, inclusions in (3.3) turn into equalities. Thus, we have (3.1). \square

REMARKS. From the above theorem, we have

$$\mathbb{R}I - J(\mathcal{S}_+^n + \mathcal{A}^n) = \mathbb{R}I - J(\mathcal{E}_{\mathcal{L}} + \mathcal{A}^n) = \mathbb{R}I - J(\mathcal{E}_{\partial(\mathcal{L})} + \mathcal{A}^n).$$

Multiplying throughout by J and noting $-\mathcal{A}^n = \mathcal{A}^n$, we get the equality of sets

$$(\mathbb{R}J - \mathcal{S}_+^n) + \mathcal{A}^n = (\mathbb{R}J - \mathcal{E}_{\mathcal{L}}) + \mathcal{A}^n = (\mathbb{R}J - \mathcal{E}_{\partial(\mathcal{L})}) + \mathcal{A}^n,$$

where each set is a sum of \mathcal{A}^n and a subset of \mathcal{S}^n . Since $\mathbb{R}^{n \times n} = \mathcal{S}^n + \mathcal{A}^n$ is an (orthogonal) direct sum decomposition, we see that

$$(3.4) \quad \mathbb{R}J - \mathcal{S}_+^n = \mathbb{R}J - \mathcal{E}_{\mathcal{L}} = \mathbb{R}J - \mathcal{E}_{\partial(\mathcal{L})}.$$

These equalities can also be established via different arguments. A result of Loewy and Schneider [14] asserts that *A symmetric matrix X is copositive on \mathcal{L} if and only if there exists $\mu \geq 0$ such that $X - \mu J \in \mathcal{S}_+^n$* . (This is essentially a consequence of the so-called S-Lemma [15]: If A and B are two symmetric matrices with $\langle Ax_0, x_0 \rangle > 0$ for some x_0 and $\langle Ax, x \rangle \geq 0 \Rightarrow \langle Bx, x \rangle \geq 0$, then there exists $\mu \geq 0$ such that $B - \mu A$ is positive semidefinite.) This result gives the equality

$$\mathcal{E}_{\mathcal{L}} = \mathcal{S}_+^n + \mathbb{R}_+ J,$$

and consequently, $\mathbb{R}J - \mathcal{S}_+^n = \mathbb{R}J - \mathcal{E}_{\mathcal{L}}$. The equality

$$\mathcal{E}_{\partial(\mathcal{L})} = \mathcal{S}_+^n + \mathbb{R}J$$

can be seen via an application of Finsler's theorem [15] that says that if A and B are two symmetric matrices with $[x \neq 0, \langle Ax, x \rangle = 0] \Rightarrow \langle Bx, x \rangle > 0$, then there exists $\mu \in \mathbb{R}$ such that $B + \mu A$ is positive semidefinite. (For $M \in \mathcal{E}_{\partial(\mathcal{L})}$ and vectors $u, v \in \mathcal{L}^\circ$, one has $\langle Jx, x \rangle = 0 \Rightarrow \langle M_k x, x \rangle > 0$, where k is a natural number and $M_k := M + \frac{1}{k} uv^\top$. When $M_k + \mu_k J$ is positive semidefinite for all k , it follows that the sequence μ_k is bounded. One can then use a limiting argument.) From this equality, one gets $\mathbb{R}J - \mathcal{S}_+^n = \mathbb{R}J - \mathcal{E}_{\partial(\mathcal{L})}$.

Our next result deals with the dual of $\mathcal{Z}(\mathcal{L})$.

THEOREM 3.2. Let \mathcal{C} denote one of \mathbb{R}^n , \mathcal{L} , or $\partial(\mathcal{L})$. Then,

$$\mathcal{Z}(\mathcal{L})^* = \{B \in \mathbb{R}^{n \times n} : \langle B, I \rangle = 0, -JB \in \mathcal{K}_{\mathcal{C}}\}.$$

In particular, (1.5) holds.

Proof. We fix \mathcal{C} . From (3.1), we see that $B \in \mathcal{Z}(\mathcal{L})^*$ if and only if

$$0 \leq \langle B, \gamma I - J(P + Q) \rangle$$

for all γ real, P in $\mathcal{E}_{\mathcal{C}}$, and Q in \mathcal{A}^n . Clearly, this holds if and only if

$$\langle B, I \rangle = 0, \quad \langle -JB, P \rangle \geq 0, \quad \text{and} \quad \langle -JB, Q \rangle = 0$$

for all γ , P , and Q specified above. Now, with the observation that a (real) matrix is orthogonal to all skew-symmetric matrices in $\mathbb{R}^{n \times n}$ if and only if it is symmetric, this further simplifies to

$$\langle B, I \rangle = 0 \quad \text{and} \quad -JB \in \mathcal{E}_{\mathcal{C}}^*,$$

where $\mathcal{E}_{\mathcal{C}}^*$ is the dual of $\mathcal{E}_{\mathcal{C}}$ computed in \mathcal{S}^n . Since $\mathcal{K}_{\mathcal{C}} = \mathcal{E}_{\mathcal{C}}^*$ in \mathcal{S}^n , we see that $B \in \mathcal{Z}(\mathcal{L})^*$ if and only if $\langle B, I \rangle = 0$ and $-JB \in \mathcal{K}_{\mathcal{C}}$. This completes the proof. \square

We remark that (1.5) can be deduced directly from (3.4) by taking the duals in \mathcal{S}^n .

In our final result, we describe the interior and boundary of $\mathcal{Z}(\mathcal{L})$. First, we recall some definitions from [9]. Let

$$\Omega := \{(x, y) \in \mathcal{L} \times \mathcal{L} : \|x\| = 1 = \|y\| \text{ and } \langle x, y \rangle = 0\}.$$

It is easy to see that Ω is compact and, from (2.1),

$$(3.5) \quad \Omega = \{(x, Jx) : x \in \partial(\mathcal{L}), \|x\| = 1\}.$$

For any $A \in \mathbb{R}^{n \times n}$, let

$$\gamma(A) := \max \{\langle Ax, y \rangle : (x, y) \in \Omega\}.$$

Note that $A \in \mathcal{Z}(\mathcal{L})$ if and only if $\gamma(A) \leq 0$. We say that $A \in \mathbb{R}^{n \times n}$ is a *strict- \mathcal{Z} -transformation on \mathcal{L}* if

$$[0 \neq x, y \in \mathcal{L}, \langle x, y \rangle = 0] \Rightarrow \langle Ax, y \rangle < 0.$$

The set of all such transformations is denoted by $str(\mathcal{Z}(\mathcal{L}))$. For $A \in \mathbb{R}^{n \times n}$, the following statements are shown in [9], Theorem 3.1:

$$\gamma(A) < 0 \iff A \in \mathcal{Z}(\mathcal{L})^\circ \iff A \in str(\mathcal{Z}(\mathcal{L}))$$

and

$$\gamma(A) = 0 \iff A \in \partial(\mathcal{Z}(\mathcal{L})).$$

Recall that $\mathcal{E}_{\mathcal{L}}$ consists of all symmetric matrices that are copositive on \mathcal{L} . We say that a symmetric matrix P is *strictly copositive on \mathcal{L}* if $0 \neq x \in \mathcal{L} \Rightarrow \langle Px, x \rangle > 0$; the set of all such matrices is denoted by $str(\mathcal{E}_{\mathcal{L}})$. Similarly, one defines $str(\mathcal{E}_{\partial(\mathcal{L})})$.

COROLLARY 3.3. *The following statements hold:*

$$\mathcal{Z}(\mathcal{L})^\circ = \text{str}(\mathcal{Z}(\mathcal{L})) = \mathbb{R}I - J \left(\text{str}(\mathcal{E}_{\partial(\mathcal{L})}) + \mathcal{A}^n \right)$$

and

$$\partial(\mathcal{Z}(\mathcal{L})) = \mathbb{R}I - J \left(\partial_*(\mathcal{E}_{\partial(\mathcal{L})}) + \mathcal{A}^n \right),$$

where $\partial_*(\mathcal{E}_{\partial(\mathcal{L})})$ denotes the boundary of $\mathcal{E}_{\partial(\mathcal{L})}$ in \mathcal{S}^n .

Proof. We first deal with the interior of $\mathcal{Z}(\mathcal{L})$. The equality

$$\{A \in \mathbb{R}^{n \times n} : \gamma(A) < 0\} = \mathcal{Z}(\mathcal{L})^\circ = \text{str}(\mathcal{Z}(\mathcal{L}))$$

has already been observed in [9], Theorem 3.1. To see the first assertion, we show that $\gamma(A) < 0$ if and only if $A = \theta I - J(P + Q)$ for some $\theta \in \mathbb{R}$, P (symmetric) strictly copositive on $\partial(\mathcal{L})$, and Q skew-symmetric. Suppose $\gamma(A) < 0$. Then, for any $\theta \in \mathbb{R}$,

$$\max \{ \langle (A - \theta I)x, y \rangle : (x, y) \in \Omega \} < 0,$$

which, from (3.5) becomes

$$\min \{ \langle J(\theta I - A)x, x \rangle : x \in \partial(\mathcal{L}), \|x\| = 1 \} > 0.$$

Now, fix θ and let $J(\theta I - A) = P + Q$, where $P \in \mathcal{S}^n$ and $Q \in \mathcal{A}^n$. As $\langle Qx, x \rangle = 0$ for any x , the above inequality implies that $\min \{ \langle Px, x \rangle : x \in \partial(\mathcal{L}), \|x\| = 1 \} > 0$. This proves that P is strictly copositive on $\partial(\mathcal{L})$. Rewriting $J(\theta I - A) = P + Q$, we see that $A = \theta I - J(P + Q)$ which is of the required form.

To see the converse, suppose $A = \theta I - J(P + Q)$, where $\theta \in \mathbb{R}$, P (symmetric) strictly copositive on $\partial(\mathcal{L})$, and Q skew-symmetric. Using (3.5), we can easily verify that $\gamma(A) < 0$. Thus, $A \in \text{str}(\mathcal{Z}(\mathcal{L}))$.

An argument similar to the above will show that $\gamma(A) = 0$ if and only if $A = \theta I - J(P + Q)$ for some $\theta \in \mathbb{R}$, $P \in \partial_*(\mathcal{E}_{\partial(\mathcal{L})})$, and Q skew-symmetric. This gives the statement regarding the boundary of $\mathcal{Z}(\mathcal{L})$. \square

We end the paper with a remark dealing with conic linear programs. Motivated by the result of Burer (mentioned in the introduction), we consider a conic linear program on a completely positive cone $\mathcal{K}_{\mathcal{C}}$ (where \mathcal{C} is a closed cone):

$$\min \{ \langle c, x \rangle : Ax = b, x \in \mathcal{K}_{\mathcal{C}} \}.$$

While such a problem is generally hard to solve, we ask: (When) can we replace $\mathcal{K}_{\mathcal{C}}$ by \mathcal{S}_+^n , and thus, reduce the above problem to the semidefinite programming problem $\min \{ \langle c, x \rangle : Ax = b, x \in \mathcal{S}_+^n \}$? Just replacing $\mathcal{K}_{\mathcal{C}}$ by \mathcal{S}_+^n without handling the constraint $Ax = b$ is not viable as $\mathcal{K}_{\mathcal{C}} = \mathcal{S}_+^n$ if and only if $\mathcal{C} \cup -\mathcal{C} = \mathbb{R}^n$ (which fails to hold when $n > 1$ and \mathcal{C} is pointed), see [11]. While we do not answer this broad question, we point out, as a consequence of (1.5), that for any $C \in \mathcal{S}^n$,

$$\min \{ \langle C, X \rangle : \langle X, J \rangle = 0, X \in \mathcal{K}_{\mathcal{L}} \} = \min \{ \langle C, X \rangle : \langle X, J \rangle = 0, X \in \mathcal{S}_+^n \}.$$

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